

2 LIMITS AND DERIVATIVES

2.1 The Tangent and Velocity Problems

1. (a) Using $P(15, 250)$, we construct the following table:

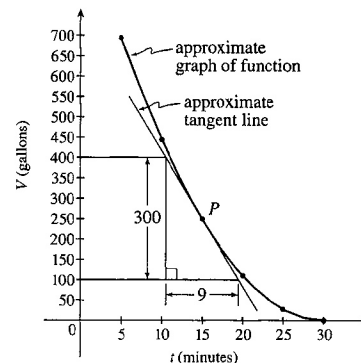
t	Q	slope = m_{PQ}
5	(5, 694)	$\frac{694-250}{5-15} = -\frac{444}{10} = -44.4$
10	(10, 444)	$\frac{444-250}{10-15} = -\frac{194}{5} = -38.8$
20	(20, 111)	$\frac{111-250}{20-15} = -\frac{139}{5} = -27.8$
25	(25, 28)	$\frac{28-250}{25-15} = -\frac{222}{10} = -22.2$
30	(30, 0)	$\frac{0-250}{30-15} = -\frac{250}{15} = -16.\bar{6}$

- (b) Using the values of t that correspond to the points closest to P ($t = 10$ and $t = 20$), we have

$$\frac{-38.8 + (-27.8)}{2} = -33.3$$

- (c) From the graph, we can estimate the slope of the tangent line at P to be

$$\frac{-300}{9} = -33.\bar{3}.$$



3. For the curve $y = x/(1+x)$ and the point $P(1, \frac{1}{2})$:

- (a)

	x	Q	m_{PQ}
(i)	0.5	(0.5, 0.333333)	0.333333
(ii)	0.9	(0.9, 0.473684)	0.263158
(iii)	0.99	(0.99, 0.497487)	0.251256
(iv)	0.999	(0.999, 0.499750)	0.250125
(v)	1.5	(1.5, 0.6)	0.2
(vi)	1.1	(1.1, 0.523810)	0.238095
(vii)	1.01	(1.01, 0.502488)	0.248756
(viii)	1.001	(1.001, 0.500250)	0.249875

- (b) The slope appears to be $\frac{1}{4}$.

$$(c) \quad y - \frac{1}{2} = \frac{1}{4}(x - 1) \quad \text{or} \\ y = \frac{1}{4}x + \frac{1}{4}.$$

5. (a) $y = y(t) = 40t - 16t^2$. At $t = 2$, $y = 40(2) - 16(2)^2 = 16$. The average velocity between times 2 and $2 + h$ is $v_{\text{ave}} = \frac{y(2+h) - y(2)}{(2+h) - 2} = \frac{[40(2+h) - 16(2+h)^2] - 16}{h} = \frac{-24h - 16h^2}{h} = -24 - 16h$, if $h \neq 0$.

(i) $[2, 2.5]: h = 0.5, v_{\text{ave}} = -32 \text{ ft/s}$

(ii) $[2, 2.1]: h = 0.1, v_{\text{ave}} = -25.6 \text{ ft/s}$

(iii) $[2, 2.05]: h = 0.05, v_{\text{ave}} = -24.8 \text{ ft/s}$

(iv) $[2, 2.01]: h = 0.01, v_{\text{ave}} = -24.16 \text{ ft/s}$

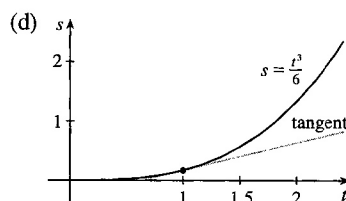
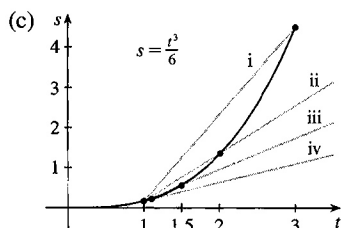
- (b) The instantaneous velocity when $t = 2$ (h approaches 0) is -24 ft/s .

- 7.
- $s = s(t) = t^3/6$
- . Average velocity between times 1 and
- $1 + h$
- is

$$v_{\text{ave}} = \frac{s(1+h) - s(1)}{(1+h) - 1} = \frac{(1+h)^3/6 - 1/6}{h} = \frac{h^3 + 3h^2 + 3h}{6h} = \frac{h^2 + 3h + 3}{6} \text{ if } h \neq 0.$$

- (a) (i) $[1, 3]: h = 2, v_{\text{ave}} = \frac{13}{6}$ ft/s (ii) $[1, 2]: h = 1, v_{\text{ave}} = \frac{7}{6}$ ft/s
 (iii) $[1, 1.5]: h = 0.5, v_{\text{ave}} = \frac{19}{24}$ ft/s (iv) $[1, 1.1]: h = 0.1, v_{\text{ave}} = \frac{331}{600}$ ft/s

(b) As h approaches 0, the velocity approaches $\frac{3}{6} = \frac{1}{2}$ ft/s.



9. For the curve
- $y = \sin(10\pi/x)$
- and the point
- $P(1, 0)$
- :

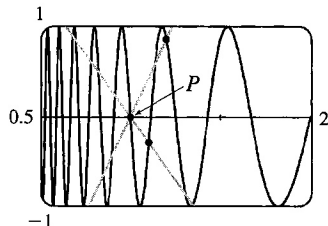
(a)

x	Q	m_{PQ}
2	(2, 0)	0
1.5	(1.5, 0.8660)	1.7321
1.4	(1.4, -0.4339)	-1.0847
1.3	(1.3, -0.8230)	-2.7433
1.2	(1.2, 0.8660)	4.3301
1.1	(1.1, -0.2817)	-2.8173

x	Q	m_{PQ}
0.5	(0.5, 0)	0
0.6	(0.6, 0.8660)	-2.1651
0.7	(0.7, 0.7818)	-2.6061
0.8	(0.8, 1)	-5
0.9	(0.9, -0.3420)	3.4202

As x approaches 1, the slopes do not appear to be approaching any particular value.

(b)



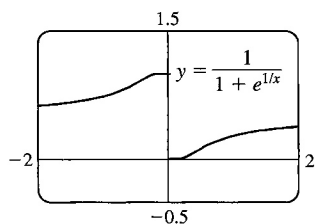
We see that problems with estimation are caused by the frequent oscillations of the graph. The tangent is so steep at P that we need to take x -values much closer to 1 in order to get accurate estimates of its slope.

- (c) If we choose $x = 1.001$, then the point Q is $(1.001, -0.0314)$ and $m_{PQ} \approx -31.3794$. If $x = 0.999$, then Q is $(0.999, 0.0314)$ and $m_{PQ} = -31.4422$. The average of these slopes is -31.4108 . So we estimate that the slope of the tangent line at P is about -31.4 .

2.2 The Limit of a Function

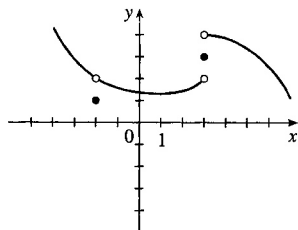
1. As x approaches 2, $f(x)$ approaches 5. [Or, the values of $f(x)$ can be made as close to 5 as we like by taking x sufficiently close to 2 (but $x \neq 2$).] Yes, the graph could have a hole at $(2, 5)$ and be defined such that $f(2) = 3$.
3. (a) $\lim_{x \rightarrow -3} f(x) = \infty$ means that the values of $f(x)$ can be made arbitrarily large (as large as we please) by taking x sufficiently close to -3 (but not equal to -3).
- (b) $\lim_{x \rightarrow 4^+} f(x) = -\infty$ means that the values of $f(x)$ can be made arbitrarily large negative by taking x sufficiently close to 4 through values larger than 4.
5. (a) $f(x)$ approaches 2 as x approaches 1 from the left, so $\lim_{x \rightarrow 1^-} f(x) = 2$.
- (b) $f(x)$ approaches 3 as x approaches 1 from the right, so $\lim_{x \rightarrow 1^+} f(x) = 3$.
- (c) $\lim_{x \rightarrow 1} f(x)$ does not exist because the limits in part (a) and part (b) are not equal.
- (d) $f(x)$ approaches 4 as x approaches 5 from the left and from the right, so $\lim_{x \rightarrow 5} f(x) = 4$.
- (e) $f(5)$ is not defined, so it doesn't exist.
7. (a) $\lim_{t \rightarrow 0^-} g(t) = -1$ (b) $\lim_{t \rightarrow 0^+} g(t) = -2$
- (c) $\lim_{t \rightarrow 0} g(t)$ does not exist because the limits in part (a) and part (b) are not equal.
- (d) $\lim_{t \rightarrow 2^-} g(t) = 2$ (e) $\lim_{t \rightarrow 2^+} g(t) = 0$
- (f) $\lim_{t \rightarrow 2} g(t)$ does not exist because the limits in part (d) and part (e) are not equal.
- (g) $g(2) = 1$ (h) $\lim_{t \rightarrow 4} g(t) = 3$
9. (a) $\lim_{x \rightarrow -7} f(x) = -\infty$ (b) $\lim_{x \rightarrow -3} f(x) = \infty$ (c) $\lim_{x \rightarrow 0} f(x) = \infty$
- (d) $\lim_{x \rightarrow 6^-} f(x) = -\infty$ (e) $\lim_{x \rightarrow 6^+} f(x) = \infty$
- (f) The equations of the vertical asymptotes are $x = -7$, $x = -3$, $x = 0$, and $x = 6$.

11.



- (a) $\lim_{x \rightarrow 0^-} f(x) = 1$
- (b) $\lim_{x \rightarrow 0^+} f(x) = 0$
- (c) $\lim_{x \rightarrow 0} f(x)$ does not exist because the limits in part (a) and part (b) are not equal.

13. $\lim_{x \rightarrow 3^+} f(x) = 4$, $\lim_{x \rightarrow 3^-} f(x) = 2$,
 $\lim_{x \rightarrow -2} f(x) = 2$, $f(3) = 3$, $f(-2) = 1$



15. For $f(x) = \frac{x^2 - 2x}{x^2 - x - 2}$:

x	$f(x)$
2.5	0.714286
2.1	0.677419
2.05	0.672131
2.01	0.667774
2.005	0.667221
2.001	0.666778

x	$f(x)$
1.9	0.655172
1.95	0.661017
1.99	0.665552
1.995	0.666110
1.999	0.666556

It appears that $\lim_{x \rightarrow 2} \frac{x^2 - 2x}{x^2 - x - 2} = 0.\bar{6} = \frac{2}{3}$.

17. For $f(x) = \frac{e^x - 1 - x}{x^2}$:

x	$f(x)$
1	0.718282
0.5	0.594885
0.1	0.517092
0.05	0.508439
0.01	0.501671

x	$f(x)$
-1	0.367879
-0.5	0.426123
-0.1	0.483742
-0.05	0.491770
-0.01	0.498337

It appears that $\lim_{x \rightarrow 0} \frac{e^x - 1 - x}{x^2} = 0.5 = \frac{1}{2}$.

19. For $f(x) = \frac{\sqrt{x+4} - 2}{x}$:

x	$f(x)$
1	0.236068
0.5	0.242641
0.1	0.248457
0.05	0.249224
0.01	0.249844

x	$f(x)$
-1	0.267949
-0.5	0.258343
-0.1	0.251582
-0.05	0.250786
-0.01	0.250156

It appears that $\lim_{x \rightarrow 0} \frac{\sqrt{x+4} - 2}{x} = 0.25 = \frac{1}{4}$.

21. For $f(x) = \frac{x^6 - 1}{x^{10} - 1}$:

x	$f(x)$
0.5	0.985337
0.9	0.719397
0.95	0.660186
0.99	0.612018
0.999	0.601200

x	$f(x)$
1.5	0.183369
1.1	0.484119
1.05	0.540783
1.01	0.588022
1.001	0.598800

It appears that $\lim_{x \rightarrow 1} \frac{x^6 - 1}{x^{10} - 1} = 0.6 = \frac{3}{5}$.

23. $\lim_{x \rightarrow 5^+} \frac{6}{x-5} = \infty$ since $(x-5) \rightarrow 0$ as $x \rightarrow 5^+$ and $\frac{6}{x-5} > 0$ for $x > 5$.
25. $\lim_{x \rightarrow 1} \frac{2-x}{(x-1)^2} = \infty$ since the numerator is positive and the denominator approaches 0 through positive values as $x \rightarrow 1$.

27. $\lim_{x \rightarrow -2^+} \frac{x-1}{x^2(x+2)} = -\infty$ since $(x+2) \rightarrow 0$ as $x \rightarrow -2^+$ and $\frac{x-1}{x^2(x+2)} < 0$ for $-2 < x < 0$.

29. $\lim_{x \rightarrow (-\pi/2)^-} \sec x = \lim_{x \rightarrow (-\pi/2)^-} (1/\cos x) = -\infty$ since $\cos x \rightarrow 0$ as $x \rightarrow (-\pi/2)^-$ and $\cos x < 0$ for $-\pi < x < -\pi/2$.

31. (a) $f(x) = 1/(x^3 - 1)$

x	$f(x)$
0.5	-1.14
0.9	-3.69
0.99	-33.7
0.999	-333.7
0.9999	-3333.7
0.99999	-33,333.7

x	$f(x)$
1.5	0.42
1.1	3.02
1.01	33.0
1.001	333.0
1.0001	3333.0
1.00001	33,333.3

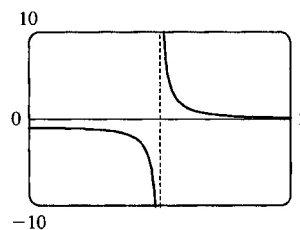
From these calculations, it seems that $\lim_{x \rightarrow 1^-} f(x) = -\infty$ and $\lim_{x \rightarrow 1^+} f(x) = \infty$.

(b) If x is slightly smaller than 1, then $x^3 - 1$ will be a negative number close to 0, and the reciprocal of $x^3 - 1$, that is, $f(x)$, will be a negative number with large absolute value. So $\lim_{x \rightarrow 1^-} f(x) = -\infty$.

If x is slightly larger than 1, then $x^3 - 1$ will be a small positive number, and its reciprocal, $f(x)$, will be a large positive number. So $\lim_{x \rightarrow 1^+} f(x) = \infty$.

(c) It appears from the graph of f that $\lim_{x \rightarrow 1^-} f(x) = -\infty$ and

$$\lim_{x \rightarrow 1^+} f(x) = \infty.$$

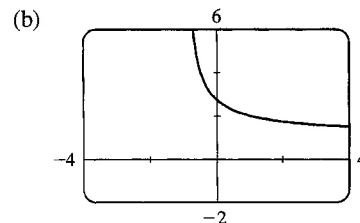


33. (a) Let $h(x) = (1+x)^{1/x}$.

x	$h(x)$
-0.001	2.71964
-0.0001	2.71842
-0.00001	2.71830
-0.000001	2.71828
0.000001	2.71828
0.00001	2.71827
0.0001	2.71815
0.001	2.71692

It appears that $\lim_{x \rightarrow 0} (1+x)^{1/x} \approx 2.71828$, which is approximately e .

In Section 3.8 we will see that the value of the limit is exactly e .



