1 □ FUNCTIONS AND MODELS

1.1 Four Ways to Represent a Function

In exercises requiring estimations or approximations, your answers may vary slightly from the answers given here.

1. (a) The point $(-1, -2)$ is on the graph of f, so $f(-1) = -2$.

(b) When $x = 2$, y is about 2.8, so $f(2) \approx 2.8$.

(c) $f(x) = 2$ is equivalent to $y = 2$. When $y = 2$, we have $x = -3$ and $x = 1$.

(d) Reasonable estimates for x when $y = 0$ are $x = -2.5$ and $x = 0.3$.

(e) The domain of f consists of all x-values on the graph of f. For this function, the domain is $-3 \le x \le 3$, or $[-3, 3]$. The range of f consists of all y-values on the graph of f. For this function, the range is $-2 \le y \le 3$, or $[-2, 3]$.

(f) As x increases from -1 to 3, y increases from -2 to 3. Thus, f is increasing on the interval $[-1, 3]$.

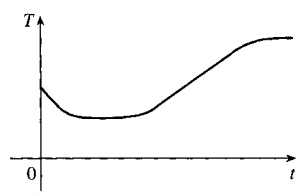
3. From Figure 1 in the text, the lowest point occurs at about $(t, a) = (12, -85)$. The highest point occurs at about $(17, 115)$. Thus, the range of the vertical ground acceleration is $-85 \le a \le 115$. In Figure 11, the range of the north-south acceleration is approximately $-325 \le a \le 485$. In Figure 12, the range of the east-west acceleration is approximately $-210 \le a \le 200$.

5. No, the curve is not the graph of a function because a vertical line intersects the curve more than once. Hence, the curve fails the Vertical Line Test.

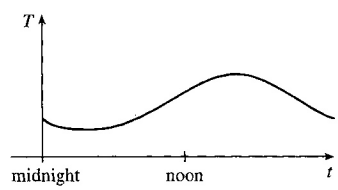
7. Yes, the curve is the graph of a function because it passes the Vertical Line Test. The domain is $[-3, 2]$ and the range is $[-3, -2) \cup [-1, 3]$.

9. The person's weight increased to about 160 pounds at age 20 and stayed fairly steady for 10 years. The person's weight dropped to about 120 pounds for the next 5 years, then increased rapidly to about 170 pounds. The next 30 years saw a gradual increase to 190 pounds. Possible reasons for the drop in weight at 30 years of age: diet, exercise, health problems.

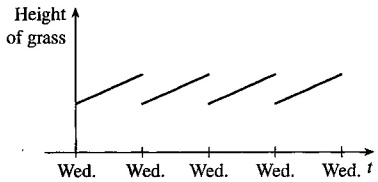
11. The water will cool down almost to freezing as the ice melts. Then, when the ice has melted, the water will slowly warm up to room temperature.



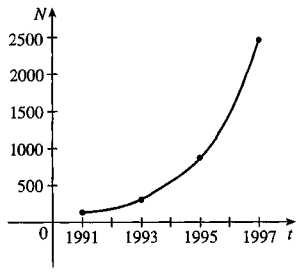
13. Of course, this graph depends strongly on the geographical location!



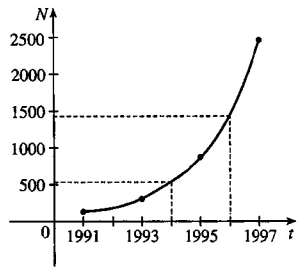
15.



17. (a)



(b) From the graph, we estimate the number of cell-phone subscribers in Malaysia to be about 540 in 1994 and 1450 in 1996.



19. $f(x) = 3x^2 - x + 2$.

$f(2) = 3(2)^2 - 2 + 2 = 12 - 2 + 2 = 12$.

$f(-2) = 3(-2)^2 - (-2) + 2 = 12 + 2 + 2 = 16$.

$f(a) = 3a^2 - a + 2$.

$f(-a) = 3(-a)^2 - (-a) + 2 = 3a^2 + a + 2$.

$f(a+1) = 3(a+1)^2 - (a+1) + 2 = 3(a^2 + 2a + 1) - a - 1 + 2 = 3a^2 + 6a + 3 - a + 1 = 3a^2 + 5a + 4$.

$2f(a) = 2 \cdot f(a) = 2(3a^2 - a + 2) = 6a^2 - 2a + 4$.

$f(2a) = 3(2a)^2 - (2a) + 2 = 3(4a^2) - 2a + 2 = 12a^2 - 2a + 2$.

$f(a^2) = 3(a^2)^2 - (a^2) + 2 = 3(a^4) - a^2 + 2 = 3a^4 - a^2 + 2$.

$$\begin{aligned}[f(a)]^2 &= \left[3a^2 - a + 2\right]^2 = \left(3a^2 - a + 2\right)\left(3a^2 - a + 2\right) \\ &= 9a^4 - 3a^3 + 6a^2 - 3a^3 + a^2 - 2a + 6a^2 - 2a + 4 = 9a^4 - 6a^3 + 13a^2 - 4a + 4.\end{aligned}$$

$f(a+h) = 3(a+h)^2 - (a+h) + 2 = 3(a^2 + 2ah + h^2) - a - h + 2 = 3a^2 + 6ah + 3h^2 - a - h + 2$.

21. $f(x) = x - x^2$, so

$f(2+h) = 2 + h - (2+h)^2 = 2 + h - (4 + 4h + h^2) = 2 + h - 4 - 4h - h^2 = -\left(h^2 + 3h + 2\right)$,

$f(x+h) = x + h - (x+h)^2 = x + h - (x^2 + 2xh + h^2) = x + h - x^2 - 2xh - h^2$, and

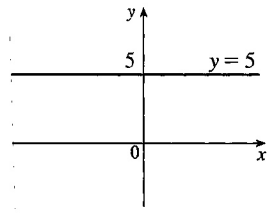
$$\frac{f(x+h) - f(x)}{h} = \frac{x + h - x^2 - 2xh - h^2 - x + x^2}{h} = \frac{h - 2xh - h^2}{h} = \frac{h(1 - 2x - h)}{h} = 1 - 2x - h.$$

23. $f(x) = x/(3x - 1)$ is defined for all x except when $0 = 3x - 1 \quad \Leftrightarrow \quad x = \frac{1}{3}$, so the domain is $\left\{x \in \mathbb{R} \mid x \neq \frac{1}{3}\right\} = \left(-\infty, \frac{1}{3}\right) \cup \left(\frac{1}{3}, \infty\right)$.

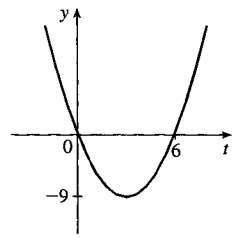
25. $f(t) = \sqrt{t} + \sqrt[3]{t}$ is defined when $t \geq 0$. These values of t give real number results for $\sqrt{t}$, whereas any value of t gives a real number result for $\sqrt[3]{t}$. The domain is $[0, \infty)$.

27. $h(x) = 1 \big/ \sqrt[4]{x^2 - 5x}$ is defined when $x^2 - 5x > 0 \quad \Leftrightarrow \quad x(x-5) > 0$. Note that $x^2 - 5x \neq 0$ since that would result in division by zero. The expression $x(x-5)$ is positive if $x < 0$ or $x > 5$. (See Appendix A for methods for solving inequalities.) Thus, the domain is $(-\infty, 0) \cup (5, \infty)$.

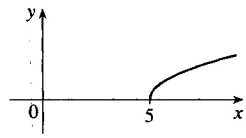
29. $f(x) = 5$ is defined for all real numbers, so the domain is $\mathbb{R}$, or $(-\infty, \infty)$. The graph of f is a horizontal line with y-intercept 5.



31. $f(t) = t^2 - 6t$ is defined for all real numbers, so the domain is $\mathbb{R}$, or $(-\infty, \infty)$. The graph of f is a parabola opening upward since the coefficient of t^2 is positive. To find the t-intercepts, let $y = 0$ and solve for t. $0 = t^2 - 6t = t(t-6) \quad \Rightarrow \quad t = 0$ and $t = 6$. The t-coordinate of the vertex is halfway between the t-intercepts, that is, at $t = 3$. Since $f(3) = 3^2 - 6 \cdot 3 = -9$, the vertex is $(3, -9)$.

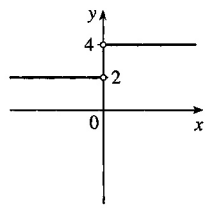


33. $g(x) = \sqrt{x-5}$ is defined when $x - 5 \geq 0$ or $x \geq 5$, so the domain is $[5, \infty)$. Since $y = \sqrt{x-5} \quad \Rightarrow \quad y^2 = x - 5 \quad \Rightarrow \quad x = y^2 + 5$, we see that g is the top half of a parabola.



35. $G(x) = \dfrac{3x + |x|}{x}$. Since $|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$, we have

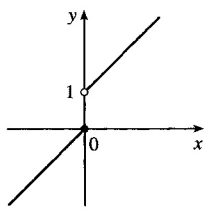
$$G(x) = \begin{cases} \dfrac{3x + x}{x} & \text{if } x > 0 \\ \dfrac{3x - x}{x} & \text{if } x < 0 \end{cases} = \begin{cases} \dfrac{4x}{x} & \text{if } x > 0 \\ \dfrac{2x}{x} & \text{if } x < 0 \end{cases} = \begin{cases} 4 & \text{if } x > 0 \\ 2 & \text{if } x < 0 \end{cases}$$



Note that G is not defined for $x = 0$. The domain is $(-\infty, 0) \cup (0, \infty)$.

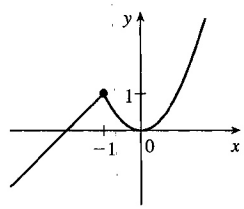
37. $f(x) = \begin{cases} x & \text{if } x \leq 0 \\ x + 1 & \text{if } x > 0 \end{cases}$

Domain is $\mathbb{R}$, or $(-\infty, \infty)$.



39. $f(x) = \begin{cases} x + 2 & \text{if } x \leq -1 \\ x^2 & \text{if } x > -1 \end{cases}$

Note that for $x = -1$, both $x + 2$ and x^2 are equal to 1. Domain is $\mathbb{R}$.



41. Recall that the slope m of a line between the two points (x_1, y_1) and (x_2, y_2) is $m = \dfrac{y_2 - y_1}{x_2 - x_1}$ and an equation of the line connecting those two points is $y - y_1 = m(x - x_1)$. The slope of this line segment is $\dfrac{-6-1}{4-(-2)} = -\dfrac{7}{6}$, so an equation is $y - 1 = -\frac{7}{6}(x+2)$. The function is $f(x) = -\frac{7}{6}x - \frac{4}{3}$, $-2 \le x \le 4$.

43. We need to solve the given equation for y. $x + (y-1)^2 = 0 \;\Leftrightarrow\; (y-1)^2 = -x \;\Leftrightarrow\; y - 1 = \pm\sqrt{-x} \;\Leftrightarrow\; y = 1 \pm \sqrt{-x}$. The expression with the positive radical represents the top half of the parabola, and the one with the negative radical represents the bottom half. Hence, we want $f(x) = 1 - \sqrt{-x}$. Note that the domain is $x \le 0$.

45. For $-1 \le x \le 2$, the graph is the line with slope 1 and y-intercept 1, that is, the line $y = x + 1$. For $2 < x \le 4$, the graph is the line with slope $-\frac{3}{2}$ and x-intercept 4 [which corresponds to the point $(4, 0)$], so $y - 0 = -\frac{3}{2}(x-4) = -\frac{3}{2}x + 6$. So the function is $f(x) = \begin{cases} x+1 & \text{if } -1 \le x \le 2 \\ -\frac{3}{2}x + 6 & \text{if } 2 < x \le 4 \end{cases}$

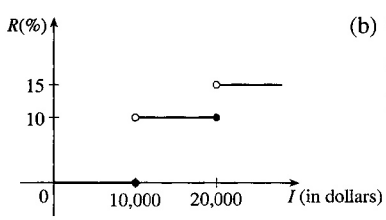
47. Let the length and width of the rectangle be L and W. Then the perimeter is $2L + 2W = 20$ and the area is $A = LW$. Solving the first equation for W in terms of L gives $W = \dfrac{20 - 2L}{2} = 10 - L$. Thus, $A(L) = L(10 - L) = 10L - L^2$. Since lengths are positive, the domain of A is $0 < L < 10$. If we further restrict L to be larger than W, then $5 < L < 10$ would be the domain.

49. Let the length of a side of the equilateral triangle be x. Then by the Pythagorean Theorem, the height y of the triangle satisfies $y^2 + \left(\frac{1}{2}x\right)^2 = x^2$, so that $y^2 = x^2 - \frac{1}{4}x^2 = \frac{3}{4}x^2$ and $y = \frac{\sqrt{3}}{2}x$. Using the formula for the area A of a triangle, $A = \frac{1}{2}(\text{base})(\text{height})$, we obtain $A(x) = \frac{1}{2}(x)\left(\frac{\sqrt{3}}{2}x\right) = \frac{\sqrt{3}}{4}x^2$, with domain $x > 0$.

51. Let each side of the base of the box have length x, and let the height of the box be h. Since the volume is 2, we know that $2 = hx^2$, so that $h = 2/x^2$, and the surface area is $S = x^2 + 4xh$. Thus, $S(x) = x^2 + 4x(2/x^2) = x^2 + (8/x)$, with domain $x > 0$.

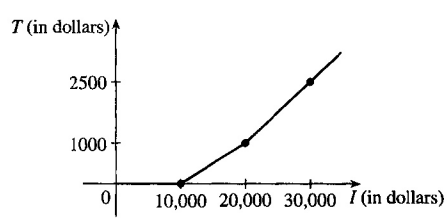
53. The height of the box is x and the length and width are $L = 20 - 2x$, $W = 12 - 2x$. Then $V = LWx$ and so $V(x) = (20 - 2x)(12 - 2x)(x) = 4(10 - x)(6 - x)(x) = 4x(60 - 16x + x^2) = 4x^3 - 64x^2 + 240x$. The sides L, W, and x must be positive. Thus, $L > 0 \;\Leftrightarrow\; 20 - 2x > 0 \;\Leftrightarrow\; x < 10$; $W > 0 \;\Leftrightarrow\; 12 - 2x > 0 \;\Leftrightarrow\; x < 6$; and $x > 0$. Combining these restrictions gives us the domain $0 < x < 6$.