35. For $f(x) = x^2 - (2^x/1000)$:

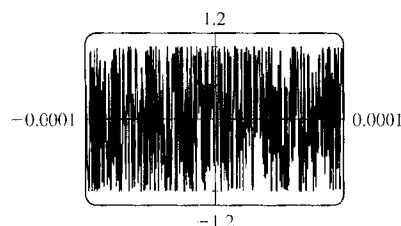
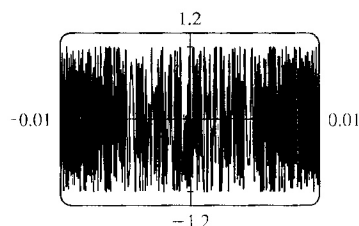
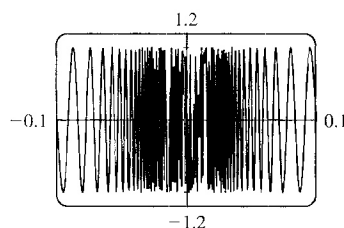
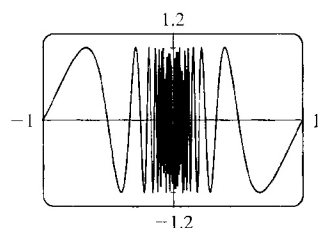
(a)

x	$f(x)$
1	0.998000
0.8	0.638259
0.6	0.358484
0.4	0.158680
0.2	0.038851
0.1	0.008928
0.05	0.001465

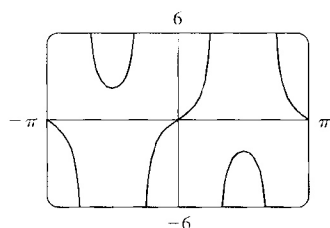
It appears that $\lim_{x \rightarrow 0} f(x) = 0$.

(b)

x	$f(x)$
0.04	0.000572
0.02	-0.000614
0.01	-0.000907
0.005	-0.000978
0.003	-0.000993
0.001	-0.001000

It appears that $\lim_{x \rightarrow 0} f(x) = -0.001$.37. No matter how many times we zoom in toward the origin, the graphs of $f(x) = \sin(\pi/x)$ appear to consist of almost-vertical lines. This indicates more and more frequent oscillations as $x \rightarrow 0$.

39.

There appear to be vertical asymptotes of the curve $y = \tan(2 \sin x)$ at $x \approx \pm 0.90$ and $x \approx \pm 2.24$. To find the exact equations of theseasymptotes, we note that the graph of the tangent function has vertical asymptotes at $x = \frac{\pi}{2} + \pi n$. Thus, we must have $2 \sin x = \frac{\pi}{2} + \pi n$,or equivalently, $\sin x = \frac{\pi}{4} + \frac{\pi}{2}n$. Since $-1 \leq \sin x \leq 1$, we musthave $\sin x = \pm \frac{\pi}{4}$ and so $x = \pm \sin^{-1} \frac{\pi}{4}$ (corresponding to $x \approx \pm 0.90$).Just as 150° is the reference angle for 30° , $\pi - \sin^{-1} \frac{\pi}{4}$ is the reference angle for $\sin^{-1} \frac{\pi}{4}$. So $x = \pm(\pi - \sin^{-1} \frac{\pi}{4})$ are also equations of the vertical asymptotes (corresponding to $x \approx \pm 2.24$).

2.3 Calculating Limits Using the Limit Laws

$$1. \text{ (a) } \lim_{x \rightarrow a} [f(x) + h(x)] = \lim_{x \rightarrow a} f(x) + \lim_{x \rightarrow a} h(x) = -3 + 8 = 5 \quad \text{(b) } \lim_{x \rightarrow a} [f(x)]^2 = \left[\lim_{x \rightarrow a} f(x) \right]^2 = (-3)^2 = 9$$

$$\text{(c) } \lim_{x \rightarrow a} \sqrt[3]{h(x)} = \sqrt[3]{\lim_{x \rightarrow a} h(x)} = \sqrt[3]{8} = 2 \quad \text{(d) } \lim_{x \rightarrow a} \frac{1}{f(x)} = \frac{1}{\lim_{x \rightarrow a} f(x)} = \frac{1}{-3} = -\frac{1}{3}$$

$$\text{(e) } \lim_{x \rightarrow a} \frac{f(x)}{h(x)} = \frac{\lim_{x \rightarrow a} f(x)}{\lim_{x \rightarrow a} h(x)} = \frac{-3}{8} = -\frac{3}{8} \quad \text{(f) } \lim_{x \rightarrow a} \frac{g(x)}{f(x)} = \frac{\lim_{x \rightarrow a} g(x)}{\lim_{x \rightarrow a} f(x)} = \frac{0}{-3} = 0$$

(g) The limit does not exist, since $\lim_{x \rightarrow a} g(x) = 0$ but $\lim_{x \rightarrow a} f(x) \neq 0$.

$$\text{(h) } \lim_{x \rightarrow a} \frac{2f(x)}{h(x) - f(x)} = \frac{2 \lim_{x \rightarrow a} f(x)}{\lim_{x \rightarrow a} h(x) - \lim_{x \rightarrow a} f(x)} = \frac{2(-3)}{8 - (-3)} = -\frac{6}{11}$$

$$\begin{aligned} 3. \lim_{x \rightarrow -2} (3x^4 + 2x^2 - x + 1) &= \lim_{x \rightarrow -2} 3x^4 + \lim_{x \rightarrow -2} 2x^2 - \lim_{x \rightarrow -2} x + \lim_{x \rightarrow -2} 1 && \text{[Limit Laws 1 and 2]} \\ &= 3 \lim_{x \rightarrow -2} x^4 + 2 \lim_{x \rightarrow -2} x^2 - \lim_{x \rightarrow -2} x + \lim_{x \rightarrow -2} 1 && [3] \\ &= 3(-2)^4 + 2(-2)^2 - (-2) + (1) && [9, 8, \text{ and } 7] \\ &= 48 + 8 + 2 + 1 = 59 \end{aligned}$$

$$\begin{aligned} 5. \lim_{x \rightarrow 3} (x^2 - 4)(x^3 + 5x - 1) &= \lim_{x \rightarrow 3} (x^2 - 4) \cdot \lim_{x \rightarrow 3} (x^3 + 5x - 1) && \text{[Limit Law 4]} \\ &= \left(\lim_{x \rightarrow 3} x^2 - \lim_{x \rightarrow 3} 4 \right) \cdot \left(\lim_{x \rightarrow 3} x^3 + 5 \lim_{x \rightarrow 3} x - \lim_{x \rightarrow 3} 1 \right) && [2, 1, \text{ and } 3] \\ &= (3^2 - 4) \cdot (3^3 + 5 \cdot 3 - 1) && [7, 8, \text{ and } 9] \\ &= 5 \cdot 41 = 205 \end{aligned}$$

$$\begin{aligned} 7. \lim_{x \rightarrow 1} \left(\frac{1 + 3x}{1 + 4x^2 + 3x^4} \right)^3 &= \left(\lim_{x \rightarrow 1} \frac{1 + 3x}{1 + 4x^2 + 3x^4} \right)^3 && [6] \\ &= \left[\frac{\lim_{x \rightarrow 1} (1 + 3x)}{\lim_{x \rightarrow 1} (1 + 4x^2 + 3x^4)} \right]^3 && [5] \\ &= \left[\frac{\lim_{x \rightarrow 1} 1 + 3 \lim_{x \rightarrow 1} x}{\lim_{x \rightarrow 1} 1 + 4 \lim_{x \rightarrow 1} x^2 + 3 \lim_{x \rightarrow 1} x^4} \right]^3 && [2, 1, \text{ and } 3] \\ &= \left[\frac{1 + 3(1)}{1 + 4(1)^2 + 3(1)^4} \right]^3 = \left[\frac{4}{8} \right]^3 = \left(\frac{1}{2} \right)^3 = \frac{1}{8} && [7, 8, \text{ and } 9] \end{aligned}$$

$$\begin{aligned} 9. \lim_{x \rightarrow 4^-} \sqrt{16 - x^2} &= \sqrt{\lim_{x \rightarrow 4^-} (16 - x^2)} && [11] \\ &= \sqrt{\lim_{x \rightarrow 4^-} 16 - \lim_{x \rightarrow 4^-} x^2} && [2] \\ &= \sqrt{16 - (4)^2} = 0 && [7 \text{ and } 9] \end{aligned}$$

$$11. \lim_{x \rightarrow 2} \frac{x^2 + x - 6}{x - 2} = \lim_{x \rightarrow 2} \frac{(x + 3)(x - 2)}{x - 2} = \lim_{x \rightarrow 2} (x + 3) = 2 + 3 = 5$$

$$13. \lim_{x \rightarrow 2} \frac{x^2 - x + 6}{x - 2} \text{ does not exist since } x - 2 \rightarrow 0 \text{ but } x^2 - x + 6 \rightarrow 8 \text{ as } x \rightarrow 2.$$

$$15. \lim_{t \rightarrow -3} \frac{t^2 - 9}{2t^2 + 7t + 3} = \lim_{t \rightarrow -3} \frac{(t+3)(t-3)}{(2t+1)(t+3)} = \lim_{t \rightarrow -3} \frac{t-3}{2t+1} = \frac{-3-3}{2(-3)+1} = \frac{-6}{-5} = \frac{6}{5}$$

$$17. \lim_{h \rightarrow 0} \frac{(4+h)^2 - 16}{h} = \lim_{h \rightarrow 0} \frac{(16+8h+h^2) - 16}{h} = \lim_{h \rightarrow 0} \frac{8h+h^2}{h} = \lim_{h \rightarrow 0} \frac{h(8+h)}{h} = \lim_{h \rightarrow 0} (8+h) = 8+0 = 8$$

$$19. \lim_{h \rightarrow 0} \frac{(1+h)^4 - 1}{h} = \lim_{h \rightarrow 0} \frac{(1+4h+6h^2+4h^3+h^4) - 1}{h} = \lim_{h \rightarrow 0} \frac{4h+6h^2+4h^3+h^4}{h} \\ = \lim_{h \rightarrow 0} \frac{h(4+6h+4h^2+h^3)}{h} = \lim_{h \rightarrow 0} (4+6h+4h^2+h^3) = 4+0+0+0 = 4$$

$$21. \lim_{t \rightarrow 9} \frac{9-t}{3-\sqrt{t}} = \lim_{t \rightarrow 9} \frac{(3+\sqrt{t})(3-\sqrt{t})}{3-\sqrt{t}} = \lim_{t \rightarrow 9} (3+\sqrt{t}) = 3+\sqrt{9} = 6$$

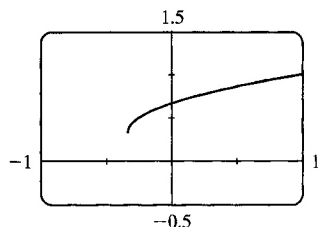
$$23. \lim_{x \rightarrow 7} \frac{\sqrt{x+2} - 3}{x-7} = \lim_{x \rightarrow 7} \frac{\sqrt{x+2} - 3}{x-7} \cdot \frac{\sqrt{x+2} + 3}{\sqrt{x+2} + 3} = \lim_{x \rightarrow 7} \frac{(x+2) - 9}{(x-7)(\sqrt{x+2} + 3)} \\ = \lim_{x \rightarrow 7} \frac{x-7}{(x-7)(\sqrt{x+2} + 3)} = \lim_{x \rightarrow 7} \frac{1}{\sqrt{x+2} + 3} = \frac{1}{\sqrt{9} + 3} = \frac{1}{6}$$

$$25. \lim_{x \rightarrow -4} \frac{\frac{1}{4} + \frac{1}{x}}{4+x} = \lim_{x \rightarrow -4} \frac{\frac{x+4}{4x}}{4+x} = \lim_{x \rightarrow -4} \frac{x+4}{4x(4+x)} = \lim_{x \rightarrow -4} \frac{1}{4x} = \frac{1}{4(-4)} = -\frac{1}{16}$$

$$27. \lim_{x \rightarrow 9} \frac{x^2 - 81}{\sqrt{x} - 3} = \lim_{x \rightarrow 9} \frac{(x-9)(x+9)}{\sqrt{x} - 3} = \lim_{x \rightarrow 9} \frac{(\sqrt{x}-3)(\sqrt{x}+3)(x+9)}{\sqrt{x} - 3} \quad \left[\begin{array}{l} \text{factor } x-9 \text{ as a} \\ \text{difference of squares} \end{array} \right] \\ = \lim_{x \rightarrow 9} [(\sqrt{x}+3)(x+9)] = (\sqrt{9}+3)(9+9) = 6 \cdot 18 = 108$$

$$29. \lim_{t \rightarrow 0} \left(\frac{1}{t\sqrt{1+t}} - \frac{1}{t} \right) = \lim_{t \rightarrow 0} \frac{1 - \sqrt{1+t}}{t\sqrt{1+t}} = \lim_{t \rightarrow 0} \frac{(1 - \sqrt{1+t})(1 + \sqrt{1+t})}{t\sqrt{1+t}(1 + \sqrt{1+t})} = \lim_{t \rightarrow 0} \frac{-t}{t\sqrt{1+t}(1 + \sqrt{1+t})} \\ = \lim_{t \rightarrow 0} \frac{-1}{\sqrt{1+t}(1 + \sqrt{1+t})} = \frac{-1}{\sqrt{1+0}(1 + \sqrt{1+0})} = -\frac{1}{2}$$

31. (a)



$$\lim_{x \rightarrow 0} \frac{x}{\sqrt{1+3x} - 1} \approx \frac{2}{3}$$

(b)

x	$f(x)$
-0.001	0.6661663
-0.0001	0.6666167
-0.00001	0.6666617
-0.000001	0.6666662
0.000001	0.6666672
0.00001	0.6666717
0.0001	0.6667167
0.001	0.6671663

The limit appears to be $\frac{2}{3}$.

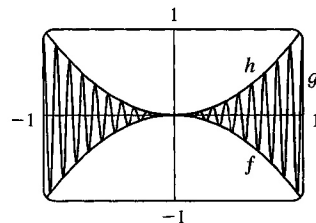
$$\begin{aligned}
 \text{(c) } \lim_{x \rightarrow 0} \left(\frac{x}{\sqrt{1+3x}-1} \cdot \frac{\sqrt{1+3x}+1}{\sqrt{1+3x}+1} \right) &= \lim_{x \rightarrow 0} \frac{x(\sqrt{1+3x}+1)}{(1+3x)-1} = \lim_{x \rightarrow 0} \frac{x(\sqrt{1+3x}+1)}{3x} \\
 &= \frac{1}{3} \lim_{x \rightarrow 0} (\sqrt{1+3x}+1) && \text{[Limit Law 3]} \\
 &= \frac{1}{3} \left[\sqrt{\lim_{x \rightarrow 0} (1+3x)} + \lim_{x \rightarrow 0} 1 \right] && \text{[1 and 11]} \\
 &= \frac{1}{3} \left(\sqrt{\lim_{x \rightarrow 0} 1 + 3 \lim_{x \rightarrow 0} x} + 1 \right) && \text{[1, 3, and 7]} \\
 &= \frac{1}{3} (\sqrt{1+3 \cdot 0} + 1) && \text{[7 and 8]} \\
 &= \frac{1}{3} (1+1) = \frac{2}{3}
 \end{aligned}$$

33. Let $f(x) = -x^2$, $g(x) = x^2 \cos 20\pi x$ and $h(x) = x^2$. Then

$$-1 \leq \cos 20\pi x \leq 1 \Rightarrow -x^2 \leq x^2 \cos 20\pi x \leq x^2 \Rightarrow$$

$f(x) \leq g(x) \leq h(x)$. So since $\lim_{x \rightarrow 0} f(x) = \lim_{x \rightarrow 0} h(x) = 0$, by the

Squeeze Theorem we have $\lim_{x \rightarrow 0} g(x) = 0$.



35. $1 \leq f(x) \leq x^2 + 2x + 2$ for all x . Now $\lim_{x \rightarrow -1} 1 = 1$ and

$$\lim_{x \rightarrow -1} (x^2 + 2x + 2) = \lim_{x \rightarrow -1} x^2 + 2 \lim_{x \rightarrow -1} x + \lim_{x \rightarrow -1} 2 = (-1)^2 + 2(-1) + 2 = 1. \text{ Therefore, by the Squeeze}$$

Theorem, $\lim_{x \rightarrow -1} f(x) = 1$.

37. $-1 \leq \cos(2/x) \leq 1 \Rightarrow -x^4 \leq x^4 \cos(2/x) \leq x^4$. Since $\lim_{x \rightarrow 0} (-x^4) = 0$ and $\lim_{x \rightarrow 0} x^4 = 0$, we have

$$\lim_{x \rightarrow 0} [x^4 \cos(2/x)] = 0 \text{ by the Squeeze Theorem.}$$

39. If $x > -4$, then $|x+4| = x+4$, so $\lim_{x \rightarrow -4^+} |x+4| = \lim_{x \rightarrow -4^+} (x+4) = -4+4 = 0$.

If $x < -4$, then $|x+4| = -(x+4)$, so $\lim_{x \rightarrow -4^-} |x+4| = \lim_{x \rightarrow -4^-} -(x+4) = -(-4+4) = 0$.

Since the right and left limits are equal, $\lim_{x \rightarrow -4} |x+4| = 0$.

41. If $x > 2$, then $|x-2| = x-2$, so $\lim_{x \rightarrow 2^+} \frac{|x-2|}{x-2} = \lim_{x \rightarrow 2^+} \frac{x-2}{x-2} = \lim_{x \rightarrow 2^+} 1 = 1$. If $x < 2$, then

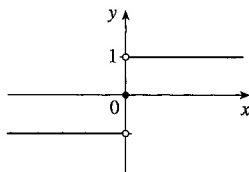
$$|x-2| = -(x-2), \text{ so } \lim_{x \rightarrow 2^-} \frac{|x-2|}{x-2} = \lim_{x \rightarrow 2^-} \frac{-(x-2)}{x-2} = \lim_{x \rightarrow 2^-} -1 = -1. \text{ The right and left limits are}$$

different, so $\lim_{x \rightarrow 2} \frac{|x-2|}{x-2}$ does not exist.

43. Since $|x| = -x$ for $x < 0$, we have $\lim_{x \rightarrow 0^-} \left(\frac{1}{x} - \frac{1}{|x|} \right) = \lim_{x \rightarrow 0^-} \left(\frac{1}{x} - \frac{1}{-x} \right) = \lim_{x \rightarrow 0^-} \frac{2}{x}$, which does not exist

since the denominator approaches 0 and the numerator does not.

45. (a)

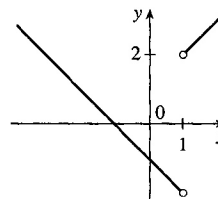
(b) (i) Since $\operatorname{sgn} x = 1$ for $x > 0$, $\lim_{x \rightarrow 0^+} \operatorname{sgn} x = \lim_{x \rightarrow 0^+} 1 = 1$.(ii) Since $\operatorname{sgn} x = -1$ for $x < 0$, $\lim_{x \rightarrow 0^-} \operatorname{sgn} x = \lim_{x \rightarrow 0^-} -1 = -1$.(iii) Since $\lim_{x \rightarrow 0^-} \operatorname{sgn} x \neq \lim_{x \rightarrow 0^+} \operatorname{sgn} x$, $\lim_{x \rightarrow 0} \operatorname{sgn} x$ does not exist.(iv) Since $|\operatorname{sgn} x| = 1$ for $x \neq 0$, $\lim_{x \rightarrow 0} |\operatorname{sgn} x| = \lim_{x \rightarrow 0} 1 = 1$.

$$47. (a) (i) \lim_{x \rightarrow 1^+} \frac{x^2 - 1}{|x - 1|} = \lim_{x \rightarrow 1^+} \frac{x^2 - 1}{x - 1} = \lim_{x \rightarrow 1^+} (x + 1) = 2$$

$$(ii) \lim_{x \rightarrow 1^-} \frac{x^2 - 1}{|x - 1|} = \lim_{x \rightarrow 1^-} \frac{x^2 - 1}{-(x - 1)} = \lim_{x \rightarrow 1^-} -(x + 1) = -2$$

(b) No, $\lim_{x \rightarrow 1} F(x)$ does not exist since $\lim_{x \rightarrow 1^+} F(x) \neq \lim_{x \rightarrow 1^-} F(x)$.

(c)



$$49. (a) (i) \lfloor x \rfloor = -2 \text{ for } -2 \leq x < -1, \text{ so } \lim_{x \rightarrow -2^+} \lfloor x \rfloor = \lim_{x \rightarrow -2^+} (-2) = -2$$

(ii) $\lfloor x \rfloor = -3$ for $-3 \leq x < -2$, so $\lim_{x \rightarrow -2^-} \lfloor x \rfloor = \lim_{x \rightarrow -2^-} (-3) = -3$. The right and left limits are different, so $\lim_{x \rightarrow -2} \lfloor x \rfloor$ does not exist.

$$(iii) \lfloor x \rfloor = -3 \text{ for } -3 \leq x < -2, \text{ so } \lim_{x \rightarrow -2.4} \lfloor x \rfloor = \lim_{x \rightarrow -2.4} (-3) = -3.$$

$$(b) (i) \lfloor x \rfloor = n - 1 \text{ for } n - 1 \leq x < n, \text{ so } \lim_{x \rightarrow n^-} \lfloor x \rfloor = \lim_{x \rightarrow n^-} (n - 1) = n - 1.$$

$$(ii) \lfloor x \rfloor = n \text{ for } n \leq x < n + 1, \text{ so } \lim_{x \rightarrow n^+} \lfloor x \rfloor = \lim_{x \rightarrow n^+} n = n.$$

(c) $\lim_{x \rightarrow a} \lfloor x \rfloor$ exists $\Leftrightarrow a$ is not an integer.

51. The graph of $f(x) = \lfloor x \rfloor + \lfloor -x \rfloor$ is the same as the graph of $g(x) = -1$ with holes at each integer, since $f(a) = 0$ for any integer a . Thus, $\lim_{x \rightarrow 2^-} f(x) = -1$ and $\lim_{x \rightarrow 2^+} f(x) = -1$, so $\lim_{x \rightarrow 2} f(x) = -1$. However, $f(2) = \lfloor 2 \rfloor + \lfloor -2 \rfloor = 2 + (-2) = 0$, so $\lim_{x \rightarrow 2} f(x) \neq f(2)$.

53. Since $p(x)$ is a polynomial, $p(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n$. Thus, by the Limit Laws,

$$\begin{aligned} \lim_{x \rightarrow a} p(x) &= \lim_{x \rightarrow a} (a_0 + a_1x + a_2x^2 + \cdots + a_nx^n) \\ &= a_0 + a_1 \lim_{x \rightarrow a} x + a_2 \lim_{x \rightarrow a} x^2 + \cdots + a_n \lim_{x \rightarrow a} x^n \\ &= a_0 + a_1a + a_2a^2 + \cdots + a_na^n = p(a) \end{aligned}$$

Thus, for any polynomial p , $\lim_{x \rightarrow a} p(x) = p(a)$.55. Observe that $0 \leq f(x) \leq x^2$ for all x , and $\lim_{x \rightarrow 0} 0 = 0 = \lim_{x \rightarrow 0} x^2$. So, by the Squeeze Theorem, $\lim_{x \rightarrow 0} f(x) = 0$.57. Let $f(x) = H(x)$ and $g(x) = 1 - H(x)$, where H is the Heaviside function defined in Exercise 1.3.59.Thus, either f or g is 0 for any value of x . Then $\lim_{x \rightarrow 0} f(x)$ and $\lim_{x \rightarrow 0} g(x)$ do not exist, but

$$\lim_{x \rightarrow 0} [f(x)g(x)] = \lim_{x \rightarrow 0} 0 = 0.$$

59. Since the denominator approaches 0 as $x \rightarrow -2$, the limit will exist only if the numerator also approaches 0 as $x \rightarrow -2$. In order for this to happen, we need $\lim_{x \rightarrow -2} (3x^2 + ax + a + 3) = 0 \Leftrightarrow$

$$3(-2)^2 + a(-2) + a + 3 = 0 \Leftrightarrow 12 - 2a + a + 3 = 0 \Leftrightarrow a = 15. \text{ With } a = 15, \text{ the limit becomes}$$

$$\lim_{x \rightarrow -2} \frac{3x^2 + 15x + 18}{x^2 + x - 2} = \lim_{x \rightarrow -2} \frac{3(x+2)(x+3)}{(x-1)(x+2)} = \lim_{x \rightarrow -2} \frac{3(x+3)}{x-1} = \frac{3(-2+3)}{-2-1} = \frac{3}{-3} = -1.$$

2.4 The Precise Definition of a Limit

1. (a) To have $5x + 3$ within a distance of 0.1 of 13, we must have $12.9 \leq 5x + 3 \leq 13.1 \Rightarrow 9.9 \leq 5x \leq 10.1 \Rightarrow 1.98 \leq x \leq 2.02$. Thus, x must be within 0.02 units of 2 so that $5x + 3$ is within 0.1 of 13.

(b) Use 0.01 in place of 0.1 in part (a) to obtain 0.002.

3. On the left side of $x = 2$, we need $|x - 2| < \left| \frac{10}{7} - 2 \right| = \frac{4}{7}$. On the right side, we need $|x - 2| < \left| \frac{10}{3} - 2 \right| = \frac{4}{3}$. For both of these conditions to be satisfied at once, we need the more restrictive of the two to hold, that is, $|x - 2| < \frac{4}{7}$. So we can choose $\delta = \frac{4}{7}$, or any smaller positive number.

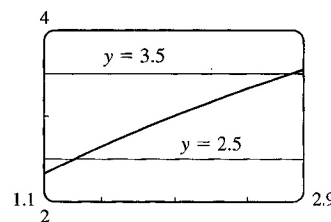
5. The leftmost question mark is the solution of $\sqrt{x} = 1.6$ and the rightmost, $\sqrt{x} = 2.4$. So the values are $1.6^2 = 2.56$ and $2.4^2 = 5.76$. On the left side, we need $|x - 4| < |2.56 - 4| = 1.44$. On the right side, we need $|x - 4| < |5.76 - 4| = 1.76$. To satisfy both conditions, we need the more restrictive condition to hold—namely, $|x - 4| < 1.44$. Thus, we can choose $\delta = 1.44$, or any smaller positive number.

7. $|\sqrt{4x+1} - 3| < 0.5 \Leftrightarrow 2.5 < \sqrt{4x+1} < 3.5$. We plot the three parts of this inequality on the same screen and identify the x -coordinates of the points of intersection using the cursor. It appears

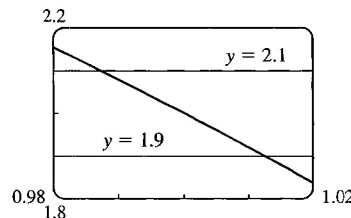
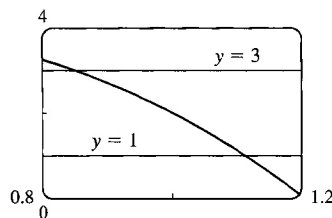
that the inequality holds for $1.3125 \leq x \leq 2.8125$. Since

$$|2 - 1.3125| = 0.6875 \text{ and } |2 - 2.8125| = 0.8125, \text{ we choose}$$

$$0 < \delta < \min\{0.6875, 0.8125\} = 0.6875.$$



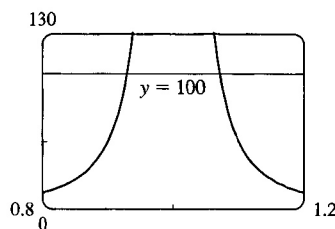
9. For $\varepsilon = 1$, the definition of a limit requires that we find δ such that $|(4 + x - 3x^3) - 2| < 1 \Leftrightarrow 1 < 4 + x - 3x^3 < 3$ whenever $0 < |x - 1| < \delta$. If we plot the graphs of $y = 1$, $y = 4 + x - 3x^3$ and $y = 3$ on the same screen, we see that we need $0.86 \leq x \leq 1.11$. So since $|1 - 0.86| = 0.14$ and $|1 - 1.11| = 0.11$, we choose $\delta = 0.11$ (or any smaller positive number). For $\varepsilon = 0.1$, we must find δ such that $|(4 + x - 3x^3) - 2| < 0.1 \Leftrightarrow 1.9 < 4 + x - 3x^3 < 2.1$ whenever $0 < |x - 1| < \delta$. From the graph, we see that we need $0.988 \leq x \leq 1.012$. So since $|1 - 0.988| = 0.012$ and $|1 - 1.012| = 0.012$, we choose $\delta = 0.012$ (or any smaller positive number) for the inequality to hold.



11. From the graph, we see that $\frac{x}{(x^2 + 1)(x - 1)^2} > 100$ whenever

$0.93 \leq x \leq 1.07$. So since $|1 - 0.93| = 0.07$ and

$|1 - 1.07| = 0.07$, we can take $\delta = 0.07$ (or any smaller positive number).



13. (a) $A = \pi r^2$ and $A = 1000 \text{ cm}^2 \Rightarrow \pi r^2 = 1000 \Rightarrow r^2 = \frac{1000}{\pi} \Rightarrow$

$$r = \sqrt{\frac{1000}{\pi}} \quad [r > 0] \approx 17.8412 \text{ cm.}$$

- (b) $|A - 1000| \leq 5 \Rightarrow -5 \leq \pi r^2 - 1000 \leq 5 \Rightarrow 1000 - 5 \leq \pi r^2 \leq 1000 + 5 \Rightarrow$

$$\sqrt{\frac{995}{\pi}} \leq r \leq \sqrt{\frac{1005}{\pi}} \Rightarrow 17.7966 \leq r \leq 17.8858. \quad \sqrt{\frac{1000}{\pi}} - \sqrt{\frac{995}{\pi}} \approx 0.04466 \text{ and}$$

$\sqrt{\frac{1005}{\pi}} - \sqrt{\frac{1000}{\pi}} \approx 0.04455$. So if the machinist gets the radius within 0.0445 cm of 17.8412, the area will be within 5 cm^2 of 1000.

- (c) x is the radius, $f(x)$ is the area, a is the target radius given in part (a), L is the target area (1000), ε is the tolerance in the area (5), and δ is the tolerance in the radius given in part (b).

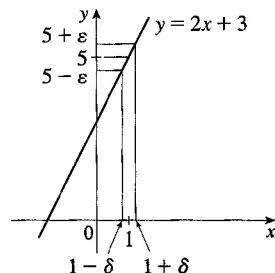
15. Given $\varepsilon > 0$, we need $\delta > 0$ such that if $0 < |x - 1| < \delta$, then

$$|(2x + 3) - 5| < \varepsilon. \text{ But } |(2x + 3) - 5| < \varepsilon \Leftrightarrow |2x - 2| < \varepsilon$$

$$\Leftrightarrow 2|x - 1| < \varepsilon \Leftrightarrow |x - 1| < \varepsilon/2. \text{ So if we choose } \delta = \varepsilon/2,$$

then $0 < |x - 1| < \delta \Rightarrow |(2x + 3) - 5| < \varepsilon$. Thus,

$\lim_{x \rightarrow 1} (2x + 3) = 5$ by the definition of a limit.



17. Given $\varepsilon > 0$, we need $\delta > 0$ such that if $0 < |x - (-3)| < \delta$, then

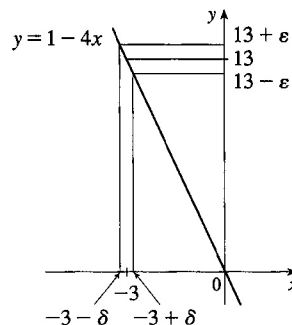
$$|(1 - 4x) - 13| < \varepsilon. \text{ But } |(1 - 4x) - 13| < \varepsilon \Leftrightarrow$$

$$|-4x - 12| < \varepsilon \Leftrightarrow |-4| |x + 3| < \varepsilon \Leftrightarrow |x - (-3)| < \varepsilon/4.$$

So if we choose $\delta = \varepsilon/4$, then $0 < |x - (-3)| < \delta \Rightarrow$

$|(1 - 4x) - 13| < \varepsilon$. Thus, $\lim_{x \rightarrow -3} (1 - 4x) = 13$ by the definition of

a limit.