55. (a)



(b) On \$14,000, tax is assessed on \$4000, and 10%(\$4000) = \$400. On \$26,000, tax is assessed on \$16,000, and 10%(\$10,000) + 15%(\$6000) = \$1000 + \$900 = \$1900.

(c) As in part (b), there is \$1000 tax assessed on \$20,000 of income, so the graph of T is a line segment from $(10{,}000, 0)$ to $(20{,}000, 1000)$. The tax on \$30,000 is \$2500, so the graph of T for $x > 20{,}000$ is the ray with initial point $(20{,}000, 1000)$ that passes through $(30{,}000, 2500)$.



57. f is an odd function because its graph is symmetric about the origin. g is an even function because its graph is symmetric with respect to the y-axis.

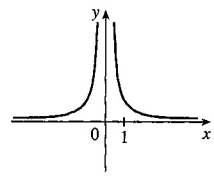
59. (a) Because an even function is symmetric with respect to the y-axis, and the point $(5, 3)$ is on the graph of this even function, the point $(-5, 3)$ must also be on its graph.

(b) Because an odd function is symmetric with respect to the origin, and the point $(5, 3)$ is on the graph of this odd function, the point $(-5, -3)$ must also be on its graph.

61. $f(x) = x^{-2}$.

$$f(-x) = (-x)^{-2} = \frac{1}{(-x)^2} = \frac{1}{x^2}$$
$$= x^{-2} = f(x)$$

So f is an even function.

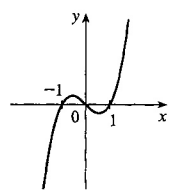


63. $f(x) = x^2 + x$, so $f(-x) = (-x)^2 + (-x) = x^2 - x$. Since this is neither $f(x)$ nor $-f(x)$, the function f is neither even nor odd.

65. $f(x) = x^3 - x$.

$$f(-x) = (-x)^3 - (-x) = -x^3 + x$$
$$= -\left(x^3 - x\right) = -f(x)$$

So f is odd.



1.2 Mathematical Models: A Catalog of Essential Functions

1. (a) $f(x) = \sqrt[5]{x}$ is a root function with $n = 5$.

(b) $g(x) = \sqrt{1 - x^2}$ is an algebraic function because it is a root of a polynomial.

(c) $h(x) = x^9 + x^4$ is a polynomial of degree 9.

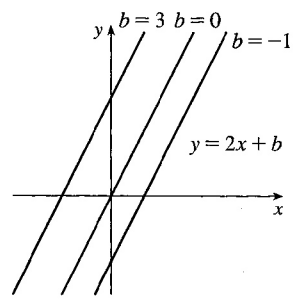
(d) $r(x) = \dfrac{x^2 + 1}{x^3 + x}$ is a rational function because it is a ratio of polynomials.

(e) $s(x) = \tan 2x$ is a trigonometric function.

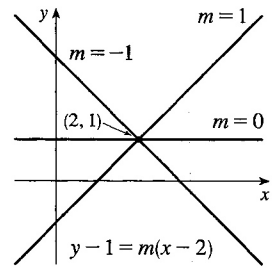
(f) $t(x) = \log_{10} x$ is a logarithmic function.

3. We notice from the figure that g and h are even functions (symmetric with respect to the y-axis) and that f is an odd function (symmetric with respect to the origin). So (b) $\left[y = x^5\right]$ must be f. Since g is flatter than h near the origin, we must have (c) $\left[y = x^8\right]$ matched with g and (a) $\left[y = x^2\right]$ matched with h.

5. (a) An equation for the family of linear functions with slope 2 is $y = f(x) = 2x + b$, where b is the y-intercept.

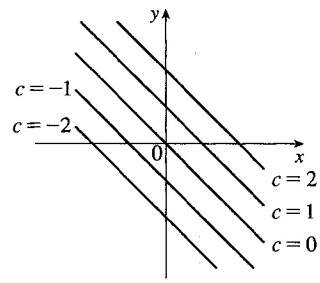


(b) $f(2) = 1$ means that the point $(2, 1)$ is on the graph of f. We can use the point-slope form of a line to obtain an equation for the family of linear functions through the point $(2, 1)$. $y - 1 = m(x - 2)$, which is equivalent to $y = mx + (1 - 2m)$ in slope-intercept form.

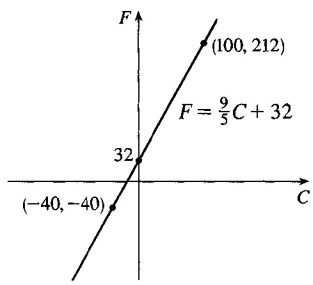


(c) To belong to both families, an equation must have slope $m = 2$, so the equation in part (b), $y = mx + (1 - 2m)$, becomes $y = 2x - 3$. It is the *only* function that belongs to both families.

7. All members of the family of linear functions $f(x) = c - x$ have graphs that are lines with slope -1. The y-intercept is c.



9. (a)



(b) The slope of $\frac{9}{5}$ means that F increases $\frac{9}{5}$ degrees for each increase of 1 °C. (Equivalently, F increases by 9 when C increases by 5 and F decreases by 9 when C decreases by 5.) The F-intercept of 32 is the Fahrenheit temperature corresponding to a Celsius temperature of 0.

11. (a) Using N in place of x and T in place of y, we find the slope to be $\dfrac{T_2 - T_1}{N_2 - N_1} = \dfrac{80 - 70}{173 - 113} = \dfrac{10}{60} = \dfrac{1}{6}$. So a linear equation is $T - 80 = \frac{1}{6}(N - 173) \;\Leftrightarrow\; T - 80 = \frac{1}{6}N - \frac{173}{6} \;\Leftrightarrow\; T = \frac{1}{6}N + \frac{307}{6}$ $\left[\frac{307}{6} = 51.1\overline{6}\right]$.

(b) The slope of $\frac{1}{6}$ means that the temperature in Fahrenheit degrees increases one-sixth as rapidly as the number of cricket chirps per minute. Said differently, each increase of 6 cricket chirps per minute corresponds to an increase of 1 °F.

(c) When $N = 150$, the temperature is given approximately by $T = \frac{1}{6}(150) + \frac{307}{6} = 76.1\overline{6}\,°\text{F} \approx 76\,°\text{F}$.

13. (a) We are given $\dfrac{\text{change in pressure}}{\text{10 feet change in depth}} = \dfrac{4.34}{10} = 0.434$. Using P for pressure and d for depth with the point $(d, P) = (0, 15)$, we have the slope-intercept form of the line, $P = 0.434d + 15$.

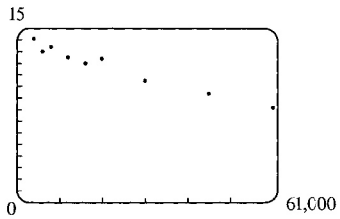
(b) When $P = 100$, then $100 = 0.434d + 15 \;\Leftrightarrow\; 0.434d = 85 \;\Leftrightarrow\; d = \frac{85}{0.434} \approx 195.85$ feet. Thus, the pressure is 100 lb/in^2 at a depth of approximately 196 feet.

15. (a) The data appear to be periodic and a sine or cosine function would make the best model. A model of the form $f(x) = a\cos(bx) + c$ seems appropriate.

(b) The data appear to be decreasing in a linear fashion. A model of the form $f(x) = mx + b$ seems appropriate.

Some values are given to many decimal places These are the results given by several computer algebra systems — rounding is left to the reader.

17. (a)

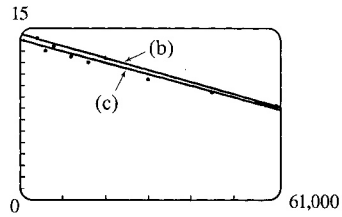


A linear model does seem appropriate.

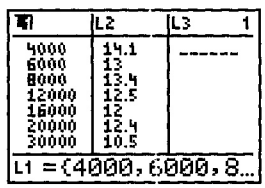
(b) Using the points $(4000, 14.1)$ and $(60{,}000, 8.2)$, we obtain

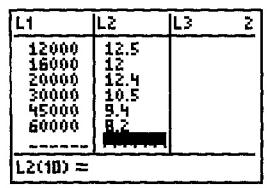
$$y - 14.1 = \frac{8.2 - 14.1}{60{,}000 - 4000}(x - 4000) \text{ or, equivalently,}$$

$$y \approx -0.000105357x + 14.521429.$$

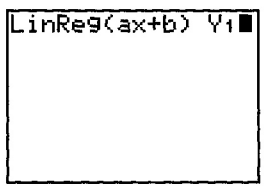


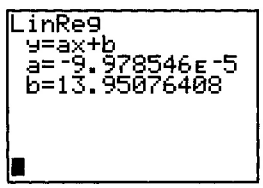
(c) Using a computing device, we obtain the least squares regression line $y = -0.0000997855x + 13.950764$. The following commands and screens illustrate how to find the least squares regression line on a TI-83 Plus. Enter the data into list one (L1) and list two (L2). Press [STAT] [1] to enter the editor.

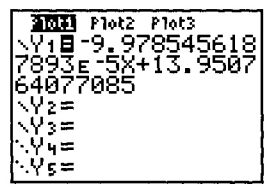




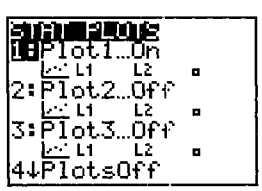
Find the regession line and store it in Y_1. Press [2nd] [QUIT] [STAT] [▸] [4] [VARS] [▸] [1] [1] [ENTER].

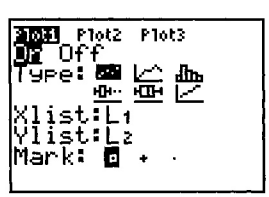




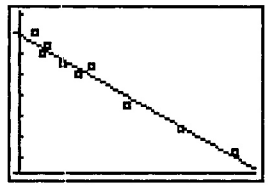


Note from the last figure that the regression line has been stored in Y_1 and that Plot1 has been turned on (Plot1 is highlighted). You can turn on Plot1 from the Y= menu by placing the cursor on Plot1 and pressing [ENTER] or by pressing [2nd] [STAT PLOT] [1] [ENTER].





Now press ZOOM 9 to produce a graph of the data and the regression line. Note that choice 9 of the ZOOM menu automatically selects a window that displays all of the data.

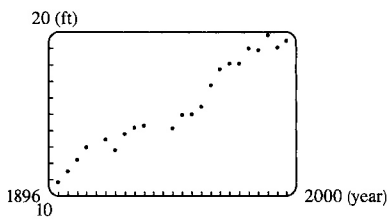


(d) When $x = 25{,}000$, $y \approx 11.456$; or about 11.5 per 100 population.

(e) When $x = 80{,}000$, $y \approx 5.968$; or about a 6% chance.

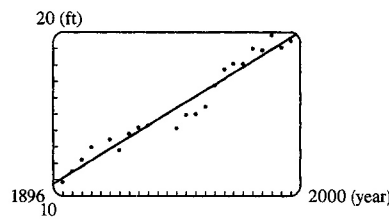
(f) When $x = 200{,}000$, y is negative, so the model does not apply.

19. (a)



A linear model does seem appropriate.

(b)

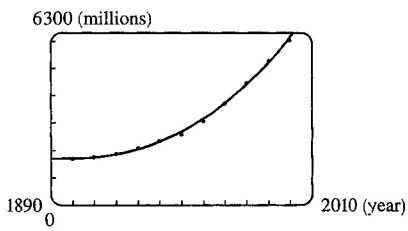


Using a computing device, we obtain the least squares regression line $y = 0.089119747x - 158.2403249$, where x is the year and y is the height in feet.

(c) When $x = 2000$, the model gives $y \approx 20.00$ ft. Note that the actual winning height for the 2000 Olympics is *less than* the winning height for 1996—so much for that prediction.

(d) When $x = 2100$, $y \approx 28.91$ ft. This would be an increase of 9.49 ft from 1996 to 2100. Even though there was an increase of 8.59 ft from 1900 to 1996, it is unlikely that a similar increase will occur over the next 100 years.

21.



Using a computing device, we obtain the cubic function $y = ax^3 + bx^2 + cx + d$ with $a = 0.0012937$, $b = -7.06142$, $c = 12{,}823$, and $d = -7{,}743{,}770$. When $x = 1925$, $y \approx 1914$ (million).

1.3 New Functions from Old Functions

1. (a) If the graph of f is shifted 3 units upward, its equation becomes $y = f(x) + 3$.

(b) If the graph of f is shifted 3 units downward, its equation becomes $y = f(x) - 3$.

(c) If the graph of f is shifted 3 units to the right, its equation becomes $y = f(x - 3)$.

(d) If the graph of f is shifted 3 units to the left, its equation becomes $y = f(x + 3)$.

(e) If the graph of f is reflected about the x-axis, its equation becomes $y = -f(x)$.

(f) If the graph of f is reflected about the y-axis, its equation becomes $y = f(-x)$.

(g) If the graph of f is stretched vertically by a factor of 3, its equation becomes $y = 3f(x)$.

(h) If the graph of f is shrunk vertically by a factor of 3, its equation becomes $y = \frac{1}{3}f(x)$.

3. (a) (graph 3) The graph of f is shifted 4 units to the right and has equation $y = f(x - 4)$.

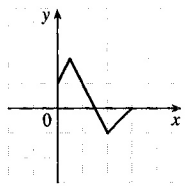
(b) (graph 1) The graph of f is shifted 3 units upward and has equation $y = f(x) + 3$.

(c) (graph 4) The graph of f is shrunk vertically by a factor of 3 and has equation $y = \frac{1}{3}f(x)$.

(d) (graph 5) The graph of f is shifted 4 units to the left and reflected about the x-axis. Its equation is $y = -f(x + 4)$.

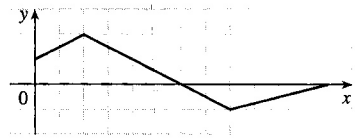
(e) (graph 2) The graph of f is shifted 6 units to the left and stretched vertically by a factor of 2. Its equation is $y = 2f(x + 6)$.

5. (a) To graph $y = f(2x)$ we shrink the graph of f horizontally by a factor of 2.



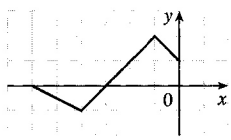
The point $(4, -1)$ on the graph of f corresponds to the point $\left(\frac{1}{2} \cdot 4, -1\right) = (2, -1)$.

(b) To graph $y = f\left(\frac{1}{2}x\right)$ we stretch the graph of f horizontally by a factor of 2.