19. Given $\varepsilon > 0$, we need $\delta > 0$ such that if $0 < |x - 3| < \delta$, then $\left| \frac{x}{5} - \frac{3}{5} \right| < \varepsilon \Leftrightarrow \frac{1}{5} |x - 3| < \varepsilon \Leftrightarrow$

$$|x - 3| < 5\varepsilon. \text{ So choose } \delta = 5\varepsilon. \text{ Then } 0 < |x - 3| < \delta \Rightarrow |x - 3| < 5\varepsilon \Rightarrow \frac{|x - 3|}{5} < \varepsilon \Rightarrow$$

$$\left| \frac{x}{5} - \frac{3}{5} \right| < \varepsilon. \text{ By the definition of a limit, } \lim_{x \rightarrow 3} \frac{x}{5} = \frac{3}{5}.$$

21. Given $\varepsilon > 0$, we need $\delta > 0$ such that if $0 < |x - (-5)| < \delta$, then $|(4 - \frac{3}{5}x) - 7| < \varepsilon \Leftrightarrow$

$$|-\frac{3}{5}x - 3| < \varepsilon \Leftrightarrow \frac{3}{5}|x + 5| < \varepsilon \Leftrightarrow |x - (-5)| < \frac{5}{3}\varepsilon. \text{ So choose } \delta = \frac{5}{3}\varepsilon. \text{ Then } |x - (-5)| < \delta \Rightarrow$$

$$|(4 - \frac{3}{5}x) - 7| < \varepsilon. \text{ Thus, } \lim_{x \rightarrow -5} (4 - \frac{3}{5}x) = 7 \text{ by the definition of a limit.}$$

23. Given $\varepsilon > 0$, we need $\delta > 0$ such that if $0 < |x - a| < \delta$, then $|x - a| < \varepsilon$. So $\delta = \varepsilon$ will work.

25. Given $\varepsilon > 0$, we need $\delta > 0$ such that if $0 < |x - 0| < \delta$, then $|x^2 - 0| < \varepsilon \Leftrightarrow x^2 < \varepsilon \Leftrightarrow |x| < \sqrt{\varepsilon}$. Take

$$\delta = \sqrt{\varepsilon}. \text{ Then } 0 < |x - 0| < \delta \Rightarrow |x^2 - 0| < \varepsilon. \text{ Thus, } \lim_{x \rightarrow 0} x^2 = 0 \text{ by the definition of a limit.}$$

27. Given $\varepsilon > 0$, we need $\delta > 0$ such that if $0 < |x - 0| < \delta$, then $||x| - 0| < \varepsilon$. But $||x|| = |x|$. So this is true if we pick $\delta = \varepsilon$. Thus, $\lim_{x \rightarrow 0} |x| = 0$ by the definition of a limit.

29. Given $\varepsilon > 0$, we need $\delta > 0$ such that if $0 < |x - 2| < \delta$, then $|(x^2 - 4x + 5) - 1| < \varepsilon \Leftrightarrow$

$$|x^2 - 4x + 4| < \varepsilon \Leftrightarrow |(x - 2)^2| < \varepsilon. \text{ So take } \delta = \sqrt{\varepsilon}. \text{ Then } 0 < |x - 2| < \delta \Leftrightarrow |x - 2| < \sqrt{\varepsilon} \Leftrightarrow$$

$$|(x - 2)^2| < \varepsilon. \text{ Thus, } \lim_{x \rightarrow 2} (x^2 - 4x + 5) = 1 \text{ by the definition of a limit.}$$

31. Given $\varepsilon > 0$, we need $\delta > 0$ such that if $0 < |x - (-2)| < \delta$, then $|(x^2 - 1) - 3| < \varepsilon$ or upon simplifying we

$$\text{need } |x^2 - 4| < \varepsilon \text{ whenever } 0 < |x + 2| < \delta. \text{ Notice that if } |x + 2| < 1, \text{ then } -1 < x + 2 < 1 \Rightarrow$$

$$-5 < x - 2 < -3 \Rightarrow |x - 2| < 5. \text{ So take } \delta = \min\{\varepsilon/5, 1\}. \text{ Then } 0 < |x + 2| < \delta \Rightarrow |x - 2| < 5 \text{ and}$$

$$|x + 2| < \varepsilon/5, \text{ so } |(x^2 - 1) - 3| = |(x + 2)(x - 2)| = |x + 2||x - 2| < (\varepsilon/5)(5) = \varepsilon. \text{ Thus, by the definition}$$

$$\text{of a limit, } \lim_{x \rightarrow -2} (x^2 - 1) = 3.$$

33. Given $\varepsilon > 0$, we let $\delta = \min\{2, \frac{\varepsilon}{8}\}$. If $0 < |x - 3| < \delta$, then $|x - 3| < 2 \Rightarrow -2 < x - 3 < 2 \Rightarrow$

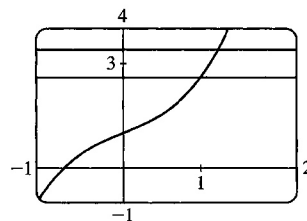
$$4 < x + 3 < 8 \Rightarrow |x + 3| < 8. \text{ Also } |x - 3| < \frac{\varepsilon}{8}, \text{ so } |x^2 - 9| = |x + 3||x - 3| < 8 \cdot \frac{\varepsilon}{8} = \varepsilon. \text{ Thus,}$$

$$\lim_{x \rightarrow 3} x^2 = 9.$$

35. (a) The points of intersection in the graph are $(x_1, 2.6)$ and $(x_2, 3.4)$

with $x_1 \approx 0.891$ and $x_2 \approx 1.093$. Thus, we can take δ to be the

smaller of $1 - x_1$ and $x_2 - 1$. So $\delta = x_2 - 1 \approx 0.093$.



(b) Solving $x^3 + x + 1 = 3 + \varepsilon$ gives us two nonreal complex roots and one real root, which is

$$x(\varepsilon) = \frac{(216 + 108\varepsilon + 12\sqrt{336 + 324\varepsilon + 81\varepsilon^2})^{2/3} - 12}{6(216 + 108\varepsilon + 12\sqrt{336 + 324\varepsilon + 81\varepsilon^2})^{1/3}}. \text{ Thus, } \delta = x(\varepsilon) - 1.$$

(c) If $\varepsilon = 0.4$, then $x(\varepsilon) \approx 1.093272342$ and $\delta = x(\varepsilon) - 1 \approx 0.093$, which agrees with our answer in part (a).

37. 1. Guessing a value for δ Given $\varepsilon > 0$, we must find $\delta > 0$ such that $|\sqrt{x} - \sqrt{a}| < \varepsilon$ whenever

$0 < |x - a| < \delta$. But $|\sqrt{x} - \sqrt{a}| = \frac{|x - a|}{\sqrt{x} + \sqrt{a}} < \varepsilon$ (from the hint). Now if we can find a positive constant C such

that $\sqrt{x} + \sqrt{a} > C$ then $\frac{|x - a|}{\sqrt{x} + \sqrt{a}} < \frac{|x - a|}{C} < \varepsilon$, and we take $|x - a| < C\varepsilon$. We can find this number by

restricting x to lie in some interval centered at a . If $|x - a| < \frac{1}{2}a$, then $-\frac{1}{2}a < x - a < \frac{1}{2}a \Rightarrow \frac{1}{2}a < x < \frac{3}{2}a$

$\Rightarrow \sqrt{x} + \sqrt{a} > \sqrt{\frac{1}{2}a} + \sqrt{a}$, and so $C = \sqrt{\frac{1}{2}a} + \sqrt{a}$ is a suitable choice for the constant. So

$|x - a| < \left(\sqrt{\frac{1}{2}a} + \sqrt{a}\right)\varepsilon$. This suggests that we let $\delta = \min\left\{\frac{1}{2}a, \left(\sqrt{\frac{1}{2}a} + \sqrt{a}\right)\varepsilon\right\}$.

2. Showing that δ works Given $\varepsilon > 0$, we let $\delta = \min\left\{\frac{1}{2}a, \left(\sqrt{\frac{1}{2}a} + \sqrt{a}\right)\varepsilon\right\}$. If $0 < |x - a| < \delta$, then

$|x - a| < \frac{1}{2}a \Rightarrow \sqrt{x} + \sqrt{a} > \sqrt{\frac{1}{2}a} + \sqrt{a}$ (as in part 1). Also $|x - a| < \left(\sqrt{\frac{1}{2}a} + \sqrt{a}\right)\varepsilon$, so

$|\sqrt{x} - \sqrt{a}| = \frac{|x - a|}{\sqrt{x} + \sqrt{a}} < \frac{\left(\sqrt{\frac{1}{2}a} + \sqrt{a}\right)\varepsilon}{\left(\sqrt{\frac{1}{2}a} + \sqrt{a}\right)} = \varepsilon$. Therefore, $\lim_{x \rightarrow a} \sqrt{x} = \sqrt{a}$ by the definition of a limit.

39. Suppose that $\lim_{x \rightarrow 0} f(x) = L$. Given $\varepsilon = \frac{1}{2}$, there exists $\delta > 0$ such that $0 < |x| < \delta \Rightarrow |f(x) - L| < \frac{1}{2}$. Take

any rational number r with $0 < |r| < \delta$. Then $f(r) = 0$, so $|0 - L| < \frac{1}{2}$, so $L < \frac{1}{2}$. Now take any irrational

number s with $0 < |s| < \delta$. Then $f(s) = 1$, so $|1 - L| < \frac{1}{2}$. Hence, $1 - L < \frac{1}{2}$, so $L > \frac{1}{2}$. This contradicts

$L < \frac{1}{2}$, so $\lim_{x \rightarrow 0} f(x)$ does not exist.

41. $\frac{1}{(x+3)^4} > 10,000 \Leftrightarrow (x+3)^4 < \frac{1}{10,000} \Leftrightarrow |x+3| < \frac{1}{\sqrt[4]{10,000}} \Leftrightarrow |x - (-3)| < \frac{1}{10}$

43. Given $M < 0$ we need $\delta > 0$ so that $\ln x < M$ whenever $0 < x < \delta$; that is, $x = e^{\ln x} < e^M$ whenever $0 < x < \delta$.

This suggests that we take $\delta = e^M$. If $0 < x < e^M$, then $\ln x < \ln e^M = M$. By the definition of a limit,

$\lim_{x \rightarrow 0^+} \ln x = -\infty$.

2.5 Continuity

1. From Definition 1, $\lim_{x \rightarrow 4} f(x) = f(4)$.

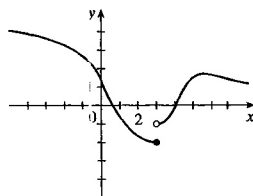
3. (a) The following are the numbers at which f is discontinuous and the type of discontinuity at that number:

-4 (removable), -2 (jump), 2 (jump), 4 (infinite).

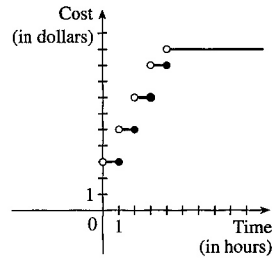
(b) f is continuous from the left at -2 since $\lim_{x \rightarrow -2^-} f(x) = f(-2)$. f is continuous from the right at 2 and 4 since

$\lim_{x \rightarrow 2^+} f(x) = f(2)$ and $\lim_{x \rightarrow 4^+} f(x) = f(4)$. It is continuous from neither side at -4 since $f(-4)$ is undefined.

5. The graph of $y = f(x)$ must have a discontinuity at $x = 3$ and must show that $\lim_{x \rightarrow 3^-} f(x) = f(3)$.



7. (a)



(b) There are discontinuities at times $t = 1, 2, 3$, and 4 . A person parking in the lot would want to keep in mind that the charge will jump at the beginning of each hour.

9. Since f and g are continuous functions,

$$\begin{aligned}\lim_{x \rightarrow 3} [2f(x) - g(x)] &= 2 \lim_{x \rightarrow 3} f(x) - \lim_{x \rightarrow 3} g(x) && \text{[by Limit Laws 2 and 3]} \\ &= 2f(3) - g(3) && \text{[by continuity of } f \text{ and } g \text{ at } x = 3\text{]} \\ &= 2 \cdot 5 - g(3) = 10 - g(3)\end{aligned}$$

Since it is given that $\lim_{x \rightarrow 3} [2f(x) - g(x)] = 4$, we have $10 - g(3) = 4$, so $g(3) = 6$.

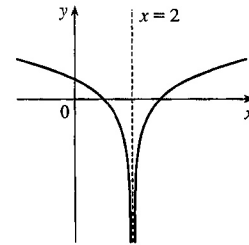
$$11. \lim_{x \rightarrow -1} f(x) = \lim_{x \rightarrow -1} (x + 2x^3)^4 = \left(\lim_{x \rightarrow -1} x + 2 \lim_{x \rightarrow -1} x^3 \right)^4 = [-1 + 2(-1)^3]^4 = (-3)^4 = 81 = f(-1).$$

By the definition of continuity, f is continuous at $a = -1$.

$$13. \text{ For } a > 2, \text{ we have } \lim_{x \rightarrow a} f(x) = \lim_{x \rightarrow a} \frac{2x+3}{x-2} = \frac{\lim_{x \rightarrow a} (2x+3)}{\lim_{x \rightarrow a} (x-2)} \quad \text{[Limit Law 5]} = \frac{2 \lim_{x \rightarrow a} x + \lim_{x \rightarrow a} 3}{\lim_{x \rightarrow a} x - \lim_{x \rightarrow a} 2}$$

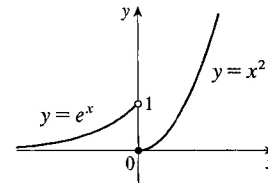
[1, 2, and 3] $= \frac{2a+3}{a-2}$ [7 and 8] $= f(a)$. Thus, f is continuous at $x = a$ for every a in $(2, \infty)$; that is, f is continuous on $(2, \infty)$.

15. $f(x) = \ln |x - 2|$ is discontinuous at 2 since $f(2) = \ln 0$ is not defined.



$$17. f(x) = \begin{cases} e^x & \text{if } x < 0 \\ x^2 & \text{if } x \geq 0 \end{cases}$$

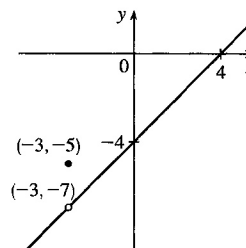
The left-hand limit of f at $a = 0$ is $\lim_{x \rightarrow 0^-} f(x) = \lim_{x \rightarrow 0^-} e^x = 1$. The right-hand limit of f at $a = 0$ is $\lim_{x \rightarrow 0^+} f(x) = \lim_{x \rightarrow 0^+} x^2 = 0$. Since these limits are not equal, $\lim_{x \rightarrow 0} f(x)$ does not exist and f is discontinuous at 0.



$$19. f(x) = \begin{cases} \frac{x^2 - x - 12}{x + 3} & \text{if } x \neq -3 \\ -5 & \text{if } x = -3 \end{cases} = \begin{cases} x - 4 & \text{if } x \neq -3 \\ -5 & \text{if } x = -3 \end{cases}$$

So $\lim_{x \rightarrow -3} f(x) = \lim_{x \rightarrow -3} (x - 4) = -7$ and $f(-3) = -5$.

Since $\lim_{x \rightarrow -3} f(x) \neq f(-3)$, f is discontinuous at -3 .



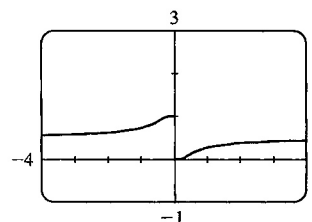
21. $F(x) = \frac{x}{x^2 + 5x + 6}$ is a rational function. So by Theorem 5 (or Theorem 7), F is continuous at every number in its domain, $\{x \mid x^2 + 5x + 6 \neq 0\} = \{x \mid (x + 3)(x + 2) \neq 0\} = \{x \mid x \neq -3, -2\}$ or $(-\infty, -3) \cup (-3, -2) \cup (-2, \infty)$.