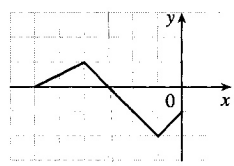
The point $(4, -1)$ on the graph of f corresponds to the point $(2 \cdot 4, -1) = (8, -1)$.

(c) To graph $y = f(-x)$ we reflect the graph of f about the y-axis.



The point $(4, -1)$ on the graph of f corresponds to the point $(-1 \cdot 4, -1) = (-4, -1)$.

(d) To graph $y = -f(-x)$ we reflect the graph of f about the y-axis, then about the x-axis.



The point $(4, -1)$ on the graph of f corresponds to the point $(-1 \cdot 4, -1 \cdot -1) = (-4, 1)$.

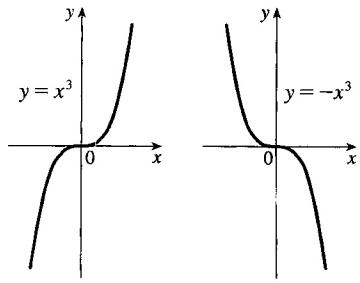
7. The graph of $y = f(x) = \sqrt{3x - x^2}$ has been shifted 4 units to the left, reflected about the x-axis, and shifted downward 1 unit. Thus, a function describing the graph is

$$y = \underbrace{-1 \cdot}_{\substack{\text{reflect}\\ \text{about}\\ x\text{-axis}}} \underbrace{f(x+4)}_{\substack{\text{shift}\\ \text{4 units}\\ \text{left}}} \underbrace{-1}_{\substack{\text{shift}\\ \text{1 unit}\\ \text{down}}}$$

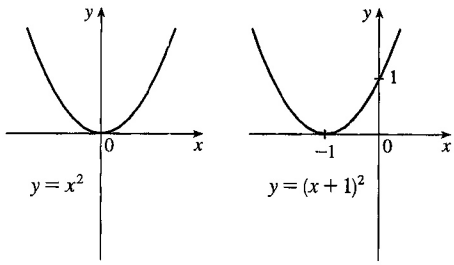
This function can be written as

$$\begin{aligned} y &= -f(x+4) - 1 = -\sqrt{3(x+4) - (x+4)^2} - 1 = -\sqrt{3x + 12 - (x^2 + 8x + 16)} - 1 \\ &= -\sqrt{-x^2 - 5x - 4} - 1 \end{aligned}$$

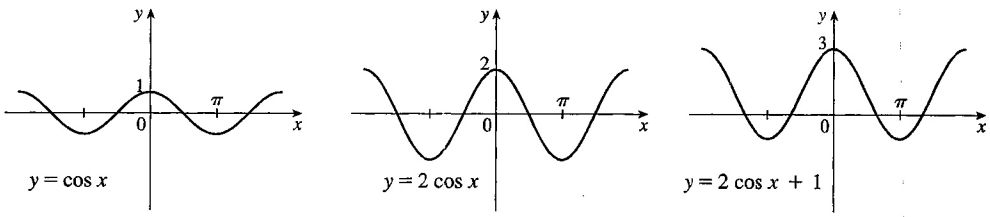
9. $y = -x^3$: Start with the graph of $y = x^3$ and reflect about the x-axis. Note: Reflecting about the y-axis gives the same result since substituting $-x$ for x gives us $y = (-x)^3 = -x^3$.



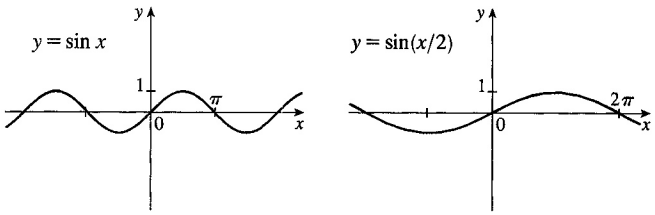
11. $y = (x + 1)^2$: Start with the graph of $y = x^2$ and shift 1 unit to the left.



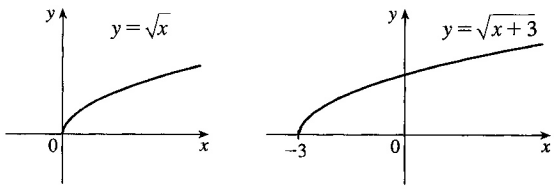
13. $y = 1 + 2\cos x$: Start with the graph of $y = \cos x$, stretch vertically by a factor of 2, and then shift 1 unit upward.



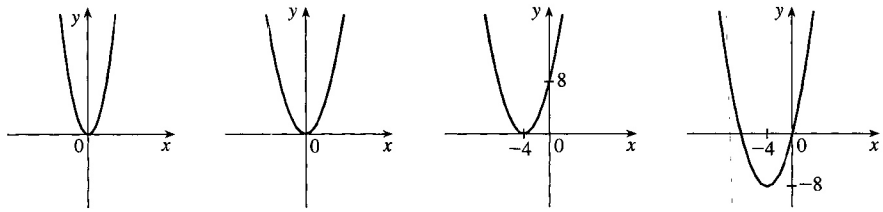
15. $y = \sin(x/2)$: Start with the graph of $y = \sin x$ and stretch horizontally by a factor of 2.



17. $y = \sqrt{x + 3}$: Start with the graph of $y = \sqrt{x}$ and shift 3 units to the left.

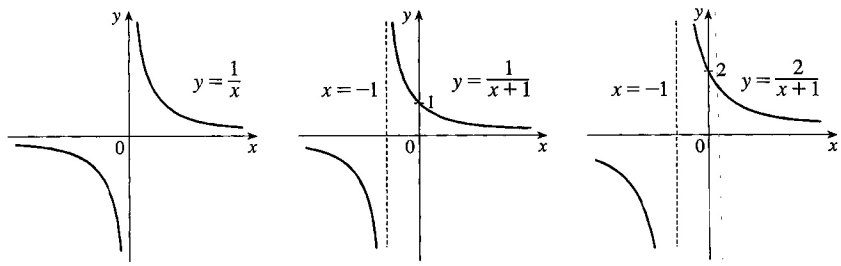


19. $y = \frac{1}{2}(x^2 + 8x) = \frac{1}{2}(x^2 + 8x + 16) - 8 = \frac{1}{2}(x + 4)^2 - 8$: Start with the graph of $y = x^2$, compress vertically by a factor of 2, shift 4 units to the left, and then shift 8 units downward.

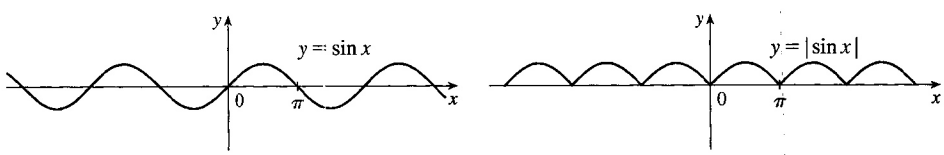


$y = x^2$ $\quad$ $y = \frac{1}{2}x^2$ $\quad$ $y = \frac{1}{2}(x + 4)^2$ $\quad$ $y = \frac{1}{2}(x + 4)^2 - 8$

21. $y = 2/(x + 1)$: Start with the graph of $y = 1/x$, shift 1 unit to the left, and then stretch vertically by a factor of 2.



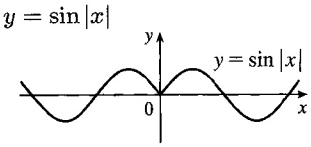
23. $y = |\sin x|$: Start with the graph of $y = \sin x$ and reflect all the parts of the graph below the x-axis about the x-axis.



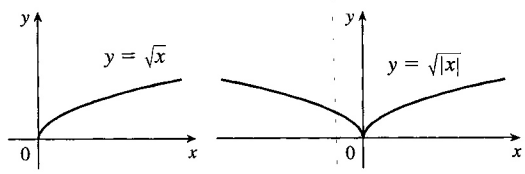
25. This is just like the solution to Example 4 except the amplitude of the curve (the 30°N curve in Figure 9 on June 21) is $14 - 12 = 2$. So the function is $L(t) := 12 + 2\sin\left[\frac{2\pi}{365}(t - 80)\right]$. March 31 is the 90th day of the year, so the model gives $L(90) \approx 12.34$ h. The daylight time (5:51 A.M. to 6:18 P.M.) is 12 hours and 27 minutes, or 12.45 h. The model value differs from the actual value by $\frac{12.45 - 12.34}{12.45} \approx 0.009$, less than 1%.

27. (a) To obtain $y = f(|x|)$, the portion of the graph of $y = f(x)$ to the right of the y-axis is reflected about the y-axis.

(b) $y = \sin|x|$

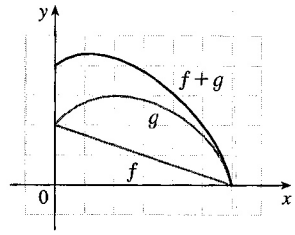


(c) $y = \sqrt{|x|}$



29. Assuming that successive horizontal and vertical gridlines are a unit apart, we can make a table of approximate values as follows.

x	0	1	2	3	4	5	6
$f(x)$	2	1.7	1.3	1.0	0.7	0.3	0
$g(x)$	2	2.7	3	2.8	2.4	1.7	0
$f(x)+g(x)$	4	4.4	4.3	3.8	3.1	2.0	0



Connecting the points $(x, f(x)+g(x))$ with a smooth curve gives an approximation to the graph of $f+g$. Extra points can be plotted between those listed above if necessary.

31. $f(x) = x^3 + 2x^2$; $g(x) = 3x^2 - 1$. $D = \mathbb{R}$ for both f and g.

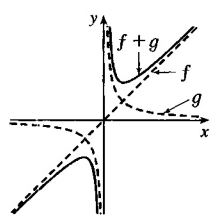
$(f+g)(x) = (x^3 + 2x^2) + (3x^2 - 1) = x^3 + 5x^2 - 1, \quad D = \mathbb{R}$.

$(f-g)(x) = (x^3 + 2x^2) - (3x^2 - 1) = x^3 - x^2 + 1, \quad D = \mathbb{R}$.

$(fg)(x) = (x^3 + 2x^2)(3x^2 - 1) = 3x^5 + 6x^4 - x^3 - 2x^2, \quad D = \mathbb{R}$.

$\left(\dfrac{f}{g}\right)(x) = \dfrac{x^3 + 2x^2}{3x^2 - 1}, \quad D = \left\{x \mid x \neq \pm\dfrac{1}{\sqrt{3}}\right\}$ since $3x^2 - 1 \neq 0$.

33. $f(x) = x, \quad g(x) = 1/x$



35. $f(x) = 2x^2 - x$; $g(x) = 3x + 2$. $D = \mathbb{R}$ for both f and g, and hence for their composites.

$(f \circ g)(x) = f(g(x)) = f(3x+2) = 2(3x+2)^2 - (3x+2) = 2(9x^2 + 12x + 4) - 3x - 2 = 18x^2 + 21x + 6$.

$(g \circ f)(x) = g(f(x)) = g(2x^2 - x) = 3(2x^2 - x) + 2 = 6x^2 - 3x + 2$.

$(f \circ f)(x) = f(f(x)) = f(2x^2 - x) = 2(2x^2 - x)^2 - (2x^2 - x) = 2(4x^4 - 4x^3 + x^2) - 2x^2 + x = 8x^4 - 8x^3 + x$.

$(g \circ g)(x) = g(g(x)) = g(3x+2) = 3(3x+2) + 2 = 9x + 6 + 2 = 9x + 8$.

37. $f(x) = \sin x, \quad D = \mathbb{R}; \quad g(x) = 1 - \sqrt{x}, \quad D = [0, \infty)$.

$(f \circ g)(x) = f(g(x)) = f(1 - \sqrt{x}) = \sin(1 - \sqrt{x}), D = [0, \infty)$.

$(g \circ f)(x) = g(f(x)) = g(\sin x) = 1 - \sqrt{\sin x}$. For $\sqrt{\sin x}$ to be defined, we must have $\sin x \geq 0 \iff x \in [0, \pi] \cup [2\pi, 3\pi] \cup [-2\pi, -\pi] \cup [4\pi, 5\pi] \cup [-4\pi, -3\pi] \cup \ldots$, so $D = \{x \mid x \in [2n\pi, \pi + 2n\pi]$, where n is an integer$\}$.

$(f \circ f)(x) = f(f(x)) = f(\sin x) = \sin(\sin x), D = \mathbb{R}$.

$(g \circ g)(x) = g(g(x)) = g(1 - \sqrt{x}) = 1 - \sqrt{1 - \sqrt{x}}$,

$D = \{x \geq 0 \mid 1 - \sqrt{x} \geq 0\} = \{x \geq 0 \mid 1 \geq \sqrt{x}\} = \{x \geq 0 \mid \sqrt{x} \leq 1\} = [0, 1]$.

39. $f(x) = x + \dfrac{1}{x}$, $D = \{x \mid x \neq 0\}$; $g(x) = \dfrac{x+1}{x+2}$, $D = \{x \mid x \neq -2\}$.

$$(f \circ g)(x) = f(g(x)) = f\left(\frac{x+1}{x+2}\right) = \frac{x+1}{x+2} + \frac{1}{\frac{x+1}{x+2}} = \frac{x+1}{x+2} + \frac{x+2}{x+1}$$

$$= \frac{(x+1)(x+1) + (x+2)(x+2)}{(x+2)(x+1)} = \frac{(x^2+2x+1) + (x^2+4x+4)}{(x+2)(x+1)} = \frac{2x^2+6x+5}{(x+2)(x+1)}$$

Since $g(x)$ is not defined for $x = -2$ and $f(g(x))$ is not defined for $x = -2$ and $x = -1$, the domain of $(f \circ g)(x)$ is $D = \{x \mid x \neq -2, -1\}$.

$$(g \circ f)(x) = g(f(x)) = g\left(x + \frac{1}{x}\right) = \frac{(x + \frac{1}{x}) + 1}{(x + \frac{1}{x}) + 2} = \frac{\dfrac{x^2+1+x}{x}}{\dfrac{x^2+1+2x}{x}} = \frac{x^2+x+1}{x^2+2x+1} = \frac{x^2+x+1}{(x+1)^2}.$$

Since $f(x)$ is not defined for $x = 0$ and $g(f(x))$ is not defined for $x = -1$, the domain of $(g \circ f)(x)$ is $D = \{x \mid x \neq -1, 0\}$.

$$(f \circ f)(x) = f(f(x)) = f\left(x + \frac{1}{x}\right) = \left(x + \frac{1}{x}\right) + \frac{1}{x + \frac{1}{x}} = x + \frac{1}{x} + \frac{1}{\frac{x^2+1}{x}} = x + \frac{1}{x} + \frac{x}{x^2+1}$$

$$= \frac{x(x)(x^2+1) + 1(x^2+1) + x(x)}{x(x^2+1)} = \frac{x^4+x^2+x^2+1+x^2}{x(x^2+1)}$$

$$= \frac{x^4+3x^2+1}{x(x^2+1)}, \quad D = \{x \mid x \neq 0\}.$$

$$(g \circ g)(x) = g(g(x)) = g\left(\frac{x+1}{x+2}\right) = \frac{\frac{x+1}{x+2} + 1}{\frac{x+1}{x+2} + 2} = \frac{\frac{x+1+1(x+2)}{x+2}}{\frac{x+1+2(x+2)}{x+2}} = \frac{x+1+x+2}{x+1+2x+4} = \frac{2x+3}{3x+5}.$$

Since $g(x)$ is not defined for $x = -2$ and $g(g(x))$ is not defined for $x = -\frac{5}{3}$, the domain of $(g \circ g)(x)$ is $D = \left\{x \mid x \neq -2, -\frac{5}{3}\right\}$.