23. By Theorem 5, the polynomials x^2 and $2x - 1$ are continuous on $(-\infty, \infty)$. By Theorem 7, the root function \sqrt{x} is continuous on $[0, \infty)$. By Theorem 9, the composite function $\sqrt{2x - 1}$ is continuous on its domain, $[\frac{1}{2}, \infty)$. By part 1 of Theorem 4, the sum $R(x) = x^2 + \sqrt{2x - 1}$ is continuous on $[\frac{1}{2}, \infty)$.

25. By Theorem 5, the polynomial $5x$ is continuous on $(-\infty, \infty)$. By Theorems 9 and 7, $\sin 5x$ is continuous on $(-\infty, \infty)$. By Theorem 7, e^x is continuous on $(-\infty, \infty)$. By part 4 of Theorem 4, the product of e^x and $\sin 5x$ is continuous at all numbers which are in both of their domains, that is, on $(-\infty, \infty)$.

27. By Theorem 5, the polynomial $t^4 - 1$ is continuous on $(-\infty, \infty)$. By Theorem 7, $\ln x$ is continuous on its domain, $(0, \infty)$. By Theorem 9, $\ln(t^4 - 1)$ is continuous on its domain, which is $\{t \mid t^4 - 1 > 0\} = \{t \mid t^4 > 1\} = \{t \mid |t| > 1\} = (-\infty, -1) \cup (1, \infty)$.

29. The function $y = \frac{1}{1 + e^{1/x}}$ is discontinuous at $x = 0$ because the left- and right-hand limits at $x = 0$ are different.



31. Because we are dealing with root functions, $5 + \sqrt{x}$ is continuous on $[0, \infty)$, $\sqrt{x + 5}$ is continuous on $[-5, \infty)$, so the quotient $f(x) = \frac{5 + \sqrt{x}}{\sqrt{5 + x}}$ is continuous on $[0, \infty)$. Since f is continuous at $x = 4$, $\lim_{x \rightarrow 4} f(x) = f(4) = \frac{7}{3}$.

33. Because $x^2 - x$ is continuous on \mathbb{R} , the composite function $f(x) = e^{x^2 - x}$ is continuous on \mathbb{R} , so $\lim_{x \rightarrow 1} f(x) = f(1) = e^{1-1} = e^0 = 1$.

$$35. f(x) = \begin{cases} x^2 & \text{if } x < 1 \\ \sqrt{x} & \text{if } x \geq 1 \end{cases}$$

By Theorem 5, since $f(x)$ equals the polynomial x^2 on $(-\infty, 1)$, f is continuous on $(-\infty, 1)$. By Theorem 7, since $f(x)$ equals the root function \sqrt{x} on $(1, \infty)$, f is continuous on $(1, \infty)$. At $x = 1$,

$\lim_{x \rightarrow 1^-} f(x) = \lim_{x \rightarrow 1^-} x^2 = 1$ and $\lim_{x \rightarrow 1^+} f(x) = \lim_{x \rightarrow 1^+} \sqrt{x} = 1$. Thus, $\lim_{x \rightarrow 1} f(x)$ exists and equals 1. Also,

$f(1) = \sqrt{1} = 1$. Thus, f is continuous at $x = 1$. We conclude that f is continuous on $(-\infty, \infty)$.

$$37. f(x) = \begin{cases} 1 + x^2 & \text{if } x \leq 0 \\ 2 - x & \text{if } 0 < x \leq 2 \\ (x - 2)^2 & \text{if } x > 2 \end{cases}$$

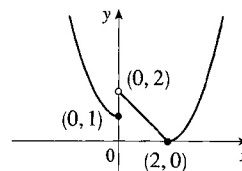
f is continuous on $(-\infty, 0)$, $(0, 2)$, and $(2, \infty)$ since it is a polynomial on

each of these intervals. Now $\lim_{x \rightarrow 0^-} f(x) = \lim_{x \rightarrow 0^-} (1 + x^2) = 1$ and

$\lim_{x \rightarrow 0^+} f(x) = \lim_{x \rightarrow 0^+} (2 - x) = 2$, so f is discontinuous at 0. Since $f(0) = 1$, f is continuous from the left at 0.

Also, $\lim_{x \rightarrow 2^-} f(x) = \lim_{x \rightarrow 2^-} (2 - x) = 0$, $\lim_{x \rightarrow 2^+} f(x) = \lim_{x \rightarrow 2^+} (x - 2)^2 = 0$, and $f(2) = 0$, so f is continuous at 2.

The only number at which f is discontinuous is 0.



$$39. f(x) = \begin{cases} x + 2 & \text{if } x < 0 \\ e^x & \text{if } 0 \leq x \leq 1 \\ 2 - x & \text{if } x > 1 \end{cases}$$

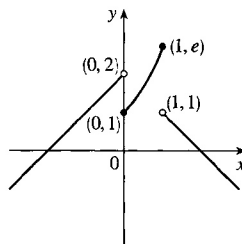
f is continuous on $(-\infty, 0)$ and $(1, \infty)$ since on each of these intervals it is

a polynomial; it is continuous on $(0, 1)$ since it is an exponential. Now

$\lim_{x \rightarrow 0^-} f(x) = \lim_{x \rightarrow 0^-} (x + 2) = 2$ and $\lim_{x \rightarrow 0^+} f(x) = \lim_{x \rightarrow 0^+} e^x = 1$, so f is

discontinuous at 0. Since $f(0) = 1$, f is continuous from the right at 0. Also $\lim_{x \rightarrow 1^-} f(x) = \lim_{x \rightarrow 1^-} e^x = e$ and

$\lim_{x \rightarrow 1^+} f(x) = \lim_{x \rightarrow 1^+} (2 - x) = 1$, so f is discontinuous at 1. Since $f(1) = e$, f is continuous from the left at 1.



41. f is continuous on $(-\infty, 3)$ and $(3, \infty)$. Now $\lim_{x \rightarrow 3^-} f(x) = \lim_{x \rightarrow 3^-} (cx + 1) = 3c + 1$ and

$\lim_{x \rightarrow 3^+} f(x) = \lim_{x \rightarrow 3^+} (cx^2 - 1) = 9c - 1$. So f is continuous $\Leftrightarrow 3c + 1 = 9c - 1 \Leftrightarrow 6c = 2 \Leftrightarrow c = \frac{1}{3}$.

Thus, for f to be continuous on $(-\infty, \infty)$, $c = \frac{1}{3}$.

43. (a) $f(x) = \frac{x^2 - 2x - 8}{x + 2} = \frac{(x - 4)(x + 2)}{x + 2}$ has a removable discontinuity at -2 because $g(x) = x - 4$ is

continuous on \mathbb{R} and $f(x) = g(x)$ for $x \neq -2$. [The discontinuity is removed by defining $f(-2) = -6$.]

(b) $f(x) = \frac{x - 7}{|x - 7|} \Rightarrow \lim_{x \rightarrow 7^-} f(x) = -1$ and $\lim_{x \rightarrow 7^+} f(x) = 1$. Thus, $\lim_{x \rightarrow 7} f(x)$ does not exist, so the

discontinuity is not removable. (It is a jump discontinuity.)

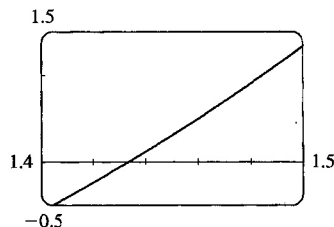
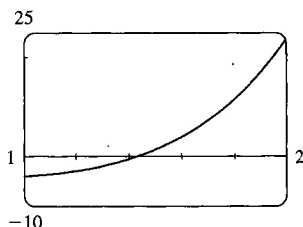
(c) $f(x) = \frac{x^3 + 64}{x + 4} = \frac{(x + 4)(x^2 - 4x + 16)}{x + 4}$ has a removable discontinuity at -4 because

$g(x) = x^2 - 4x + 16$ is continuous on \mathbb{R} and $f(x) = g(x)$ for $x \neq -4$. [The discontinuity is removed by defining $f(-4) = 48$.]

(d) $f(x) = \frac{3 - \sqrt{x}}{9 - x} = \frac{3 - \sqrt{x}}{(3 - \sqrt{x})(3 + \sqrt{x})}$ has a removable discontinuity at 9 because $g(x) = \frac{1}{3 + \sqrt{x}}$ is

continuous on \mathbb{R} and $f(x) = g(x)$ for $x \neq 9$. [The discontinuity is removed by defining $f(9) = \frac{1}{6}$.]

45. $f(x) = x^3 - x^2 + x$ is continuous on the interval $[2, 3]$, $f(2) = 6$, and $f(3) = 21$. Since $6 < 10 < 21$, there is a number c in $(2, 3)$ such that $f(c) = 10$ by the Intermediate Value Theorem.
47. $f(x) = x^4 + x - 3$ is continuous on the interval $[1, 2]$, $f(1) = -1$, and $f(2) = 15$. Since $-1 < 0 < 15$, there is a number c in $(1, 2)$ such that $f(c) = 0$ by the Intermediate Value Theorem. Thus, there is a root of the equation $x^4 + x - 3 = 0$ in the interval $(1, 2)$.
49. $f(x) = \cos x - x$ is continuous on the interval $[0, 1]$, $f(0) = 1$, and $f(1) = \cos 1 - 1 \approx -0.46$. Since $-0.46 < 0 < 1$, there is a number c in $(0, 1)$ such that $f(c) = 0$ by the Intermediate Value Theorem. Thus, there is a root of the equation $\cos x - x = 0$, or $\cos x = x$, in the interval $(0, 1)$.
51. (a) $f(x) = e^x + x - 2$ is continuous on the interval $[0, 1]$, $f(0) = -1 < 0$, and $f(1) = e - 1 \approx 1.72 > 0$. Since $-1 < 0 < 1.72$, there is a number c in $(0, 1)$ such that $f(c) = 0$ by the Intermediate Value Theorem. Thus, there is a root of the equation $e^x + x - 2 = 0$, or $e^x = 2 - x$, in the interval $(0, 1)$.
- (b) $f(0.44) \approx -0.007 < 0$ and $f(0.45) \approx 0.018 > 0$, so there is a root between 0.44 and 0.45.
53. (a) Let $f(x) = x^5 - x^2 - 4$. Then $f(1) = 1^5 - 1^2 - 4 = -4 < 0$ and $f(2) = 2^5 - 2^2 - 4 = 24 > 0$. So by the Intermediate Value Theorem, there is a number c in $(1, 2)$ such that $f(c) = c^5 - c^2 - 4 = 0$.
- (b) We can see from the graphs that, correct to three decimal places, the root is $x \approx 1.434$.



55. (\Rightarrow) If f is continuous at a , then by Theorem 8 with $g(h) = a + h$, we have

$$\lim_{h \rightarrow 0} f(a + h) = f\left(\lim_{h \rightarrow 0} (a + h)\right) = f(a).$$

(\Leftarrow) Let $\varepsilon > 0$. Since $\lim_{h \rightarrow 0} f(a + h) = f(a)$, there exists $\delta > 0$ such that $0 < |h| < \delta \Rightarrow$

$$|f(a + h) - f(a)| < \varepsilon. \text{ So if } 0 < |x - a| < \delta, \text{ then } |f(x) - f(a)| = |f(a + (x - a)) - f(a)| < \varepsilon.$$

Thus, $\lim_{x \rightarrow a} f(x) = f(a)$ and so f is continuous at a .

57. As in the previous exercise, we must show that $\lim_{h \rightarrow 0} \cos(a + h) = \cos a$ to prove that the cosine function is continuous.

$$\begin{aligned} \lim_{h \rightarrow 0} \cos(a + h) &= \lim_{h \rightarrow 0} (\cos a \cos h - \sin a \sin h) \\ &= \lim_{h \rightarrow 0} (\cos a \cos h) - \lim_{h \rightarrow 0} (\sin a \sin h) \\ &= \left(\lim_{h \rightarrow 0} \cos a\right) \left(\lim_{h \rightarrow 0} \cos h\right) - \left(\lim_{h \rightarrow 0} \sin a\right) \left(\lim_{h \rightarrow 0} \sin h\right) \\ &= (\cos a)(1) - (\sin a)(0) = \cos a \end{aligned}$$

59. $f(x) = \begin{cases} 0 & \text{if } x \text{ is rational} \\ 1 & \text{if } x \text{ is irrational} \end{cases}$ is continuous nowhere. For, given any number a and any $\delta > 0$, the interval

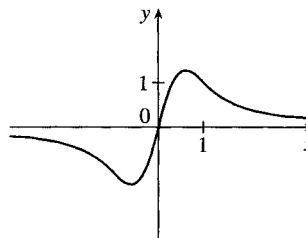
$(a - \delta, a + \delta)$ contains both infinitely many rational and infinitely many irrational numbers. Since $f(a) = 0$ or 1 , there are infinitely many numbers x with $0 < |x - a| < \delta$ and $|f(x) - f(a)| = 1$. Thus, $\lim_{x \rightarrow a} f(x) \neq f(a)$.

[In fact, $\lim_{x \rightarrow a} f(x)$ does not even exist.]

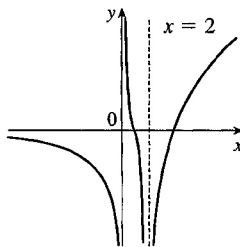
61. If there is such a number, it satisfies the equation $x^3 + 1 = x \Leftrightarrow x^3 - x + 1 = 0$. Let the left-hand side of this equation be called $f(x)$. Now $f(-2) = -5 < 0$, and $f(-1) = 1 > 0$. Note also that $f(x)$ is a polynomial, and thus continuous. So by the Intermediate Value Theorem, there is a number c between -2 and -1 such that $f(c) = 0$, so that $c = c^3 + 1$.
63. Define $u(t)$ to be the monk's distance from the monastery, as a function of time, on the first day, and define $d(t)$ to be his distance from the monastery, as a function of time, on the second day. Let D be the distance from the monastery to the top of the mountain. From the given information we know that $u(0) = 0$, $u(12) = D$, $d(0) = D$ and $d(12) = 0$. Now consider the function $u - d$, which is clearly continuous. We calculate that $(u - d)(0) = -D$ and $(u - d)(12) = D$. So by the Intermediate Value Theorem, there must be some time t_0 between 0 and 12 such that $(u - d)(t_0) = 0 \Leftrightarrow u(t_0) = d(t_0)$. So at time t_0 after 7:00 A.M., the monk will be at the same place on both days.