41. $(f \circ g \circ h)(x) = f(g(h(x))) = f(g(x-1)) = f(2(x-1))$
$= 2(x-1) + 1 = 2x - 1$

43. $(f \circ g \circ h)(x) = f(g(h(x))) = f(g(x+3)) = f((x+3)^2 + 2)$
$= f(x^2+6x+11) = \sqrt{(x^2+6x+11) - 1} = \sqrt{x^2+6x+10}$

45. Let $g(x) = x^2 + 1$ and $f(x) = x^{10}$. Then $(f \circ g)(x) = f(g(x)) = (x^2+1)^{10} = F(x)$.

47. Let $g(x) = x^2$ and $f(x) = \dfrac{x}{x+4}$. Then $(f \circ g)(x) = f(g(x)) = \dfrac{x^2}{x^2+4} = G(x)$.

49. Let $g(t) = \cos t$ and $f(t) = \sqrt{t}$. Then $(f \circ g)(t) = f(g(t)) = \sqrt{\cos t} = u(t)$.

51. Let $h(x) = x^2$, $g(x) = 3^x$, and $f(x) = 1 - x$. Then $(f \circ g \circ h)(x) = 1 - 3^{x^2} = H(x)$.

53. Let $h(x) = \sqrt{x}$, $g(x) = \sec x$, and $f(x) = x^4$. Then $(f \circ g \circ h)(x) = (\sec\sqrt{x})^4 = \sec^4(\sqrt{x}) = H(x)$.

55. (a) $g(2) = 5$, because the point $(2, 5)$ is on the graph of g. Thus, $f(g(2)) = f(5) = 4$, because the point $(5, 4)$ is on the graph of f.

(b) $g(f(0)) = g(0) = 3$

(c) $(f \circ g)(0) = f(g(0)) = f(3) = 0$

(d) $(g \circ f)(6) = g(f(6)) = g(6)$. This value is not defined, because there is no point on the graph of g that has x-coordinate 6.

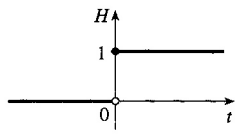
(e) $(g \circ g)(-2) = g(g(-2)) = g(1) = 4$

(f) $(f \circ f)(4) = f(f(4)) = f(2) = -2$

57. (a) Using the relationship *distance* = *rate* · *time* with the radius r as the distance, we have $r(t) = 60t$.

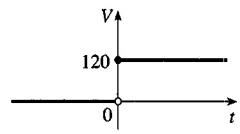
(b) $A = \pi r^2 \quad \Rightarrow \quad (A \circ r)(t) = A(r(t)) = \pi(60t)^2 = 3600\pi t^2$. This formula gives us the extent of the rippled area (in cm^2) at any time t.

59. (a)



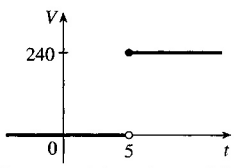
$$H(t) = \begin{cases} 0 & \text{if } t < 0 \\ 1 & \text{if } t \geq 0 \end{cases}$$

(b)



$$V(t) = \begin{cases} 0 & \text{if } t < 0 \\ 120 & \text{if } t \geq 0 \end{cases} \quad \text{so } V(t) = 120H(t).$$

(c)



Starting with the formula in part (b), we replace 120 with 240 to reflect the different voltage. Also, because we are starting 5 units to the right of $t = 0$, we replace t with $t - 5$. Thus, the formula is $V(t) = 240H(t - 5)$.

61. (a) By examining the variable terms in g and h, we deduce that we must square g to get the terms $4x^2$ and $4x$ in h. If we let $f(x) = x^2 + c$, then $(f \circ g)(x) = f(g(x)) = f(2x + 1) = (2x + 1)^2 + c = 4x^2 + 4x + (1 + c)$. Since $h(x) = 4x^2 + 4x + 7$, we must have $1 + c = 7$. So $c = 6$ and $f(x) = x^2 + 6$.

(b) We need a function g so that $f(g(x)) = 3(g(x)) + 5 = h(x)$. But $h(x) = 3x^2 + 3x + 2 = 3(x^2 + x) + 2 = 3(x^2 + x - 1) + 5$, so we see that $g(x) = x^2 + x - 1$.

63. We need to examine $h(-x)$.

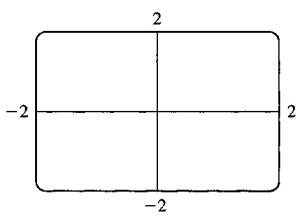
$$h(-x) = (f \circ g)(-x) = f(g(-x)) = f(g(x)) \quad \text{[because } g \text{ is even]} \quad = h(x)$$

Because $h(-x) = h(x)$, h is an even function.

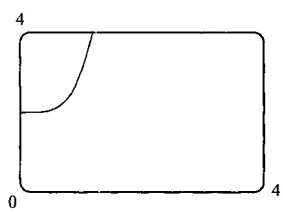
1.4 Graphing Calculators and Computers

1. $f(x) = x^4 + 2$

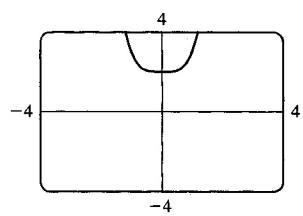
(a) $[-2, 2]$ by $[-2, 2]$



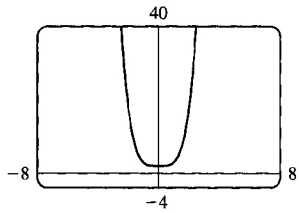
(b) $[0, 4]$ by $[0, 4]$



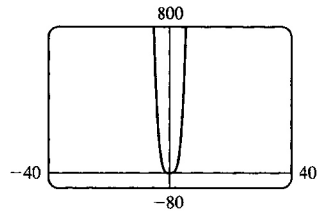
(c) $[-4, 4]$ by $[-4, 4]$



(d) $[-8, 8]$ by $[-4, 40]$



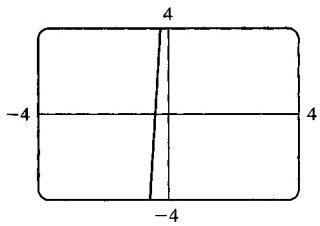
(e) $[-40, 40]$ by $[-80, 800]$



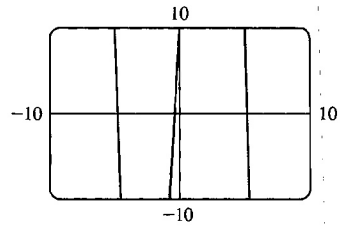
The most appropriate graph is produced in viewing rectangle (d).

3. $f(x) = 10 + 25x - x^3$

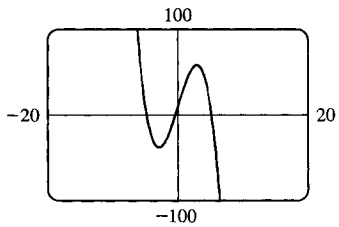
(a) $[-4, 4]$ by $[-4, 4]$



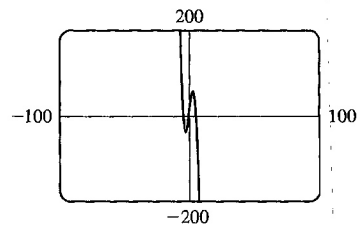
(b) $[-10, 10]$ by $[-10, 10]$



(c) $[-20, 20]$ by $[-100, 100]$

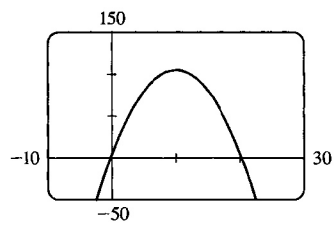


(d) $[-100, 100]$ by $[-200, 200]$

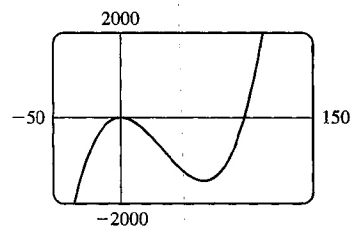


The most appropriate graph is produced in viewing rectangle (c) because the maximum and minimum points are fairly easy to see and estimate.

5. Since the graph of $f(x) = 5 + 20x - x^2$ is a parabola opening downward, an appropriate viewing rectangle should include the maximum point.



7. $f(x) = 0.01x^3 - x^2 + 5$. Graphing f in a standard viewing rectangle, $[-10, 10]$ by $[-10, 10]$, shows us what appears to be a parabola. But since this is a cubic polynomial, we know that a larger viewing rectangle will reveal a minimum point as well as the maximum point. After some trial and error, we choose the viewing rectangle $[-50, 150]$ by $[-2000, 2000]$.

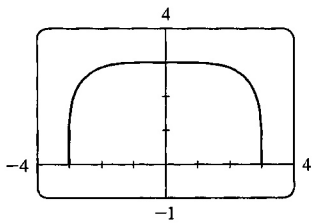


9. $f(x) = \sqrt[4]{81 - x^4}$ is defined when

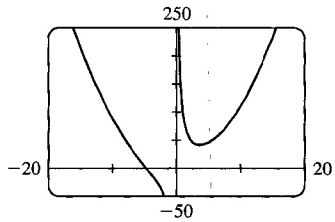
$81 - x^4 \geq 0 \quad \Leftrightarrow \quad x^4 \leq 81 \quad \Leftrightarrow \quad |x| \leq 3$, so

the domain of f is $[-3, 3]$. Also

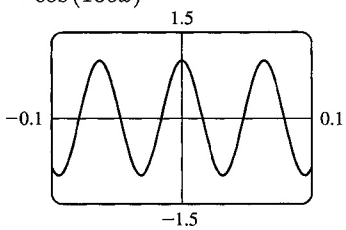
$0 \leq \sqrt[4]{81 - x^4} \leq \sqrt[4]{81} = 3$, so the range is $[0, 3]$.



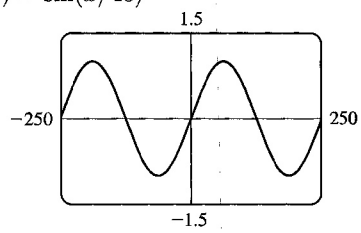
11. The graph of $f(x) = x^2 + (100/x)$ has a vertical asymptote of $x = 0$. As you zoom out, the graph of f looks more and more like that of $y = x^2$.



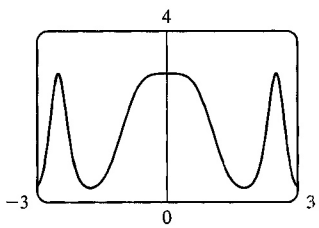
13. $f(x) = \cos(100x)$

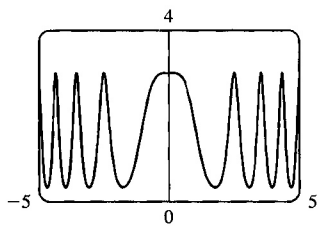


15. $f(x) = \sin(x/40)$



17. $y = 3^{\cos(x^2)}$

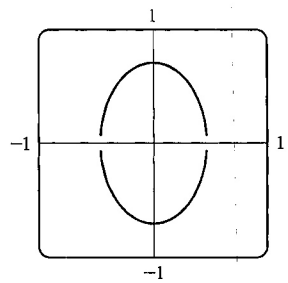




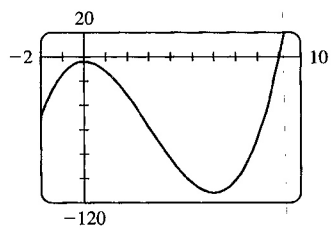
19. We must solve the given equation for y to obtain equations for the upper and lower halves of the ellipse.

$$4x^2 + 2y^2 = 1 \quad \Leftrightarrow \quad 2y^2 = 1 - 4x^2 \quad \Leftrightarrow \quad y^2 = \frac{1 - 4x^2}{2}$$

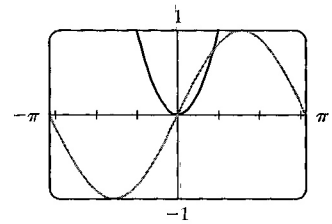
$$\Leftrightarrow \quad y = \pm\sqrt{\frac{1 - 4x^2}{2}}$$



21. From the graph of $f(x) = x^3 - 9x^2 - 4$, we see that there is one solution of the equation $f(x) = 0$ and it is slightly larger than 9. By zooming in or using a root or zero feature, we obtain $x \approx 9.05$.

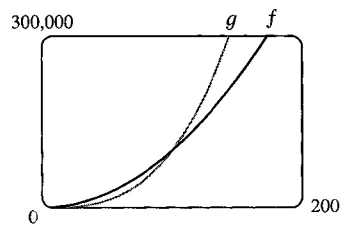


23. We see that the graphs of $f(x) = x^2$ and $g(x) = \sin x$ intersect twice. One solution is $x = 0$. The other solution of $f = g$ is the x-coordinate of the point of intersection in the first quadrant. Using an intersect feature or zooming in, we find this value to be approximately 0.88. Alternatively, we could find that value by finding the positive zero of $h(x) = x^2 - \sin x$.

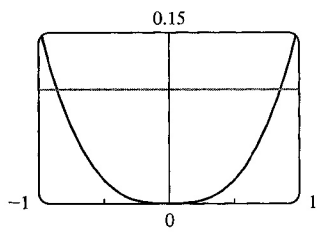


Note: After producing the graph on a TI-83 Plus, we can find the approximate value 0.88 by using the following keystrokes: [2nd] [CALC] [5] [ENTER] [ENTER] 1 [ENTER]. The "1" is just a guess for 0.88.

25. $g(x) = x^3/10$ is larger than $f(x) = 10x^2$ whenever $x > 100$.

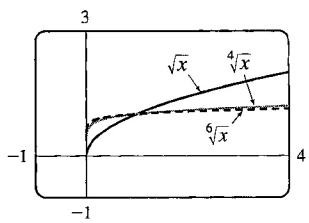


27.

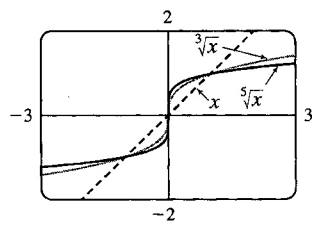


We see from the graphs of $y = |\sin x - x|$ and $y = 0.1$ that there are two solutions to the equation $|\sin x - x| = 0.1$: $x \approx -0.85$ and $x \approx 0.85$. The condition $|\sin x - x| < 0.1$ holds for any x lying between these two values.

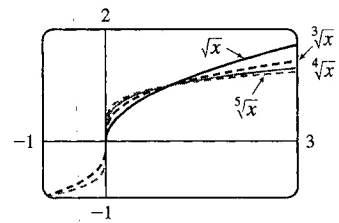
29. (a) The root functions $y = \sqrt{x}$, $y = \sqrt[4]{x}$ and $y = \sqrt[6]{x}$



(b) The root functions $y = x$, $y = \sqrt[3]{x}$ and $y = \sqrt[5]{x}$

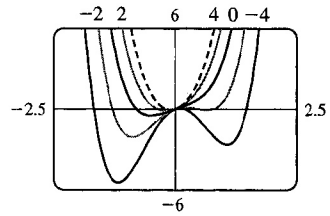


(c) The root functions $y = \sqrt{x}$, $y = \sqrt[3]{x}$, $y = \sqrt[4]{x}$ and $y = \sqrt[5]{x}$

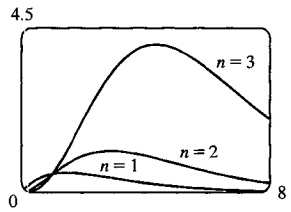


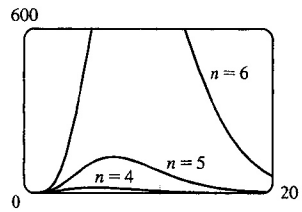
(d)
- For any n, the nth root of 0 is 0 and the nth root of 1 is 1; that is, all nth root functions pass through the points $(0, 0)$ and $(1, 1)$.
- For odd n, the domain of the nth root function is $\mathbb{R}$, while for even n, it is $\{x \in \mathbb{R} \mid x \geq 0\}$.
- Graphs of even root functions look similar to that of $\sqrt{x}$, while those of odd root functions resemble that of $\sqrt[3]{x}$.
- As n increases, the graph of $\sqrt[n]{x}$ becomes steeper near 0 and flatter for $x > 1$.

31. $f(x) = x^4 + cx^2 + x$. If $c < 0$, there are three humps: two minimum points and a maximum point. These humps get flatter as c increases, until at $c = 0$ two of the humps disappear and there is only one minimum point. This single hump then moves to the right and approaches the origin as c increases.



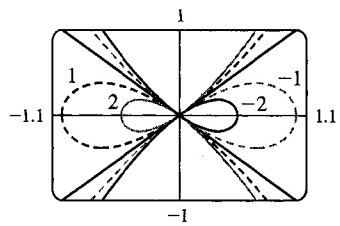
33. $y = x^n 2^{-x}$. As n increases, the maximum of the function moves further from the origin, and gets larger. Note, however, that regardless of n, the function approaches 0 as $x \to \infty$.





35. $y^2 = cx^3 + x^2$

If $c < 0$, the loop is to the right of the origin, and if c is positive, it is to the left. In both cases, the closer c is to 0, the larger the loop is. (In the limiting case, $c = 0$, the loop is "infinite", that is, it doesn't close.) Also, the larger $|c|$ is, the steeper the slope is on the loopless side of the origin.



37. The graphing window is 95 pixels wide and we want to start with $x = 0$ and end with $x = 2\pi$. Since there are 94 "gaps" between pixels, the distance between pixels is $\frac{2\pi - 0}{94}$. Thus, the x-values that the calculator actually plots are $x = 0 + \frac{2\pi}{94} \cdot n$, where $n = 0, 1, 2, \ldots, 93, 94$. For $y = \sin 2x$, the actual points plotted by the calculator are $\left(\frac{2\pi}{94} \cdot n, \sin\left(2 \cdot \frac{2\pi}{94} \cdot n\right)\right)$ for $n = 0, 1, \ldots, 94$. For $y = \sin 96x$, the points plotted are $\left(\frac{2\pi}{94} \cdot n, \sin\left(96 \cdot \frac{2\pi}{94} \cdot n\right)\right)$ for $n = 0, 1, \ldots, 94$. But

$$\begin{aligned} \sin\left(96 \cdot \tfrac{2\pi}{94} \cdot n\right) &= \sin\left(94 \cdot \tfrac{2\pi}{94} \cdot n + 2 \cdot \tfrac{2\pi}{94} \cdot n\right) = \sin\left(2\pi n + 2 \cdot \tfrac{2\pi}{94} \cdot n\right) \\ &= \sin\left(2 \cdot \tfrac{2\pi}{94} \cdot n\right) \quad \text{[by periodicity of sine]}, \quad n = 0, 1, \ldots, 94 \end{aligned}$$

So the y-values, and hence the points, plotted for $y = \sin 96x$ are identical to those plotted for $y = \sin 2x$.

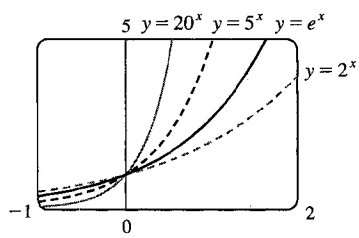
Note: Try graphing $y = \sin 94x$. Can you see why all the y-values are zero?

1.5 Exponential Functions

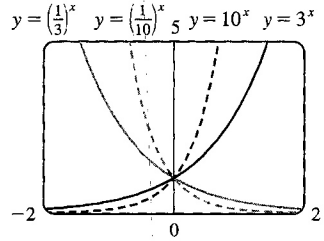
1. (a) $f(x) = a^x, \ a > 0$ (b) $\mathbb{R}$

(c) $(0, \infty)$ (d) See Figures 6(c), 6(b), and 6(a), respectively.

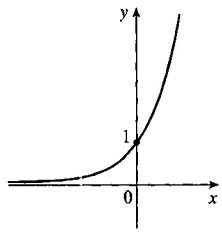
3. All of these graphs approach 0 as $x \to -\infty$, all of them pass through the point $(0, 1)$, and all of them are increasing and approach ∞ as $x \to \infty$. The larger the base, the faster the function increases for $x > 0$, and the faster it approaches 0 as $x \to -\infty$.



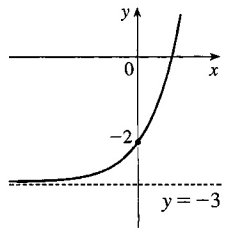
5. The functions with bases greater than 1 (3^x and 10^x) are increasing, while those with bases less than 1 $\left[\left(\frac{1}{3}\right)^x \text{ and } \left(\frac{1}{10}\right)^x\right]$ are decreasing. The graph of $\left(\frac{1}{3}\right)^x$ is the reflection of that of 3^x about the y-axis, and the graph of $\left(\frac{1}{10}\right)^x$ is the reflection of that of 10^x about the y-axis. The graph of 10^x increases more quickly than that of 3^x for $x > 0$, and approaches 0 faster as $x \to -\infty$.



7. We start with the graph of $y = 4^x$ (Figure 3) and then shift 3 units downward. This shift doesn't affect the domain, but the range of $y = 4^x - 3$ is $(-3, \infty)$. There is a horizontal asymptote of $y = -3$.

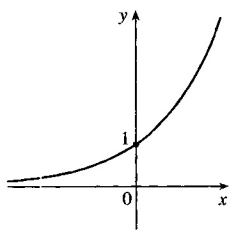


$y = 4^x$

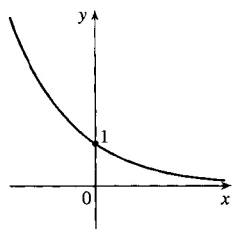


$y = 4^x - 3$

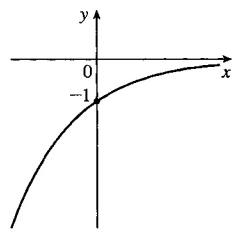
9. We start with the graph of $y = 2^x$ (Figure 2), reflect it about the y-axis, and then about the x-axis (or just rotate 180° to handle both reflections) to obtain the graph of $y = -2^{-x}$. In each graph, $y = 0$ is the horizontal asymptote.



$y = 2^x$

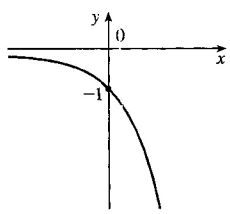


$y = 2^{-x}$

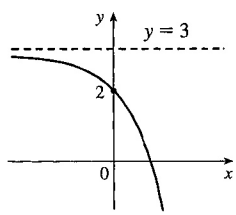


$y = -2^{-x}$

11. We start with the graph of $y = e^x$ (Figure 13), reflect it about the x-axis, and then shift 3 units upward. Note the horizontal asymptote of $y = 3$.



$y = -e^x$



$y = 3 - e^x$

13. (a) To find the equation of the graph that results from shifting the graph of $y = e^x$ 2 units downward, we subtract 2 from the original function to get $y = e^x - 2$.

(b) To find the equation of the graph that results from shifting the graph of $y = e^x$ 2 units to the right, we replace x with $x - 2$ in the original function to get $y = e^{(x-2)}$.

(c) To find the equation of the graph that results from reflecting the graph of $y = e^x$ about the x-axis, we multiply the original function by -1 to get $y = -e^x$.

(d) To find the equation of the graph that results from reflecting the graph of $y = e^x$ about the y-axis, we replace x with $-x$ in the original function to get $y = e^{-x}$.

(e) To find the equation of the graph that results from reflecting the graph of $y = e^x$ about the x-axis and then about the y-axis, we first multiply the original function by -1 (to get $y = -e^x$) and then replace x with $-x$ in this equation to get $y = -e^{-x}$.

15. (a) The denominator $1 + e^x$ is never equal to zero because $e^x > 0$, so the domain of $f(x) = 1/(1 + e^x)$ is $\mathbb{R}$.

(b) $1 - e^x = 0 \;\Leftrightarrow\; e^x = 1 \;\Leftrightarrow\; x = 0$, so the domain of $f(x) = 1/(1 - e^x)$ is $(-\infty, 0) \cup (0, \infty)$.

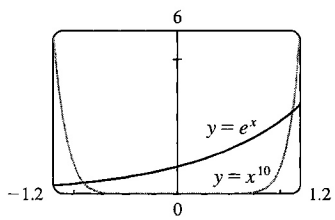
17. Use $y = Ca^x$ with the points $(1, 6)$ and $(3, 24)$. $6 = Ca^1 \quad \left[C = \frac{6}{a}\right]$ and $24 = Ca^3 \;\Rightarrow\; 24 = \left(\frac{6}{a}\right)a^3 \;\Rightarrow$

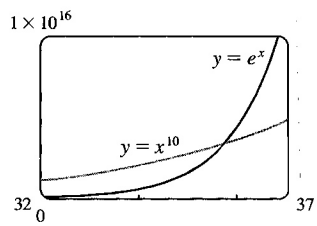
$4 = a^2 \;\Rightarrow\; a = 2$ [since $a > 0$] and $C = \frac{6}{2} = 3$. The function is $f(x) = 3 \cdot 2^x$.

19. If $f(x) = 5^x$, then $\dfrac{f(x+h) - f(x)}{h} = \dfrac{5^{x+h} - 5^x}{h} = \dfrac{5^x 5^h - 5^x}{h} = \dfrac{5^x(5^h - 1)}{h} = 5^x\left(\dfrac{5^h - 1}{h}\right)$.

21. 2 ft = 24 in, $f(24) = 24^2$ in = 576 in = 48 ft. $g(24) = 2^{24}$ in $= 2^{24}/(12 \cdot 5280)$ mi ≈ 265 mi

23. The graph of g finally surpasses that of f at $x \approx 35.8$.



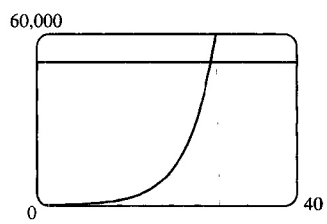


25. (a) Fifteen hours represents 5 doubling periods (one doubling period is three hours).
$100 \cdot 2^5 = 3200$

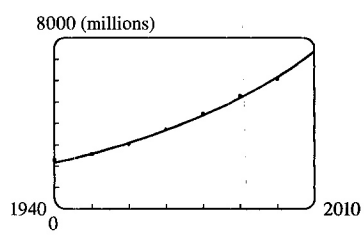
(b) In t hours, there will be $t/3$ doubling periods. The initial population is 100, so the population y at time t is $y = 100 \cdot 2^{t/3}$.

(c) $t = 20 \;\Rightarrow\; y = 100 \cdot 2^{20/3} \approx 10{,}159$

(d) We graph $y_1 = 100 \cdot 2^{x/3}$ and $y_2 = 50{,}000$. The two curves intersect at $x \approx 26.9$, so the population reaches 50,000 in about 26.9 hours.



27. An exponential model is $y = ab^t$, where $a = 3.154832569 \times 10^{-12}$ and $b = 1.017764706$. This model gives $y(1993) \approx 5498$ million and $y(2010) \approx 7417$ million.



1.6 Inverse Functions and Logarithms

1. (a) See Definition 1.

(b) It must pass the Horizontal Line Test.

3. f is not one-to-one because $2 \neq 6$, but $f(2) = 2.0 = f(6)$.

5. No horizontal line intersects the graph of f more than once. Thus, by the Horizontal Line Test, f is one-to-one.

7. The horizontal line $y = 0$ (the x-axis) intersects the graph of f in more than one point. Thus, by the Horizontal Line Test, f is not one-to-one.