2.6 Limits at Infinity; Horizontal Asymptotes

1. (a) As x becomes large, the values of $f(x)$ approach 5.
 (b) As x becomes large negative, the values of $f(x)$ approach 3.
3. (a) $\lim_{x \rightarrow 2} f(x) = \infty$ (b) $\lim_{x \rightarrow -1^-} f(x) = \infty$
 (c) $\lim_{x \rightarrow -1^+} f(x) = -\infty$ (d) $\lim_{x \rightarrow \infty} f(x) = 1$
 (e) $\lim_{x \rightarrow -\infty} f(x) = 2$ (f) Vertical: $x = -1$, $x = 2$; Horizontal: $y = 1$, $y = 2$
5. $f(0) = 0$, $f(1) = 1$, $\lim_{x \rightarrow \infty} f(x) = 0$,
 f is odd



$$\begin{aligned}
 7. \quad \lim_{x \rightarrow 2} f(x) &= -\infty, \quad \lim_{x \rightarrow \infty} f(x) = \infty, \\
 \lim_{x \rightarrow -\infty} f(x) &= 0, \quad \lim_{x \rightarrow 0^+} f(x) = \infty, \\
 \lim_{x \rightarrow 0^-} f(x) &= -\infty
 \end{aligned}$$



9. If $f(x) = x^2/2^x$, then a calculator gives $f(0) = 0$, $f(1) = 0.5$, $f(2) = 1$, $f(3) = 1.125$, $f(4) = 1$,
 $f(5) = 0.78125$, $f(6) = 0.5625$, $f(7) = 0.3828125$, $f(8) = 0.25$, $f(9) = 0.158203125$, $f(10) = 0.09765625$,
 $f(20) \approx 0.00038147$, $f(50) \approx 2.2204 \times 10^{-12}$, $f(100) \approx 7.8886 \times 10^{-27}$.
 It appears that $\lim_{x \rightarrow \infty} (x^2/2^x) = 0$.

$$11. \quad \lim_{x \rightarrow \infty} \frac{3x^2 - x + 4}{2x^2 + 5x - 8} = \lim_{x \rightarrow \infty} \frac{(3x^2 - x + 4)/x^2}{(2x^2 + 5x - 8)/x^2}$$

[divide both the numerator and denominator by x^2
 (the highest power of x that
 appears in the denominator)]

$$= \frac{\lim_{x \rightarrow \infty} (3 - 1/x + 4/x^2)}{\lim_{x \rightarrow \infty} (2 + 5/x - 8/x^2)}$$

[Limit Law 5]

$$= \frac{\lim_{x \rightarrow \infty} 3 - \lim_{x \rightarrow \infty} (1/x) + \lim_{x \rightarrow \infty} (4/x^2)}{\lim_{x \rightarrow \infty} 2 + \lim_{x \rightarrow \infty} (5/x) - \lim_{x \rightarrow \infty} (8/x^2)}$$

[Limit Laws 1 and 2]

$$= \frac{3 - \lim_{x \rightarrow \infty} (1/x) + 4 \lim_{x \rightarrow \infty} (1/x^2)}{2 + 5 \lim_{x \rightarrow \infty} (1/x) - 8 \lim_{x \rightarrow \infty} (1/x^2)}$$

[Limit Laws 7 and 3]

$$= \frac{3 - 0 + 4(0)}{2 + 5(0) - 8(0)}$$

[Theorem 5 of Section 2.5]

$$= \frac{3}{2}$$

$$13. \quad \lim_{x \rightarrow \infty} \frac{1}{2x + 3} = \lim_{x \rightarrow \infty} \frac{1/x}{(2x + 3)/x} = \frac{\lim_{x \rightarrow \infty} (1/x)}{\lim_{x \rightarrow \infty} (2 + 3/x)} = \frac{\lim_{x \rightarrow \infty} (1/x)}{\lim_{x \rightarrow \infty} 2 + 3 \lim_{x \rightarrow \infty} (1/x)} = \frac{0}{2 + 3(0)} = \frac{0}{2} = 0$$

$$\begin{aligned}
 15. \quad \lim_{x \rightarrow -\infty} \frac{1 - x - x^2}{2x^2 - 7} &= \lim_{x \rightarrow -\infty} \frac{(1 - x - x^2)/x^2}{(2x^2 - 7)/x^2} = \frac{\lim_{x \rightarrow -\infty} (1/x^2 - 1/x - 1)}{\lim_{x \rightarrow -\infty} (2 - 7/x^2)} \\
 &= \frac{\lim_{x \rightarrow -\infty} (1/x^2) - \lim_{x \rightarrow -\infty} (1/x) - \lim_{x \rightarrow -\infty} 1}{\lim_{x \rightarrow -\infty} 2 - 7 \lim_{x \rightarrow -\infty} (1/x^2)} = \frac{0 - 0 - 1}{2 - 7(0)} = -\frac{1}{2}
 \end{aligned}$$

17. Divide both the numerator and denominator by x^3 (the highest power of x that occurs in the denominator).

$$\begin{aligned}\lim_{x \rightarrow \infty} \frac{x^3 + 5x}{2x^3 - x^2 + 4} &= \lim_{x \rightarrow \infty} \frac{\frac{x^3 + 5x}{x^3}}{\frac{2x^3 - x^2 + 4}{x^3}} = \lim_{x \rightarrow \infty} \frac{1 + \frac{5}{x^2}}{2 - \frac{1}{x} + \frac{4}{x^3}} = \frac{\lim_{x \rightarrow \infty} \left(1 + \frac{5}{x^2}\right)}{\lim_{x \rightarrow \infty} \left(2 - \frac{1}{x} + \frac{4}{x^3}\right)} \\ &= \frac{\lim_{x \rightarrow \infty} 1 + 5 \lim_{x \rightarrow \infty} \frac{1}{x^2}}{\lim_{x \rightarrow \infty} 2 - \lim_{x \rightarrow \infty} \frac{1}{x} + 4 \lim_{x \rightarrow \infty} \frac{1}{x^3}} = \frac{1 + 5(0)}{2 - 0 + 4(0)} = \frac{1}{2}\end{aligned}$$

19. First, multiply the factors in the denominator. Then divide both the numerator and denominator by u^4 .

$$\begin{aligned}\lim_{u \rightarrow \infty} \frac{4u^4 + 5}{(u^2 - 2)(2u^2 - 1)} &= \lim_{u \rightarrow \infty} \frac{4u^4 + 5}{2u^4 - 5u^2 + 2} = \lim_{u \rightarrow \infty} \frac{\frac{4u^4 + 5}{u^4}}{\frac{2u^4 - 5u^2 + 2}{u^4}} = \lim_{u \rightarrow \infty} \frac{4 + \frac{5}{u^4}}{2 - \frac{5}{u^2} + \frac{2}{u^4}} \\ &= \frac{\lim_{u \rightarrow \infty} \left(4 + \frac{5}{u^4}\right)}{\lim_{u \rightarrow \infty} \left(2 - \frac{5}{u^2} + \frac{2}{u^4}\right)} = \frac{\lim_{u \rightarrow \infty} 4 + 5 \lim_{u \rightarrow \infty} \frac{1}{u^4}}{\lim_{u \rightarrow \infty} 2 - 5 \lim_{u \rightarrow \infty} \frac{1}{u^2} + 2 \lim_{u \rightarrow \infty} \frac{1}{u^4}} = \frac{4 + 5(0)}{2 - 5(0) + 2(0)} \\ &= \frac{4}{2} = 2\end{aligned}$$

$$\begin{aligned}21. \lim_{x \rightarrow \infty} \frac{\sqrt{9x^6 - x}}{x^3 + 1} &= \lim_{x \rightarrow \infty} \frac{\sqrt{9x^6 - x}/x^3}{(x^3 + 1)/x^3} = \frac{\lim_{x \rightarrow \infty} \sqrt{(9x^6 - x)/x^6}}{\lim_{x \rightarrow \infty} (1 + 1/x^3)} \quad [\text{since } x^3 = \sqrt{x^6} \text{ for } x > 0] \\ &= \frac{\lim_{x \rightarrow \infty} \sqrt{9 - 1/x^5}}{\lim_{x \rightarrow \infty} 1 + \lim_{x \rightarrow \infty} (1/x^3)} = \frac{\sqrt{\lim_{x \rightarrow \infty} 9 - \lim_{x \rightarrow \infty} (1/x^5)}}{1 + 0} \\ &= \sqrt{9 - 0} = 3\end{aligned}$$

$$\begin{aligned}23. \lim_{x \rightarrow \infty} (\sqrt{9x^2 + x} - 3x) &= \lim_{x \rightarrow \infty} \frac{(\sqrt{9x^2 + x} - 3x)(\sqrt{9x^2 + x} + 3x)}{\sqrt{9x^2 + x} + 3x} = \lim_{x \rightarrow \infty} \frac{(\sqrt{9x^2 + x})^2 - (3x)^2}{\sqrt{9x^2 + x} + 3x} \\ &= \lim_{x \rightarrow \infty} \frac{(9x^2 + x) - 9x^2}{\sqrt{9x^2 + x} + 3x} = \lim_{x \rightarrow \infty} \frac{x}{\sqrt{9x^2 + x} + 3x} \cdot \frac{1/x}{1/x} \\ &= \lim_{x \rightarrow \infty} \frac{x/x}{\sqrt{9x^2/x^2 + x/x^2} + 3x/x} = \lim_{x \rightarrow \infty} \frac{1}{\sqrt{9 + 1/x} + 3} = \frac{1}{\sqrt{9 + 0} + 3} = \frac{1}{3 + 3} = \frac{1}{6}\end{aligned}$$

$$\begin{aligned}25. \lim_{x \rightarrow \infty} (\sqrt{x^2 + ax} - \sqrt{x^2 + bx}) &= \lim_{x \rightarrow \infty} \frac{(\sqrt{x^2 + ax} - \sqrt{x^2 + bx})(\sqrt{x^2 + ax} + \sqrt{x^2 + bx})}{\sqrt{x^2 + ax} + \sqrt{x^2 + bx}} \\ &= \lim_{x \rightarrow \infty} \frac{(x^2 + ax) - (x^2 + bx)}{\sqrt{x^2 + ax} + \sqrt{x^2 + bx}} = \lim_{x \rightarrow \infty} \frac{[(a - b)x]/x}{(\sqrt{x^2 + ax} + \sqrt{x^2 + bx})/\sqrt{x^2}} \\ &= \lim_{x \rightarrow \infty} \frac{a - b}{\sqrt{1 + a/x} + \sqrt{1 + b/x}} = \frac{a - b}{\sqrt{1 + 0} + \sqrt{1 + 0}} = \frac{a - b}{2}\end{aligned}$$

27. \sqrt{x} is large when x is large, so $\lim_{x \rightarrow \infty} \sqrt{x} = \infty$.

29. $\lim_{x \rightarrow \infty} (x - \sqrt{x}) = \lim_{x \rightarrow \infty} \sqrt{x}(\sqrt{x} - 1) = \infty$ since $\sqrt{x} \rightarrow \infty$ and $\sqrt{x} - 1 \rightarrow \infty$ as $x \rightarrow \infty$.

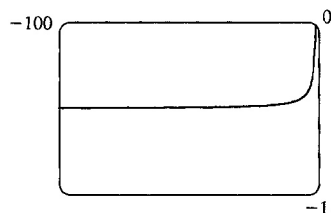
31. $\lim_{x \rightarrow -\infty} (x^4 + x^5) = \lim_{x \rightarrow -\infty} x^5(\frac{1}{x} + 1)$ [factor out the largest power of x] $= -\infty$ because $x^5 \rightarrow -\infty$ and $1/x + 1 \rightarrow 1$ as $x \rightarrow -\infty$.

33. $\lim_{x \rightarrow \infty} \frac{x + x^3 + x^5}{1 - x^2 + x^4} = \lim_{x \rightarrow \infty} \frac{(x + x^3 + x^5)/x^4}{(1 - x^2 + x^4)/x^4}$ [divide by the highest power of x in the denominator]

$$= \lim_{x \rightarrow \infty} \frac{1/x^3 + 1/x + x}{1/x^4 - 1/x^2 + 1} = \infty$$

because $(1/x^3 + 1/x + x) \rightarrow \infty$ and $(1/x^4 - 1/x^2 + 1) \rightarrow 1$ as $x \rightarrow \infty$.

35. (a)



From the graph of $f(x) = \sqrt{x^2 + x + 1} + x$, we estimate the value of $\lim_{x \rightarrow -\infty} f(x)$ to be -0.5 .

(b)

x	$f(x)$
-10,000	-0.4999625
-100,000	-0.4999962
-1,000,000	-0.4999996

From the table, we estimate the limit to be -0.5 .

(c) $\lim_{x \rightarrow -\infty} (\sqrt{x^2 + x + 1} + x) = \lim_{x \rightarrow -\infty} (\sqrt{x^2 + x + 1} + x) \left[\frac{\sqrt{x^2 + x + 1} - x}{\sqrt{x^2 + x + 1} - x} \right] = \lim_{x \rightarrow -\infty} \frac{(x^2 + x + 1) - x^2}{\sqrt{x^2 + x + 1} - x}$

$$= \lim_{x \rightarrow -\infty} \frac{(x+1)(1/x)}{(\sqrt{x^2 + x + 1} - x)(1/x)} = \lim_{x \rightarrow -\infty} \frac{1 + (1/x)}{-\sqrt{1 + (1/x) + (1/x^2)} - 1}$$

$$= \frac{1 + 0}{-\sqrt{1 + 0 + 0} - 1} = -\frac{1}{2}$$

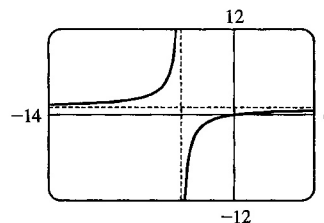
Note that for $x < 0$, we have $\sqrt{x^2} = |x| = -x$, so when we divide the radical by x , with $x < 0$, we get

$$\frac{1}{x} \sqrt{x^2 + x + 1} = -\frac{1}{\sqrt{x^2}} \sqrt{x^2 + x + 1} = -\sqrt{1 + (1/x) + (1/x^2)}.$$

37. $\lim_{x \rightarrow \pm\infty} \frac{x}{x+4} = \lim_{x \rightarrow \pm\infty} \frac{1}{1+4/x} = \frac{1}{1+0} = 1$, so $y = 1$ is a horizontal

asymptote. $\lim_{x \rightarrow -4^-} \frac{x}{x+4} = \infty$ and $\lim_{x \rightarrow -4^+} \frac{x}{x+4} = -\infty$, so $x = -4$ is

a vertical asymptote. The graph confirms these calculations.



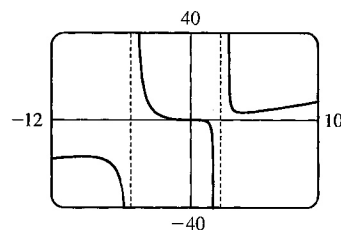
39. $\lim_{x \rightarrow \pm\infty} \frac{x^3}{x^2 + 3x - 10} = \lim_{x \rightarrow \pm\infty} \frac{x}{1 + (3/x) - (10/x^2)} = \pm\infty$, so there is no horizontal asymptote.

$$\lim_{x \rightarrow 2^+} \frac{x^3}{x^2 + 3x - 10} = \lim_{x \rightarrow 2^+} \frac{x^3}{(x+5)(x-2)} = \infty, \text{ since}$$

$$\frac{x^3}{(x+5)(x-2)} > 0 \text{ for } x > 2. \text{ Similarly, } \lim_{x \rightarrow 2^-} \frac{x^3}{x^2 + 3x - 10} = -\infty,$$

$$\lim_{x \rightarrow -5^-} \frac{x^3}{x^2 + 3x - 10} = -\infty, \text{ and } \lim_{x \rightarrow -5^+} \frac{x^3}{x^2 + 3x - 10} = \infty, \text{ so}$$

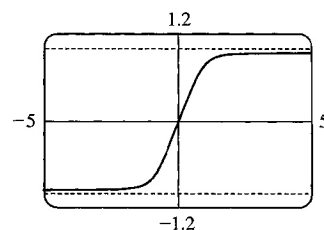
$x = 2$ and $x = -5$ are vertical asymptotes. The graph confirms these calculations.



41. $\lim_{x \rightarrow \infty} \frac{x}{\sqrt[4]{x^4 + 1}} \cdot \frac{1/x}{1/\sqrt[4]{x^4}} = \lim_{x \rightarrow \infty} \frac{1}{\sqrt[4]{1 + \frac{1}{x^4}}} = \frac{1}{\sqrt[4]{1 + 0}} = 1$ and

$$\lim_{x \rightarrow -\infty} \frac{x}{\sqrt[4]{x^4 + 1}} \cdot \frac{1/x}{-1/\sqrt[4]{x^4}} = \lim_{x \rightarrow -\infty} \frac{1}{-\sqrt[4]{1 + \frac{1}{x^4}}} = -\frac{1}{\sqrt[4]{1 + 0}} = -1,$$

so $y = \pm 1$ are horizontal asymptotes. There is no vertical asymptote.