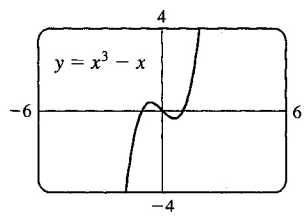
9. The graph of $f(x) = \frac{1}{2}(x+5)$ is a line with slope $\frac{1}{2}$. It passes the Horizontal Line Test, so f is one-to-one.

Algebraic solution: If $x_1 \neq x_2$, then $x_1 + 5 \neq x_2 + 5 \;\Rightarrow\; \frac{1}{2}(x_1+5) \neq \frac{1}{2}(x_2+5) \;\Rightarrow\; f(x_1) \neq f(x_2)$, so f is one-to-one.

11. $g(x) = |x| \;\Rightarrow\; g(-1) = 1 = g(1)$, so g is not one-to-one.

13. A football will attain every height h up to its maximum height twice: once on the way up, and again on the way down. Thus, even if t_1 does not equal t_2, $f(t_1)$ may equal $f(t_2)$, so f is not 1-1.

15. f does not pass the Horizontal Line Test, so f is not 1-1.



17. Since $f(2) = 9$ and f is 1-1, we know that $f^{-1}(9) = 2$. Remember, if the point $(2, 9)$ is on the graph of f, then the point $(9, 2)$ is on the graph of f^{-1}.

19. First, we must determine x such that $g(x) = 4$. By inspection, we see that if $x = 0$, then $g(x) = 4$. Since g is 1-1 (g is an increasing function), it has an inverse, and $g^{-1}(4) = 0$.

21. We solve $C = \frac{5}{9}(F - 32)$ for F: $\frac{9}{5}C = F - 32 \;\Rightarrow\; F = \frac{9}{5}C + 32$. This gives us a formula for the inverse function, that is, the Fahrenheit temperature F as a function of the Celsius temperature C. $F \geq -459.67 \;\Rightarrow\; \frac{9}{5}C + 32 \geq -459.67 \;\Rightarrow\; \frac{9}{5}C \geq -491.67 \;\Rightarrow\; C \geq -273.15$, the domain of the inverse function.

23. $f(x) = \sqrt{10 - 3x} \;\Rightarrow\; y = \sqrt{10 - 3x} \;\; (y \geq 0) \;\Rightarrow\; y^2 = 10 - 3x \;\Rightarrow\; 3x = 10 - y^2 \;\Rightarrow\; x = -\frac{1}{3}y^2 + \frac{10}{3}$. Interchange x and y: $y = -\frac{1}{3}x^2 + \frac{10}{3}$. So $f^{-1}(x) = -\frac{1}{3}x^2 + \frac{10}{3}$. Note that the domain of f^{-1} is $x \geq 0$.

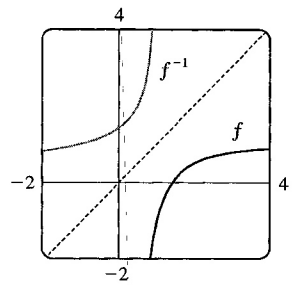
25. $f(x) = e^{x^3} \;\Rightarrow\; y = e^{x^3} \;\Rightarrow\; \ln y = x^3 \;\Rightarrow\; x = \sqrt[3]{\ln y}$. Interchange x and y: $y = \sqrt[3]{\ln x}$. So $f^{-1}(x) = \sqrt[3]{\ln x}$.

27. $y = \ln(x+3) \;\Rightarrow\; x + 3 = e^y \;\Rightarrow\; x = e^y - 3$. Interchange x and y: $y = e^x - 3$. So $f^{-1}(x) = e^x - 3$.

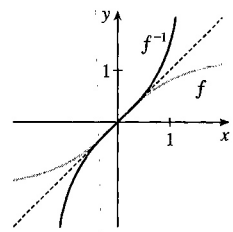
29. $y = f(x) = 1 - \dfrac{2}{x^2} \;\Rightarrow\; 1 - y = \dfrac{2}{x^2} \;\Rightarrow\; x^2 = \dfrac{2}{1-y} \;\Rightarrow$

$x = \sqrt{\dfrac{2}{1-y}}$, since $x > 0$. Interchange x and y: $y = \sqrt{\dfrac{2}{1-x}}$.

So $f^{-1}(x) = \sqrt{\dfrac{2}{1-x}}$.



31. The function f is one-to-one, so its inverse exists and the graph of its inverse can be obtained by reflecting the graph of f about the line $y = x$.



33. (a) It is defined as the inverse of the exponential function with base a, that is, $\log_a x = y \;\Leftrightarrow\; a^y = x$.

(b) $(0, \infty)$ (c) $\mathbb{R}$ (d) See Figure 11.

35. (a) $\log_2 64 = 6$ since $2^6 = 64$. (b) $\log_6 \frac{1}{36} = -2$ since $6^{-2} = \frac{1}{36}$.

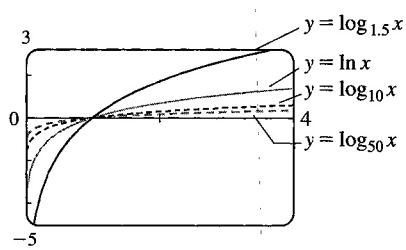
37. (a) $\log_{10} 1.25 + \log_{10} 80 = \log_{10}(1.25 \cdot 80) = \log_{10} 100 = \log_{10} 10^2 = 2$

(b) $\log_5 10 + \log_5 20 - 3\log_5 2 = \log_5(10 \cdot 20) - \log_5 2^3 = \log_5 \frac{200}{8} = \log_5 25 = \log_5 5^2 = 2$

39. $2\ln 4 - \ln 2 = \ln 4^2 - \ln 2 = \ln 16 - \ln 2 = \ln \frac{16}{2} = \ln 8$

41. $\ln(1+x^2) + \frac{1}{2}\ln x - \ln\sin x = \ln(1+x^2) + \ln x^{1/2} - \ln\sin x = \ln[(1+x^2)\sqrt{x}] - \ln\sin x = \ln\dfrac{(1+x^2)\sqrt{x}}{\sin x}$

43. To graph these functions, we use $\log_{1.5} x = \dfrac{\ln x}{\ln 1.5}$ and $\log_{50} x = \dfrac{\ln x}{\ln 50}$. These graphs all approach $-\infty$ as $x \to 0^+$, and they all pass through the point $(1, 0)$. Also, they are all increasing, and all approach ∞ as $x \to \infty$. The functions with larger bases increase extremely slowly, and the ones with smaller bases do so somewhat more quickly. The functions with large bases approach the y-axis more closely as $x \to 0^+$.

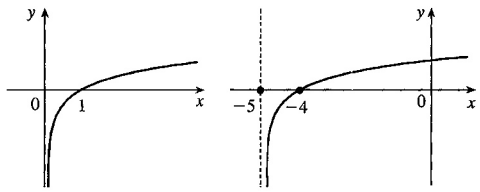


45. 3 ft = 36 in, so we need x such that $\log_2 x = 36 \;\Leftrightarrow\; x = 2^{36} = 68{,}719{,}476{,}736$. In miles, this is

$68{,}719{,}476{,}736 \text{ in} \cdot \dfrac{1 \text{ ft}}{12 \text{ in}} \cdot \dfrac{1 \text{ mi}}{5280 \text{ ft}} \approx 1{,}084{,}587.7 \text{ mi}.$

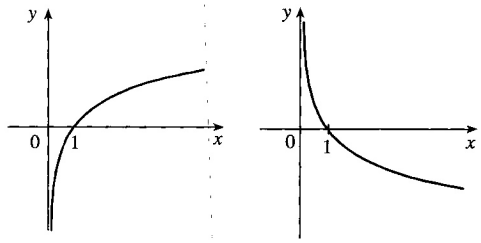
47. (a) Shift the graph of $y = \log_{10} x$ five units to the left to obtain the graph of $y = \log_{10}(x+5)$. Note the vertical asymptote of $x = -5$.

$y = \log_{10} x$ $\qquad$ $y = \log_{10}(x+5)$



(b) Reflect the graph of $y = \ln x$ about the x-axis to obtain the graph of $y = -\ln x$.

$y = \ln x$ $\qquad$ $y = -\ln x$



49. (a) $2\ln x = 1 \;\Rightarrow\; \ln x = \frac{1}{2} \;\Rightarrow\; x = e^{1/2} = \sqrt{e}$

(b) $e^{-x} = 5 \;\Rightarrow\; -x = \ln 5 \;\Rightarrow\; x = -\ln 5$

51. (a) $2^{x-5} = 3 \;\Leftrightarrow\; \log_2 3 = x - 5 \;\Leftrightarrow\; x = 5 + \log_2 3$.

Or: $2^{x-5} = 3 \;\Leftrightarrow\; \ln\left(2^{x-5}\right) = \ln 3 \;\Leftrightarrow\; (x-5)\ln 2 = \ln 3 \;\Leftrightarrow\; x - 5 = \dfrac{\ln 3}{\ln 2} \;\Leftrightarrow\; x = 5 + \dfrac{\ln 3}{\ln 2}$

(b) $\ln x + \ln(x-1) = \ln(x(x-1)) = 1 \;\Leftrightarrow\; x(x-1) = e^1 \;\Leftrightarrow\; x^2 - x - e = 0$. The quadratic formula (with $a = 1$, $b = -1$, and $c = -e$) gives $x = \frac{1}{2}\left(1 \pm \sqrt{1+4e}\right)$, but we reject the negative root since the natural logarithm is not defined for $x < 0$. So $x = \frac{1}{2}\left(1 + \sqrt{1+4e}\right)$.

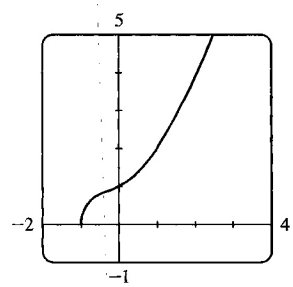
53. (a) $e^x < 10 \;\Rightarrow\; \ln e^x < \ln 10 \;\Rightarrow\; x < \ln 10 \;\Rightarrow\; x \in (-\infty, \ln 10)$

(b) $\ln x > -1 \;\Rightarrow\; e^{\ln x} > e^{-1} \;\Rightarrow\; x > e^{-1} \;\Rightarrow\; x \in (1/e, \infty)$

55. (a) For $f(x) = \sqrt{3 - e^{2x}}$, we must have $3 - e^{2x} \geq 0 \;\Rightarrow\; e^{2x} \leq 3 \;\Rightarrow\; 2x \leq \ln 3 \;\Rightarrow\; x \leq \frac{1}{2}\ln 3$. Thus, the domain of f is $(-\infty, \frac{1}{2}\ln 3]$.

(b) $y = f(x) = \sqrt{3 - e^{2x}}$ [note that $y \geq 0$] $\;\Rightarrow\; y^2 = 3 - e^{2x} \;\Rightarrow\; e^{2x} = 3 - y^2 \;\Rightarrow\; 2x = \ln(3 - y^2)$ $\;\Rightarrow\; x = \frac{1}{2}\ln(3 - y^2)$. Interchange x and y: $y = \frac{1}{2}\ln(3 - x^2)$. So $f^{-1}(x) = \frac{1}{2}\ln(3 - x^2)$. For the domain of f^{-1}, we must have $3 - x^2 > 0 \;\Rightarrow\; x^2 < 3 \;\Rightarrow\; |x| < \sqrt{3} \;\Rightarrow\; -\sqrt{3} < x < \sqrt{3} \;\Rightarrow\; 0 \leq x < \sqrt{3}$ since $x \geq 0$. Note that the domain of f^{-1}, $[0, \sqrt{3})$, equals the range of f.

57. We see that the graph of $y = f(x) = \sqrt{x^3 + x^2 + x + 1}$ is increasing, so f is 1-1. Enter $x = \sqrt{y^3 + y^2 + y + 1}$ and use your CAS to solve the equation for y. Using Derive, we get two (irrelevant) solutions involving imaginary expressions, as well as one which can be simplified to the following:



$$y = f^{-1}(x) = -\frac{\sqrt[3]{4}}{6}\left(\sqrt[3]{D - 27x^2 + 20} - \sqrt[3]{D + 27x^2 - 20} + \sqrt[3]{2}\right)$$

where $D = 3\sqrt{3}\sqrt{27x^4 - 40x^2 + 16}$. Maple and Mathematica each give two complex expressions and one real expression, and the real expression is equivalent to that given by Derive. For example, Maple's expression simplifies to $\dfrac{1}{6}\dfrac{M^{2/3} - 8 - 2M^{1/3}}{2M^{1/3}}$, where $M = 108x^2 + 12\sqrt{48 - 120x^2 + 81x^4} - 80$.

59. (a) $n = 100 \cdot 2^{t/3} \quad \Rightarrow \quad \dfrac{n}{100} = 2^{t/3} \quad \Rightarrow \quad \log_2\left(\dfrac{n}{100}\right) = \dfrac{t}{3} \quad \Rightarrow \quad t = 3\log_2\left(\dfrac{n}{100}\right)$. Using formula (10), we can write this as $t = 3 \cdot \dfrac{\ln(n/100)}{\ln 2}$. This function tells us how long it will take to obtain n bacteria (given the number n).

(b) $n = 50{,}000 \quad \Rightarrow \quad t = 3\log_2 \dfrac{50{,}000}{100} = 3\log_2 500 = 3\left(\dfrac{\ln 500}{\ln 2}\right) \approx 26.9$ hours

61. (a) To find the equation of the graph that results from shifting the graph of $y = \ln x$ 3 units upward, we add 3 to the original function to get $y = \ln x + 3$.

(b) To find the equation of the graph that results from shifting the graph of $y = \ln x$ 3 units to the left, we replace x with $x + 3$ in the original function to get $y = \ln(x + 3)$.

(c) To find the equation of the graph that results from reflecting the graph of $y = \ln x$ about the x-axis, we multiply the original equation by -1 to get $y = -\ln x$.

(d) To find the equation of the graph that results from reflecting the graph of $y = \ln x$ about the y-axis, we replace x with $-x$ in the original equation to get $y = \ln(-x)$.

(e) To find the equation of the graph that results from reflecting the graph of $y = \ln x$ about the line $y = x$, we interchange x and y in the original equation to get $x = \ln y \quad \Leftrightarrow \quad y = e^x$.