43. Let's look for a rational function.

(1) $\lim_{x \rightarrow \pm\infty} f(x) = 0 \Rightarrow$ degree of numerator < degree of denominator

(2) $\lim_{x \rightarrow 0} f(x) = -\infty \Rightarrow$ there is a factor of x^2 in the denominator (not just x , since that would produce a sign change at $x = 0$), and the function is negative near $x = 0$.

(3) $\lim_{x \rightarrow 3^-} f(x) = \infty$ and $\lim_{x \rightarrow 3^+} f(x) = -\infty \Rightarrow$ vertical asymptote at $x = 3$; there is a factor of $(x - 3)$ in the denominator.

(4) $f(2) = 0 \Rightarrow$ 2 is an x -intercept; there is at least one factor of $(x - 2)$ in the numerator.

Combining all of this information and putting in a negative sign to give us the desired left- and right-hand limits

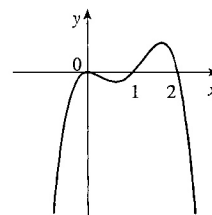
gives us $f(x) = \frac{2 - x}{x^2(x - 3)}$ as one possibility.

45. $y = f(x) = x^2(x - 2)(1 - x)$. The y -intercept is $f(0) = 0$, and the x -intercepts

occur when $y = 0 \Rightarrow x = 0, 1, \text{ and } 2$. Notice that, since x^2 is always positive, the graph does not cross the x -axis at 0, but does cross the x -axis at 1 and 2.

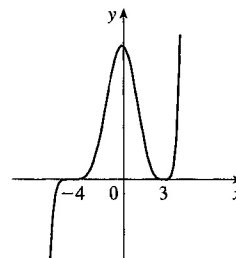
$\lim_{x \rightarrow \infty} x^2(x - 2)(1 - x) = -\infty$, since the first two factors are large positive and the third large negative when x is large positive. $\lim_{x \rightarrow -\infty} x^2(x - 2)(1 - x) = -\infty$ because

the first and third factors are large positive and the second large negative as $x \rightarrow -\infty$.



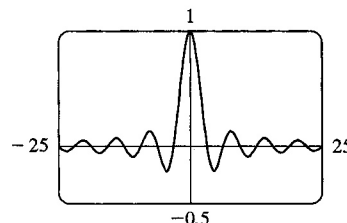
47. $y = f(x) = (x+4)^5(x-3)^4$. The y -intercept is $f(0) = 4^5(-3)^4 = 82,944$.

The x -intercepts occur when $y = 0 \Rightarrow x = -4, 3$. Notice that the graph does not cross the x -axis at 3 because $(x-3)^4$ is always positive, but does cross the x -axis at -4 . $\lim_{x \rightarrow \infty} (x+4)^5(x-3)^4 = \infty$ since both factors are large positive when x is large positive. $\lim_{x \rightarrow -\infty} (x+4)^5(x-3)^4 = -\infty$ since the first factor is large negative and the second factor is large positive when x is large negative.



49. (a) Since $-1 \leq \sin x \leq 1$ for all x , $-\frac{1}{x} \leq \frac{\sin x}{x} \leq \frac{1}{x}$ for $x > 0$. As $x \rightarrow \infty$, $-1/x \rightarrow 0$ and $1/x \rightarrow 0$, so by the Squeeze Theorem, $(\sin x)/x \rightarrow 0$. Thus, $\lim_{x \rightarrow \infty} \frac{\sin x}{x} = 0$.

(b) From part (a), the horizontal asymptote is $y = 0$. The function $y = (\sin x)/x$ crosses the horizontal asymptote whenever $\sin x = 0$; that is, at $x = \pi n$ for every integer n . Thus, the graph crosses the asymptote an infinite number of times.



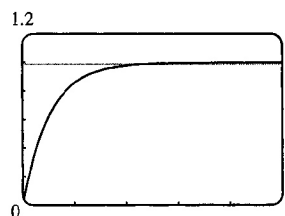
51. Divide the numerator and the denominator by the highest power of x in $Q(x)$.

(a) If $\deg P < \deg Q$, then the numerator $\rightarrow 0$ but the denominator doesn't. So $\lim_{x \rightarrow \infty} [P(x)/Q(x)] = 0$.
 (b) If $\deg P > \deg Q$, then the numerator $\rightarrow \pm\infty$ but the denominator doesn't, so $\lim_{x \rightarrow \infty} [P(x)/Q(x)] = \pm\infty$ (depending on the ratio of the leading coefficients of P and Q).

53. $\lim_{x \rightarrow \infty} \frac{4x-1}{x} = \lim_{x \rightarrow \infty} \left(4 - \frac{1}{x}\right) = 4$, and $\lim_{x \rightarrow \infty} \frac{4x^2+3x}{x^2} = \lim_{x \rightarrow \infty} \left(4 + \frac{3}{x}\right) = 4$. Therefore, by the Squeeze Theorem, $\lim_{x \rightarrow \infty} f(x) = 4$.

55. (a) $\lim_{t \rightarrow \infty} v(t) = \lim_{t \rightarrow \infty} v^* (1 - e^{-gt/v^*}) = v^* (1 - 0) = v^*$

(b) We graph $v(t) = 1 - e^{-9.8t}$ and $v(t) = 0.99v^*$, or in this case, $v(t) = 0.99$. Using an intersect feature or zooming in on the point of intersection, we find that $t \approx 0.47$ s.

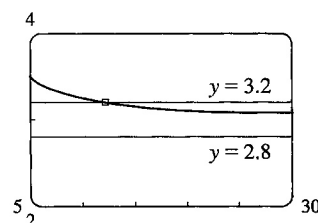


57. $\left| \frac{6x^2+5x-3}{2x^2-1} - 3 \right| < 0.2 \Leftrightarrow 2.8 < \frac{6x^2+5x-3}{2x^2-1} < 3.2$. So

we graph the three parts of this inequality on the same screen, and

find that the curve $y = \frac{6x^2+5x-3}{2x^2-1}$ seems to lie between the lines

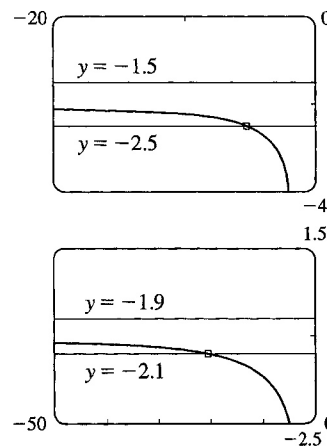
$y = 2.8$ and $y = 3.2$ whenever $x > 12.8$. So we can choose $N = 13$ (or any larger number) so that the inequality holds whenever $x \geq N$.



59. For $\varepsilon = 0.5$, we need to find N such that $\left| \frac{\sqrt{4x^2 + 1}}{x + 1} - (-2) \right| < 0.5$

$\Leftrightarrow -2.5 < \frac{\sqrt{4x^2 + 1}}{x + 1} < -1.5$ whenever $x \leq N$. We graph the three parts of this inequality on the same screen, and see that the inequality holds for $x \leq -6$. So we choose $N = -6$ (or any smaller number).

For $\varepsilon = 0.1$, we need $-2.1 < \frac{\sqrt{4x^2 + 1}}{x + 1} < -1.9$ whenever $x \leq N$. From the graph, it seems that this inequality holds for $x \leq -22$. So we choose $N = -22$ (or any smaller number).



61. (a) $1/x^2 < 0.0001 \Leftrightarrow x^2 > 1/0.0001 = 10,000 \Leftrightarrow x > 100 \quad (x > 0)$

(b) If $\varepsilon > 0$ is given, then $1/x^2 < \varepsilon \Leftrightarrow x^2 > 1/\varepsilon \Leftrightarrow x > 1/\sqrt{\varepsilon}$. Let $N = 1/\sqrt{\varepsilon}$.

$$\text{Then } x > N \Rightarrow x > \frac{1}{\sqrt{\varepsilon}} \Rightarrow \left| \frac{1}{x^2} - 0 \right| = \frac{1}{x^2} < \varepsilon, \text{ so } \lim_{x \rightarrow \infty} \frac{1}{x^2} = 0.$$

63. For $x < 0$, $|1/x - 0| = -1/x$. If $\varepsilon > 0$ is given, then $-1/x < \varepsilon \Leftrightarrow x < -1/\varepsilon$.

Take $N = -1/\varepsilon$. Then $x < N \Rightarrow x < -1/\varepsilon \Rightarrow |(1/x) - 0| = -1/x < \varepsilon$, so $\lim_{x \rightarrow -\infty} (1/x) = 0$.

65. Given $M > 0$, we need $N > 0$ such that $x > N \Rightarrow e^x > M$. Now $e^x > M \Leftrightarrow x > \ln M$, so take $N = \max(1, \ln M)$. (This ensures that $N > 0$.) Then $x > N = \max(1, \ln M) \Rightarrow e^x > \max(e, M) \geq M$, so $\lim_{x \rightarrow \infty} e^x = \infty$.

67. Suppose that $\lim_{x \rightarrow \infty} f(x) = L$. Then for every $\varepsilon > 0$ there is a corresponding positive number N such that

$$|f(x) - L| < \varepsilon \text{ whenever } x > N. \text{ If } t = 1/x, \text{ then } x > N \Leftrightarrow 0 < 1/x < 1/N \Leftrightarrow 0 < t < 1/N. \text{ Thus,}$$

for every $\varepsilon > 0$ there is a corresponding $\delta > 0$ (namely $1/N$) such that $|f(1/t) - L| < \varepsilon$ whenever $0 < t < \delta$.

This proves that $\lim_{t \rightarrow 0^+} f(1/t) = L = \lim_{x \rightarrow \infty} f(x)$.

Now suppose that $\lim_{x \rightarrow -\infty} f(x) = L$. Then for every $\varepsilon > 0$ there is a corresponding negative number N such that

$$|f(x) - L| < \varepsilon \text{ whenever } x < N. \text{ If } t = 1/x, \text{ then } x < N \Leftrightarrow 1/N < 1/x < 0 \Leftrightarrow 1/N < t < 0. \text{ Thus, for}$$

every $\varepsilon > 0$ there is a corresponding $\delta > 0$ (namely $-1/N$) such that $|f(1/t) - L| < \varepsilon$ whenever $-\delta < t < 0$.

This proves that $\lim_{t \rightarrow 0^-} f(1/t) = L = \lim_{x \rightarrow -\infty} f(x)$.

2.7 Tangents, Velocities, and Other Rates of Change

1. (a) This is just the slope of the line through two points: $m_{PQ} = \frac{\Delta y}{\Delta x} = \frac{f(x) - f(3)}{x - 3}$.

(b) This is the limit of the slope of the secant line PQ as Q approaches P : $m = \lim_{x \rightarrow 3} \frac{f(x) - f(3)}{x - 3}$.

3. The slope at D is the largest positive slope, followed by the positive slope at E . The slope at C is zero. The slope at B is steeper than at A (both are negative). In decreasing order, we have the slopes at: D, E, C, A , and B .

5. (a) (i) Using Definition 1,

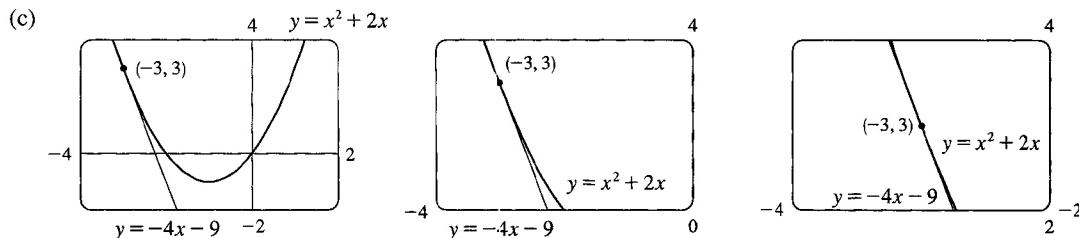
$$\begin{aligned} m &= \lim_{x \rightarrow a} \frac{f(x) - f(a)}{x - a} = \lim_{x \rightarrow -3} \frac{f(x) - f(-3)}{x - (-3)} = \lim_{x \rightarrow -3} \frac{(x^2 + 2x) - (3)}{x - (-3)} = \lim_{x \rightarrow -3} \frac{(x+3)(x-1)}{x+3} \\ &= \lim_{x \rightarrow -3} (x-1) = -4 \end{aligned}$$

(ii) Using Equation 2,

$$\begin{aligned} m &= \lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h} = \lim_{h \rightarrow 0} \frac{f(-3+h) - f(-3)}{h} = \lim_{h \rightarrow 0} \frac{[(-3+h)^2 + 2(-3+h)] - (3)}{h} \\ &= \lim_{h \rightarrow 0} \frac{9 - 6h + h^2 - 6 + 2h - 3}{h} = \lim_{h \rightarrow 0} \frac{h(h-4)}{h} = \lim_{h \rightarrow 0} (h-4) = -4 \end{aligned}$$

(b) Using the point-slope form of the equation of a line, an equation of the tangent line is $y - 3 = -4(x + 3)$.

Solving for y gives us $y = -4x - 9$, which is the slope-intercept form of the equation of the tangent line.



7. Using (2) with $f(x) = 1 + 2x - x^3$ and $P(1, 2)$,

$$\begin{aligned} m &= \lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h} = \lim_{h \rightarrow 0} \frac{f(1+h) - f(1)}{h} = \lim_{h \rightarrow 0} \frac{[1 + 2(1+h) - (1+h)^3] - 2}{h} \\ &= \lim_{h \rightarrow 0} \frac{1 + 2 + 2h - (1 + 3h + 3h^2 + h^3) - 2}{h} = \lim_{h \rightarrow 0} \frac{-h^3 - 3h^2 - h}{h} \\ &= \lim_{h \rightarrow 0} \frac{h(-h^2 - 3h - 1)}{h} = \lim_{h \rightarrow 0} (-h^2 - 3h - 1) = -1 \end{aligned}$$

Tangent line: $y - 2 = -1(x - 1) \Leftrightarrow y - 2 = -x + 1 \Leftrightarrow y = -x + 3$

9. Using (1) with $f(x) = \frac{x-1}{x-2}$ and $P(3, 2)$,

$$\begin{aligned} m &= \lim_{x \rightarrow a} \frac{f(x) - f(a)}{x - a} = \lim_{x \rightarrow 3} \frac{\frac{x-1}{x-2} - 2}{x - 3} = \lim_{x \rightarrow 3} \frac{\frac{x-1-2(x-2)}{x-2}}{x-3} = \lim_{x \rightarrow 3} \frac{3-x}{(x-2)(x-3)} \\ &= \lim_{x \rightarrow 3} \frac{-1}{x-2} = \frac{-1}{1} = -1. \end{aligned}$$

Tangent line: $y - 2 = -1(x - 3) \Leftrightarrow y - 2 = -x + 3 \Leftrightarrow y = -x + 5$

$$\begin{aligned}
 11. (a) \quad m &= \lim_{x \rightarrow a} \frac{f(x) - f(a)}{x - a} = \lim_{x \rightarrow a} \frac{2/(x+3) - 2/(a+3)}{x - a} = \lim_{x \rightarrow a} \frac{2(a+3) - 2(x+3)}{(x-a)(x+3)(a+3)} \\
 &= \lim_{x \rightarrow a} \frac{2(a-x)}{(x-a)(x+3)(a+3)} = \lim_{x \rightarrow a} \frac{-2}{(x+3)(a+3)} = \frac{-2}{(a+3)^2}
 \end{aligned}$$

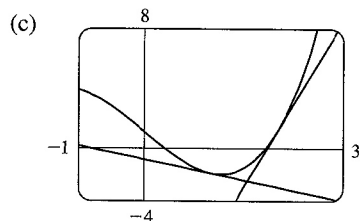
$$(b) \quad (i) \quad a = -1 \Rightarrow m = \frac{-2}{(-1+3)^2} = -\frac{1}{2} \qquad (ii) \quad a = 0 \Rightarrow m = \frac{-2}{(0+3)^2} = -\frac{2}{9}$$

$$(iii) \quad a = 1 \Rightarrow m = \frac{-2}{(1+3)^2} = -\frac{1}{8}$$

13. (a) Using (1),

$$\begin{aligned}
 m &= \lim_{x \rightarrow a} \frac{(x^3 - 4x + 1) - (a^3 - 4a + 1)}{x - a} = \lim_{x \rightarrow a} \frac{(x^3 - a^3) - 4(x - a)}{x - a} \\
 &= \lim_{x \rightarrow a} \frac{(x-a)(x^2 + ax + a^2) - 4(x-a)}{x - a} = \lim_{x \rightarrow a} (x^2 + ax + a^2 - 4) = 3a^2 - 4
 \end{aligned}$$

(b) At $(1, -2)$: $m = 3(1)^2 - 4 = -1$, so an equation of the tangent line is $y - (-2) = -1(x - 1) \Leftrightarrow y = -x - 1$. At $(2, 1)$: $m = 3(2)^2 - 4 = 8$, so an equation of the tangent line is $y - 1 = 8(x - 2) \Leftrightarrow y = 8x - 15$.