(f) To find the equation of the graph that results from reflecting the graph of $y = \ln x$ about the x-axis and then about the line $y = x$, we first multiply the original equation by -1 [to get $y = -\ln x$] and then interchange x and y in this equation to get $x = -\ln y \quad \Leftrightarrow \quad \ln y = -x \quad \Leftrightarrow \quad y = e^{-x}$.

(g) To find the equation of the graph that results from reflecting the graph of $y = \ln x$ about the y-axis and then about the line $y = x$, we first replace x with $-x$ in the original equation [to get $y = \ln(-x)$] and then interchange x and y to get $x = \ln(-y) \quad \Leftrightarrow \quad -y = e^x \quad \Leftrightarrow \quad y = -e^x$.

(h) To find the equation of the graph that results from shifting the graph of $y = \ln x$ 3 units to the left and then reflecting it about the line $y = x$, we first replace x with $x + 3$ in the original equation [to get $y = \ln(x + 3)$] and then interchange x and y in this equation to get $x = \ln(y + 3) \quad \Leftrightarrow \quad y + 3 = e^x \quad \Leftrightarrow \quad y = e^x - 3$.

63. (a) $\sin^{-1}\left(\frac{\sqrt{3}}{2}\right) = \frac{\pi}{3}$ since $\sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$ and $\frac{\pi}{3}$ is in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

(b) $\cos^{-1}(-1) = \pi$ since $\cos\pi = -1$ and π is in $[0, \pi]$.

65. (a) $\tan^{-1}\sqrt{3} = \frac{\pi}{3}$ since $\tan\frac{\pi}{3} = \sqrt{3}$ and $\frac{\pi}{3}$ is in $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

(b) $\arcsin\left(-\frac{1}{\sqrt{2}}\right) = -\frac{\pi}{4}$ since $\sin\left(-\frac{\pi}{4}\right) = -\frac{1}{\sqrt{2}}$ and $-\frac{\pi}{4}$ is in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

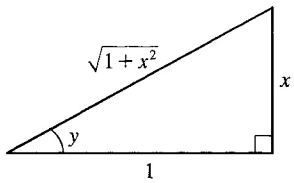
67. (a) $\sin(\sin^{-1} 0.7) = 0.7$ since 0.7 is in $[-1, 1]$.

(b) $\tan^{-1}\left(\tan\frac{4\pi}{3}\right) = \tan^{-1}\sqrt{3} = \frac{\pi}{3}$ since $\frac{\pi}{3}$ is in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

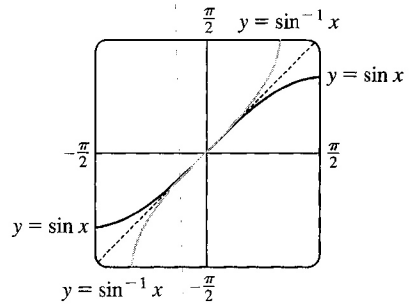
69. Let $y = \sin^{-1} x$. Then $-\frac{\pi}{2} \le y \le \frac{\pi}{2} \quad \Rightarrow \quad \cos y \ge 0$, so $\cos(\sin^{-1} x) = \cos y = \sqrt{1 - \sin^2 y} = \sqrt{1 - x^2}$

71. Let $y = \tan^{-1} x$. Then $\tan y = x$, so from the triangle we see that

$$\sin(\tan^{-1} x) = \sin y = \frac{x}{\sqrt{1+x^2}}.$$



73.



The graph of $\sin^{-1} x$ is the reflection of the graph of $\sin x$ about the line $y = x$.

75. $g(x) = \sin^{-1}(3x+1)$.

Domain $(g) = \{x \mid -1 \le 3x+1 \le 1\} = \{x \mid -2 \le 3x \le 0\} = \{x \mid -\frac{2}{3} \le x \le 0\} = [-\frac{2}{3}, 0]$.

Range $(g) = \{y \mid -\frac{\pi}{2} \le y \le \frac{\pi}{2}\} = [-\frac{\pi}{2}, \frac{\pi}{2}]$.

1 Review

CONCEPT CHECK

1. (a) A **function** f is a rule that assigns to each element x in a set A exactly one element, called $f(x)$, in a set B. The set A is called the **domain** of the function. The **range** of f is the set of all possible values of $f(x)$ as x varies throughout the domain.

(b) If f is a function with domain A, then its **graph** is the set of ordered pairs $\{(x, f(x)) \mid x \in A\}$.

(c) Use the Vertical Line Test on page 17.

2. The four ways to represent a function are: verbally, numerically, visually, and algebraically. An example of each is given below.

Verbally: An assignment of students to chairs in a classroom (a description in words)

Numerically: A tax table that assigns an amount of tax to an income (a table of values)

Visually: A graphical history of the Dow Jones average (a graph)

Algebraically: A relationship between distance, rate, and time: $d = rt$ (an explicit formula)

3. (a) An **even function** f satisfies $f(-x) = f(x)$ for every number x in its domain. It is symmetric with respect to the y-axis.

(b) An **odd function** g satisfies $g(-x) = -g(x)$ for every number x in its domain. It is symmetric with respect to the origin.

4. A function f is called **increasing** on an interval I if $f(x_1) < f(x_2)$ whenever $x_1 < x_2$ in I.

5. A **mathematical model** is a mathematical description (often by means of a function or an equation) of a real-world phenomenon.

6. (a) Linear function: $f(x) = 2x + 1$, $f(x) = ax + b$

(b) Power function: $f(x) = x^2$, $f(x) = x^a$

(c) Exponential function: $f(x) = 2^x$, $f(x) = a^x$

(d) Quadratic function: $f(x) = x^2 + x + 1$,

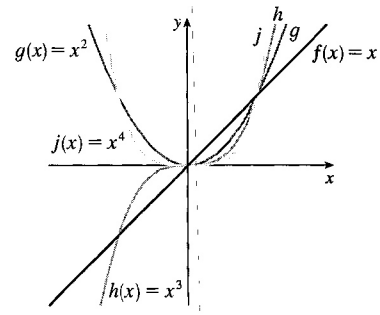
$f(x) = ax^2 + bx + c$

(e) Polynomial of degree 5: $f(x) = x^5 + 2$

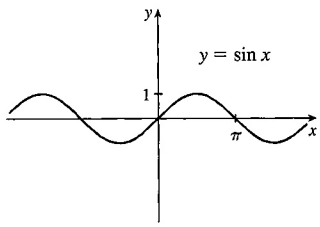
(f) Rational function: $f(x) = \dfrac{x}{x+2}$, $f(x) = \dfrac{P(x)}{Q(x)}$ where

$P(x)$ and $Q(x)$ are polynomials

7.



8. (a)



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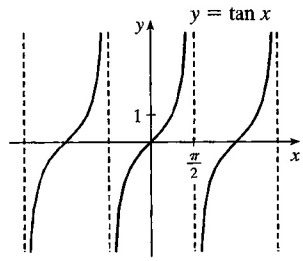

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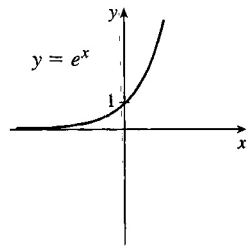

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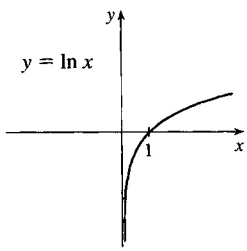

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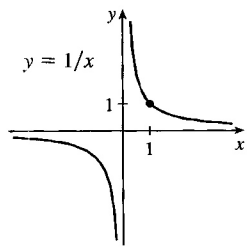

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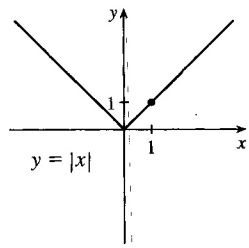

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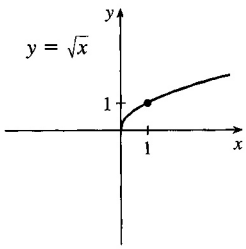

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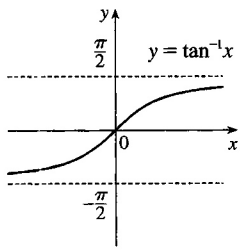


9. (a) The domain of $f + g$ is the intersection of the domain of f and the domain of g; that is, $A \cap B$.

(b) The domain of fg is also $A \cap B$.

(c) The domain of f/g must exclude values of x that make g equal to 0; that is, $\{x \in A \cap B \mid g(x) \neq 0\}$.

10. Given two functions f and g, the **composite** function $f \circ g$ is defined by $(f \circ g)(x) = f(g(x))$. The domain of $f \circ g$ is the set of all x in the domain of g such that $g(x)$ is in the domain of f.

11. (a) If the graph of f is shifted 2 units upward, its equation becomes $y = f(x) + 2$.

(b) If the graph of f is shifted 2 units downward, its equation becomes $y = f(x) - 2$.

(c) If the graph of f is shifted 2 units to the right, its equation becomes $y = f(x - 2)$.

(d) If the graph of f is shifted 2 units to the left, its equation becomes $y = f(x + 2)$.

(e) If the graph of f is reflected about the x-axis, its equation becomes $y = -f(x)$.

(f) If the graph of f is reflected about the y-axis, its equation becomes $y = f(-x)$.

(g) If the graph of f is stretched vertically by a factor of 2, its equation becomes $y = 2f(x)$.

(h) If the graph of f is shrunk vertically by a factor of 2, its equation becomes $y = \frac{1}{2}f(x)$.

(i) If the graph of f is stretched horizontally by a factor of 2, its equation becomes $y = f(\frac{1}{2})x$.

(j) If the graph of f is shrunk horizontally by a factor of 2, its equation becomes $y = f(2x)$.

12. (a) A function f is called a *one-to-one function* if it never takes on the same value twice; that is, if $f(x_1) \neq f(x_2)$ whenever $x_1 \neq x_2$. (Or, f is 1-1 if each output corresponds to only one input.)
Use the Horizontal Line Test: A function is one-to-one if and only if no horizontal line intersects its graph more than once.

(b) If f is a one-to-one function with domain A and range B, then its *inverse function* f^{-1} has domain B and range A and is defined by

$$f^{-1}(y) = x \quad \Leftrightarrow \quad f(x) = y$$

for any y in B. The graph of f^{-1} is obtained by reflecting the graph of f about the line $y = x$.

13. (a) The inverse sine function $f(x) = \sin^{-1} x$ is defined as follows:

$$\sin^{-1} x = y \quad \Leftrightarrow \quad \sin y = x \qquad \textit{and} \qquad -\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$$

Its domain is $-1 \leq x \leq 1$ and its range is $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$.

(b) The inverse cosine function $f(x) = \cos^{-1} x$ is defined as follows:

$$\cos^{-1} x = y \quad \Leftrightarrow \quad \cos y = x \qquad \textit{and} \qquad 0 \leq y \leq \pi$$

Its domain is $-1 \leq x \leq 1$ and its range is $0 \leq y \leq \pi$.

(c) The inverse tangent function $f(x) = \tan^{-1} x$ is defined as follows:

$$\tan^{-1} x = y \quad \Leftrightarrow \quad \tan y = x \qquad \textit{and} \qquad -\frac{\pi}{2} < y < \frac{\pi}{2}$$

Its domain is $\mathbb{R}$ and its range is $-\frac{\pi}{2} < y < \frac{\pi}{2}$.

TRUE-FALSE QUIZ

1. False. Let $f(x) = x^2$, $s = -1$, and $t = 1$. Then $f(s + t) = (-1 + 1)^2 = 0^2 = 0$, but $f(s) + f(t) = (-1)^2 + 1^2 = 2 \neq 0 = f(s + t)$.

3. False. Let $f(x) = x^2$. Then $f(3x) = (3x)^2 = 9x^2$ and $3f(x) = 3x^2$. So $f(3x) \neq 3f(x)$.

5. True. See the Vertical Line Test.

7. False. Let $f(x) = x^3$. Then f is one-to-one and $f^{-1}(x) = \sqrt[3]{x}$. But $1/f(x) = 1/x^3$, which is not equal to $f^{-1}(x)$.

9. True. The function $\ln x$ is an increasing function on $(0, \infty)$.

11. False. Let $x = e^2$ and $a = e$. Then $\dfrac{\ln x}{\ln a} = \dfrac{\ln e^2}{\ln e} = \dfrac{2 \ln e}{\ln e} = 2$ and $\ln \dfrac{x}{a} = \ln \dfrac{e^2}{e} = \ln e = 1$, so in general the statement is false. What *is* true, however, is that $\ln \dfrac{x}{a} = \ln x - \ln a$.

EXERCISES

1. (a) When $x = 2$, $y \approx 2.7$. Thus, $f(2) \approx 2.7$.

(b) $f(x) = 3 \quad \Rightarrow \quad x \approx 2.3, 5.6$

(c) The domain of f is $-6 \le x \le 6$, or $[-6, 6]$.

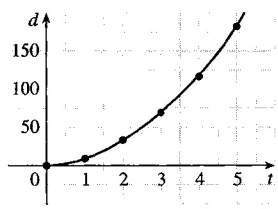
(d) The range of f is $-4 \le y \le 4$, or $[-4, 4]$.

(e) f is increasing on $[-4, 4]$, that is, on $-4 \le x \le 4$.

(f) f is not one-to-one since it fails the Horizontal Line Test.

(g) f is odd since its graph is symmetric about the origin.

3. (a)



(b) From the graph, we see that the distance traveled after 4.5 seconds is slightly less than 150 feet.

5. $f(x) = \sqrt{4 - 3x^2}$. Domain: $4 - 3x^2 \ge 0 \quad \Rightarrow \quad 3x^2 \le 4 \quad \Rightarrow \quad x^2 \le \frac{4}{3} \quad \Rightarrow \quad |x| \le \frac{2}{\sqrt{3}}$. Range: $y \ge 0$ and $y \le \sqrt{4} \quad \Rightarrow \quad 0 \le y \le 2$.

7. $y = 1 + \sin x$. Domain: $\mathbb{R}$. Range: $-1 \le \sin x \le 1 \quad \Rightarrow \quad 0 \le 1 + \sin x \le 2 \quad \Rightarrow \quad 0 \le y \le 2$.

9. (a) To obtain the graph of $y = f(x) + 8$, we shift the graph of $y = f(x)$ up 8 units.

(b) To obtain the graph of $y = f(x + 8)$, we shift the graph of $y = f(x)$ left 8 units.

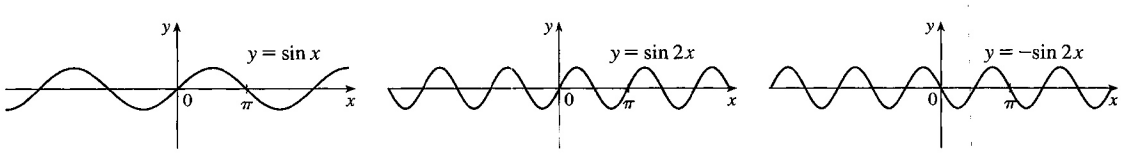
(c) To obtain the graph of $y = 1 + 2f(x)$, we stretch the graph of $y = f(x)$ vertically by a factor of 2, and then shift the resulting graph 1 unit upward.

(d) To obtain the graph of $y = f(x - 2) - 2$, we shift the graph of $y = f(x)$ right 2 units (for the "-2" inside the parentheses), and then shift the resulting graph 2 units downward.

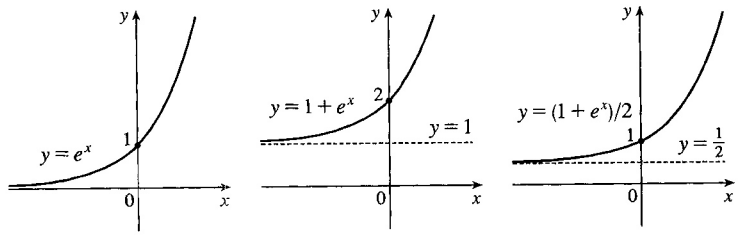
(e) To obtain the graph of $y = -f(x)$, we reflect the graph of $y = f(x)$ about the x-axis.

(f) To obtain the graph of $y = f^{-1}(x)$, we reflect the graph of $y = f(x)$ about the line $y = x$ (assuming f is one-to-one).

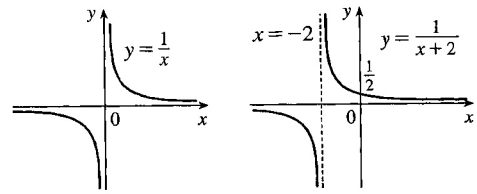
11. $y = -\sin 2x$: Start with the graph of $y = \sin x$, compress horizontally by a factor of 2, and reflect about the x-axis.



13. $y = (1 + e^x)/2$: Start with the graph of $y = e^x$, shift 1 unit upward, and compress vertically by a factor of 2.



15. $f(x) = \dfrac{1}{x+2}$: Start with the graph of $f(x) = 1/x$ and shift 2 units to the left.



17. (a) The terms of f are a mixture of odd and even powers of x, so f is neither even nor odd.

(b) The terms of f are all odd powers of x, so f is odd.

(c) $f(-x) = e^{-(-x)^2} = e^{-x^2} = f(x)$, so f is even.

(d) $f(-x) = 1 + \sin(-x) = 1 - \sin x$. Now $f(-x) \neq f(x)$ and $f(-x) \neq -f(x)$, so f is neither even nor odd.

19. $f(x) = \ln x, \quad D = (0, \infty); \quad g(x) = x^2 - 9, \quad D = \mathbb{R}$.

$(f \circ g)(x) = f(g(x)) = f(x^2 - 9) = \ln(x^2 - 9)$.

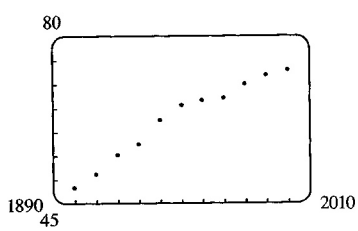
Domain: $x^2 - 9 > 0 \;\Rightarrow\; x^2 > 9 \;\Rightarrow\; |x| > 3 \;\Rightarrow\; x \in (-\infty, -3) \cup (3, \infty)$

$(g \circ f)(x) = g(f(x)) = g(\ln x) = (\ln x)^2 - 9$. Domain: $x > 0$, or $(0, \infty)$

$(f \circ f)(x) = f(f(x)) = f(\ln x) = \ln(\ln x)$. Domain: $\ln x > 0 \;\Rightarrow\; x > e^0 = 1$, or $(1, \infty)$

$(g \circ g)(x) = g(g(x)) = g(x^2 - 9) = (x^2 - 9)^2 - 9$. Domain: $x \in \mathbb{R}$, or $(-\infty, \infty)$

21.



Many models appear to be plausible. Your choice depends on whether you think medical advances will keep increasing life expectancy, or if there is bound to be a natural leveling-off of life expectancy. A linear model, $y = 0.2493x - 423.4818$ gives us an estimate of 77.6 years for the year 2010.

23. We need to know the value of x such that $f(x) = 2x + \ln x = 2$. Since $x = 1$ gives us $y = 2$, $f^{-1}(2) = 1$.

25. (a) $e^{2\ln 3} = (e^{\ln 3})^2 = 3^2 = 9$

(b) $\log_{10} 25 + \log_{10} 4 = \log_{10}(25 \cdot 4) = \log_{10} 100 = \log_{10} 10^2 = 2$

(c) $\tan(\arcsin \frac{1}{2}) = \tan \frac{\pi}{6} = \frac{1}{\sqrt{3}}$

(d) Let $\theta = \cos^{-1} \frac{4}{5}$, so $\cos\theta = \frac{4}{5}$. Then $\sin(\cos^{-1} \frac{4}{5}) = \sin\theta = \sqrt{1 - \cos^2\theta} = \sqrt{1 - (\frac{4}{5})^2} = \sqrt{\frac{9}{25}} = \frac{3}{5}$.

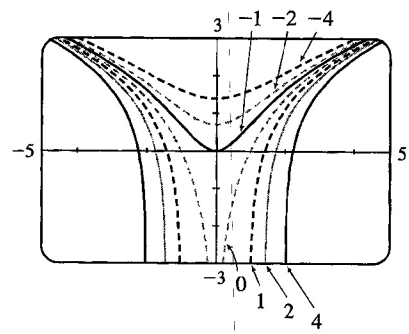
27. (a) After 4 days, $\frac{1}{2}$ gram remains; after 8 days, $\frac{1}{4}$ g; after 12 days, $\frac{1}{8}$ g; after 16 days, $\frac{1}{16}$ g.

(b) $m(4) = \dfrac{1}{2}$, $m(8) = \dfrac{1}{2^2}$, $m(12) = \dfrac{1}{2^3}$, $m(16) = \dfrac{1}{2^4}$. From the pattern, we see that $m(t) = \dfrac{1}{2^{t/4}}$, or $2^{-t/4}$.

(c) $m = 2^{-t/4} \quad \Rightarrow \quad \log_2 m = -t/4 \quad \Rightarrow \quad t = -4\log_2 m$; this is the time elapsed when there are m grams of ^{100}Pd.

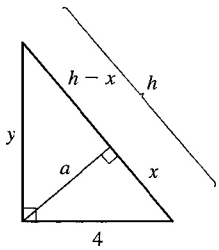
(d) $m = 0.01 \quad \Rightarrow \quad t = -4\log_2 0.01 = -4\left(\dfrac{\ln 0.01}{\ln 2}\right) \approx 26.6$ days

29. $f(x) = \ln(x^2 - c)$. If $c < 0$, the domain of f is $\mathbb{R}$. If $c = 0$, the domain of f is $(-\infty, 0) \cup (0, \infty)$. If $c > 0$, the domain of f is $(-\infty, -\sqrt{c}) \cup (\sqrt{c}, \infty)$. As c increases, the dip at $x = 0$ becomes deeper. For $c \geq 0$, the graph has asymptotes at $x = \pm\sqrt{c}$.



☐ PRINCIPLES OF PROBLEM SOLVING

1.



By using the area formula for a triangle, $\frac{1}{2}$ (base) (height), in two ways, we see that $\frac{1}{2}(4)(y) = \frac{1}{2}(h)(a)$, so $a = \dfrac{4y}{h}$. Since $4^2 + y^2 = h^2$, $y = \sqrt{h^2 - 16}$, and $a = \dfrac{4\sqrt{h^2 - 16}}{h}$.

3. $|2x - 1| = \begin{cases} 2x - 1 & \text{if } x \geq \frac{1}{2} \\ 1 - 2x & \text{if } x < \frac{1}{2} \end{cases}$ and $|x + 5| = \begin{cases} x + 5 & \text{if } x \geq -5 \\ -x - 5 & \text{if } x < -5 \end{cases}$

Therefore, we consider the three cases $x < -5$, $-5 \leq x < \frac{1}{2}$, and $x \geq \frac{1}{2}$.

If $x < -5$, we must have $1 - 2x - (-x - 5) = 3 \iff x = 3$, which is false, since we are considering $x < -5$.

If $-5 \leq x < \frac{1}{2}$, we must have $1 - 2x - (x + 5) = 3 \iff x = -\frac{7}{3}$.

If $x \geq \frac{1}{2}$, we must have $2x - 1 - (x + 5) = 3 \iff x = 9$.

So the two solutions of the equation are $x = -\frac{7}{3}$ and $x = 9$.

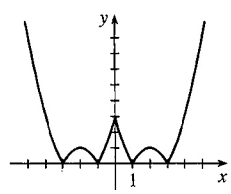
5. $f(x) = |x^2 - 4|x| + 3|$. If $x \geq 0$, then $f(x) = |x^2 - 4x + 3| = |(x - 1)(x - 3)|$.

Case (i): If $0 < x \leq 1$, then $f(x) = x^2 - 4x + 3$.

Case (ii): If $1 < x \leq 3$, then $f(x) = -(x^2 - 4x + 3) = -x^2 + 4x - 3$.

Case (iii): If $x > 3$, then $f(x) = x^2 - 4x + 3$.

This enables us to sketch the graph for $x \geq 0$. Then we use the fact that f is an even function to reflect this part of the graph about the y-axis to obtain the entire graph. Or, we could consider also the cases $x < -3$, $-3 \leq x < -1$, and $-1 \leq x < 0$.



7. Remember that $|a| = a$ if $a \geq 0$ and that $|a| = -a$ if $a < 0$. Thus,

$$x + |x| = \begin{cases} 2x & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases} \quad \text{and} \quad y + |y| = \begin{cases} 2y & \text{if } y \geq 0 \\ 0 & \text{if } y < 0 \end{cases}$$

We will consider the equation $x + |x| = y + |y|$ in four cases.

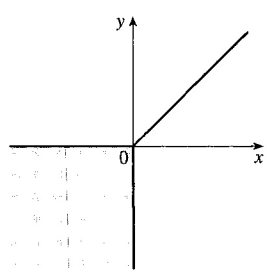
(1) $x \geq 0, y \geq 0$	(2) $x \geq 0, y < 0$	(3) $x < 0, y \geq 0$	(4) $x < 0, y < 0$
$2x = 2y$	$2x = 0$	$0 = 2y$	$0 = 0$
$x = y$	$x = 0$	$0 = y$	

Case 1 gives us the line $y = x$ with nonnegative x and y.

Case 2 gives us the portion of the y-axis with y negative.

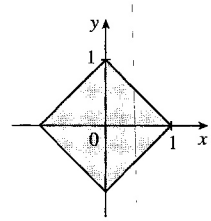
Case 3 gives us the portion of the x-axis with x negative.

Case 4 gives us the entire third quadrant.



9. $|x| + |y| \le 1$. The boundary of the region has equation $|x| + |y| = 1$.

In quadrants I, II, III, and IV, this becomes the lines $x + y = 1$, $-x + y = 1$, $-x - y = 1$, and $x - y = 1$ respectively.



11. $(\log_2 3)(\log_3 4)(\log_4 5)\cdots(\log_{31} 32) = \left(\frac{\ln 3}{\ln 2}\right)\left(\frac{\ln 4}{\ln 3}\right)\left(\frac{\ln 5}{\ln 4}\right)\cdots\left(\frac{\ln 32}{\ln 31}\right) = \frac{\ln 32}{\ln 2} = \frac{\ln 2^5}{\ln 2} = \frac{5\ln 2}{\ln 2} = 5$

13. $\ln(x^2 - 2x - 2) \le 0 \;\Rightarrow\; x^2 - 2x - 2 \le e^0 = 1 \;\Rightarrow\; x^2 - 2x - 3 \le 0 \;\Rightarrow\; (x-3)(x+1) \le 0 \;\Rightarrow$ $x \in [-1, 3]$. Since the argument must be positive, $x^2 - 2x - 2 > 0 \;\Rightarrow\; \left[x - \left(1 - \sqrt{3}\right)\right]\left[x - \left(1 + \sqrt{3}\right)\right] > 0$ $\Rightarrow\; x \in \left(-\infty, 1 - \sqrt{3}\right) \cup \left(1 + \sqrt{3}, \infty\right)$. The intersection of these intervals is $\left[-1, 1 - \sqrt{3}\right) \cup \left(1 + \sqrt{3}, 3\right]$.

15. Let d be the distance traveled on each half of the trip. Let t_1 and t_2 be the times taken for the first and second halves of the trip.

For the first half of the trip we have $t_1 = d/30$ and for the second half we have $t_2 = d/60$. Thus, the average speed for the entire trip is $\frac{\text{total distance}}{\text{total time}} = \frac{2d}{t_1 + t_2} = \frac{2d}{\frac{d}{30} + \frac{d}{60}} \cdot \frac{60}{60} = \frac{120d}{2d + d} = \frac{120d}{3d} = 40$. The average speed for the entire trip is 40 mi/h.

17. Let S_n be the statement that $7^n - 1$ is divisible by 6.

- S_1 is true because $7^1 - 1 = 6$ is divisible by 6.
- Assume S_k is true, that is, $7^k - 1$ is divisible by 6. In other words, $7^k - 1 = 6m$ for some positive integer m. Then $7^{k+1} - 1 = 7^k \cdot 7 - 1 = (6m + 1) \cdot 7 - 1 = 42m + 6 = 6(7m + 1)$, which is divisible by 6, so S_{k+1} is true.
- Therefore, by mathematical induction, $7^n - 1$ is divisible by 6 for every positive integer n.

19. $f_0(x) = x^2$ and $f_{n+1}(x) = f_0(f_n(x))$ for $n = 0, 1, 2, \ldots$.

$f_1(x) = f_0(f_0(x)) = f_0(x^2) = (x^2)^2 = x^4$, $f_2(x) = f_0(f_1(x)) = f_0(x^4) = (x^4)^2 = x^8$,

$f_3(x) = f_0(f_2(x)) = f_0(x^8) = (x^8)^2 = x^{16}, \ldots$. Thus, a general formula is $f_n(x) = x^{2^{n+1}}$.