15. (a) Since the slope of the tangent at $t = 0$ is 0, the car's initial velocity was 0.

(b) The slope of the tangent is greater at C than at B , so the car was going faster at C .

(c) Near A , the tangent lines are becoming steeper as x increases, so the velocity was increasing, so the car was speeding up. Near B , the tangent lines are becoming less steep, so the car was slowing down. The steepest tangent near C is the one at C , so at C the car had just finished speeding up, and was about to start slowing down.

(d) Between D and E , the slope of the tangent is 0, so the car did not move during that time.

17. Let $s(t) = 40t - 16t^2$.

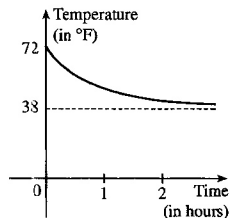
$$\begin{aligned}
 v(2) &= \lim_{t \rightarrow 2} \frac{s(t) - s(2)}{t - 2} = \lim_{t \rightarrow 2} \frac{(40t - 16t^2) - 16}{t - 2} = \lim_{t \rightarrow 2} \frac{-16t^2 + 40t - 16}{t - 2} = \lim_{t \rightarrow 2} \frac{-8(2t^2 - 5t + 2)}{t - 2} \\
 &= \lim_{t \rightarrow 2} \frac{-8(t-2)(2t-1)}{t-2} = -8 \lim_{t \rightarrow 2} (2t-1) = -8(3) = -24
 \end{aligned}$$

Thus, the instantaneous velocity when $t = 2$ is -24 ft/s.

$$\begin{aligned}
 19. \quad v(a) &= \lim_{h \rightarrow 0} \frac{s(a+h) - s(a)}{h} = \lim_{h \rightarrow 0} \frac{4(a+h)^3 + 6(a+h) + 2 - (4a^3 + 6a + 2)}{h} \\
 &= \lim_{h \rightarrow 0} \frac{4a^3 + 12a^2h + 12ah^2 + 4h^3 + 6a + 6h + 2 - 4a^3 - 6a - 2}{h} \\
 &= \lim_{h \rightarrow 0} \frac{12a^2h + 12ah^2 + 4h^3 + 6h}{h} = \lim_{h \rightarrow 0} (12a^2 + 12ah + 4h^2 + 6) = (12a^2 + 6) \text{ m/s}
 \end{aligned}$$

So $v(1) = 12(1)^2 + 6 = 18$ m/s, $v(2) = 12(2)^2 + 6 = 54$ m/s, and $v(3) = 12(3)^2 + 6 = 114$ m/s.

21. The sketch shows the graph for a room temperature of 72° and a refrigerator temperature of 38° . The initial rate of change is greater in magnitude than the rate of change after an hour.



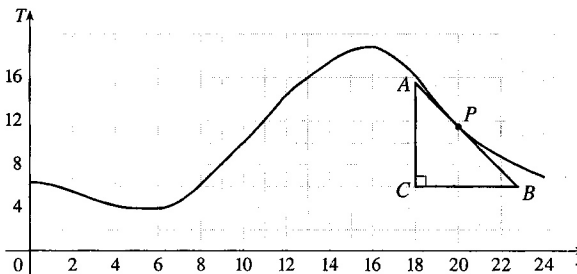
23. (a) (i) $[20, 23]: \frac{7.9 - 11.5}{23 - 20} = -1.2^\circ\text{C/h}$

(ii) $[20, 22]: \frac{9.0 - 11.5}{22 - 20} = -1.25^\circ\text{C/h}$

(iii) $[20, 21]: \frac{10.2 - 11.5}{21 - 20} = -1.3^\circ\text{C/h}$

- (b) In the figure, we estimate A to be $(18, 15.5)$ and B as $(23, 6)$. So the slope is

$$\frac{6 - 15.5}{23 - 18} = -1.9^\circ\text{C/h at 8:00 P.M.}$$



25. (a) (i) $[1995, 1997]: \frac{N(1997) - N(1995)}{1997 - 1995} = \frac{2461 - 873}{2} = \frac{1588}{2} = 794 \text{ thousand/year}$

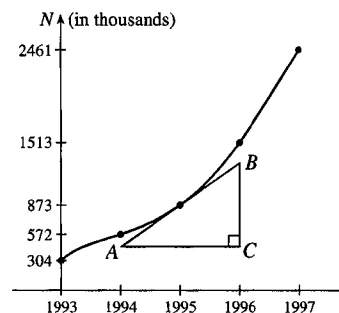
(ii) $[1995, 1996]: \frac{N(1996) - N(1995)}{1996 - 1995} = \frac{1513 - 873}{1} = 640 \text{ thousand/year}$

(iii) $[1994, 1995]: \frac{N(1995) - N(1994)}{1995 - 1994} = \frac{873 - 572}{1} = 301 \text{ thousand/year}$

- (b) Using the values from (ii) and (iii), we have $\frac{640 + 301}{2} = \frac{941}{2} = 470.5 \text{ thousand/year}$.

- (c) Estimating A as $(1994, 420)$ and B as $(1996, 1275)$, the slope

at 1995 is $\frac{1275 - 420}{1996 - 1994} = \frac{855}{2} = 427.5 \text{ thousand/year}$



27. (a) (i) $\frac{\Delta C}{\Delta x} = \frac{C(105) - C(100)}{105 - 100} = \frac{6601.25 - 6500}{5} = \$20.25/\text{unit}$.

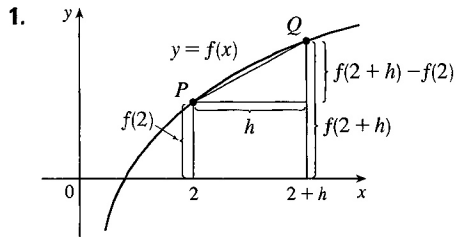
(ii) $\frac{\Delta C}{\Delta x} = \frac{C(101) - C(100)}{101 - 100} = \frac{6520.05 - 6500}{1} = \$20.05/\text{unit}$.

(b)
$$\frac{C(100 + h) - C(100)}{h} = \frac{[5000 + 10(100 + h) + 0.05(100 + h)^2] - 6500}{h} = \frac{20h + 0.05h^2}{h}$$

$$= 20 + 0.05h, h \neq 0$$

So the instantaneous rate of change is $\lim_{h \rightarrow 0} \frac{C(100 + h) - C(100)}{h} = \lim_{h \rightarrow 0} (20 + 0.05h) = \$20/\text{unit}$.

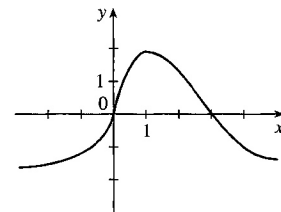
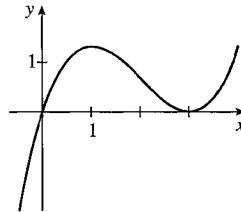
2.8 Derivatives



The line from $P(2, f(2))$ to $Q(2 + h, f(2 + h))$ is the line that has slope $\frac{f(2 + h) - f(2)}{h}$.

3. $g'(0)$ is the only negative value. The slope at $x = 4$ is smaller than the slope at $x = 2$ and both are smaller than the slope at $x = -2$. Thus, $g'(0) < 0 < g'(4) < g'(2) < g'(-2)$.

5. We begin by drawing a curve through the origin at a slope of 3 to satisfy $f(0) = 0$ and $f'(0) = 3$. Since $f'(1) = 0$, we will round off our figure so that there is a horizontal tangent directly over $x = 1$. Lastly, we make sure that the curve has a slope of -1 as we pass over $x = 2$. Two of the many possibilities are shown.



7. Using Definition 2 with $f(x) = 3x^2 - 5x$ and the point $(2, 2)$, we have

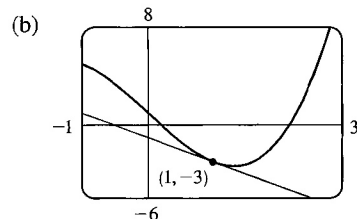
$$\begin{aligned} f'(2) &= \lim_{h \rightarrow 0} \frac{f(2+h) - f(2)}{h} = \lim_{h \rightarrow 0} \frac{[3(2+h)^2 - 5(2+h)] - 2}{h} \\ &= \lim_{h \rightarrow 0} \frac{(12 + 12h + 3h^2 - 10 - 5h) - 2}{h} = \lim_{h \rightarrow 0} \frac{3h^2 + 7h}{h} = \lim_{h \rightarrow 0} (3h + 7) = 7. \end{aligned}$$

So an equation of the tangent line at $(2, 2)$ is $y - 2 = 7(x - 2)$ or $y = 7x - 12$.

9. (a) Using Definition 2 with $F(x) = x^3 - 5x + 1$ and the point $(1, -3)$, we have

$$\begin{aligned} F'(1) &= \lim_{h \rightarrow 0} \frac{F(1+h) - F(1)}{h} = \lim_{h \rightarrow 0} \frac{[(1+h)^3 - 5(1+h) + 1] - (-3)}{h} \\ &= \lim_{h \rightarrow 0} \frac{(1 + 3h + 3h^2 + h^3 - 5 - 5h + 1) + 3}{h} = \lim_{h \rightarrow 0} \frac{h^3 + 3h^2 - 2h}{h} \\ &= \lim_{h \rightarrow 0} \frac{h(h^2 + 3h - 2)}{h} = \lim_{h \rightarrow 0} (h^2 + 3h - 2) = -2 \end{aligned}$$

So an equation of the tangent line at $(1, -3)$ is $y - (-3) = -2(x - 1) \Leftrightarrow y = -2x - 1$.



11. (a) $f'(1) = \lim_{h \rightarrow 0} \frac{f(1+h) - f(1)}{h} = \lim_{h \rightarrow 0} \frac{3^{1+h} - 3^1}{h}.$

So let $F(h) = \frac{3^{1+h} - 3}{h}$. We calculate:

h	$F(h)$	h	$F(h)$
0.1	3.484	-0.1	3.121
0.01	3.314	-0.01	3.278
0.001	3.298	-0.001	3.294
0.0001	3.296	-0.0001	3.296

We estimate that $f'(1) \approx 3.296$.

13. Use Definition 2 with $f(x) = 3 - 2x + 4x^2$.

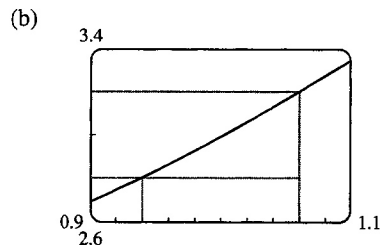
$$\begin{aligned} f'(a) &= \lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h} = \lim_{h \rightarrow 0} \frac{[3 - 2(a+h) + 4(a+h)^2] - (3 - 2a + 4a^2)}{h} \\ &= \lim_{h \rightarrow 0} \frac{(3 - 2a - 2h + 4a^2 + 8ah + 4h^2) - (3 - 2a + 4a^2)}{h} \\ &= \lim_{h \rightarrow 0} \frac{-2h + 8ah + 4h^2}{h} = \lim_{h \rightarrow 0} \frac{h(-2 + 8a + 4h)}{h} = \lim_{h \rightarrow 0} (-2 + 8a + 4h) = -2 + 8a \end{aligned}$$

15. Use Definition 2 with $f(t) = (2t+1)/(t+3)$.

$$\begin{aligned} f'(a) &= \lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h} = \lim_{h \rightarrow 0} \frac{\frac{2(a+h)+1}{(a+h)+3} - \frac{2a+1}{a+3}}{h} \\ &= \lim_{h \rightarrow 0} \frac{(2a+2h+1)(a+3) - (2a+1)(a+h+3)}{h(a+h+3)(a+3)} \\ &= \lim_{h \rightarrow 0} \frac{(2a^2 + 6a + 2ah + 6h + a + 3) - (2a^2 + 2ah + 6a + a + h + 3)}{h(a+h+3)(a+3)} \\ &= \lim_{h \rightarrow 0} \frac{5h}{h(a+h+3)(a+3)} = \lim_{h \rightarrow 0} \frac{5}{(a+h+3)(a+3)} = \frac{5}{(a+3)^2} \end{aligned}$$

17. Use Definition 2 with $f(x) = 1/\sqrt{x+2}$.

$$\begin{aligned} f'(a) &= \lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h} = \lim_{h \rightarrow 0} \frac{\frac{1}{\sqrt{(a+h)+2}} - \frac{1}{\sqrt{a+2}}}{h} \\ &= \lim_{h \rightarrow 0} \frac{\frac{\sqrt{a+2} - \sqrt{a+h+2}}{\sqrt{a+h+2}\sqrt{a+2}}}{h} = \lim_{h \rightarrow 0} \left[\frac{\sqrt{a+2} - \sqrt{a+h+2}}{h\sqrt{a+h+2}\sqrt{a+2}} \cdot \frac{\sqrt{a+2} + \sqrt{a+h+2}}{\sqrt{a+2} + \sqrt{a+h+2}} \right] \\ &= \lim_{h \rightarrow 0} \frac{(a+2) - (a+h+2)}{h\sqrt{a+h+2}\sqrt{a+2}(\sqrt{a+2} + \sqrt{a+h+2})} \\ &= \lim_{h \rightarrow 0} \frac{-h}{h\sqrt{a+h+2}\sqrt{a+2}(\sqrt{a+2} + \sqrt{a+h+2})} \\ &= \lim_{h \rightarrow 0} \frac{-1}{\sqrt{a+h+2}\sqrt{a+2}(\sqrt{a+2} + \sqrt{a+h+2})} \\ &= \frac{-1}{(\sqrt{a+2})^2(2\sqrt{a+2})} = -\frac{1}{2(a+2)^{3/2}} \end{aligned}$$



From the graph, we estimate that the slope of the tangent is about

$$\frac{3.2 - 2.8}{1.06 - 0.94} = \frac{0.4}{0.12} \approx 3.3.$$

Note that the answers to Exercises 19–24 are not unique.

19. By Definition 2, $\lim_{h \rightarrow 0} \frac{(1+h)^{10} - 1}{h} = f'(1)$, where $f(x) = x^{10}$ and $a = 1$.

Or: By Definition 2, $\lim_{h \rightarrow 0} \frac{(1+h)^{10} - 1}{h} = f'(0)$, where $f(x) = (1+x)^{10}$ and $a = 0$.

21. By Equation 3, $\lim_{x \rightarrow 5} \frac{2^x - 32}{x - 5} = f'(5)$, where $f(x) = 2^x$ and $a = 5$.

23. By Definition 2, $\lim_{h \rightarrow 0} \frac{\cos(\pi + h) + 1}{h} = f'(\pi)$, where $f(x) = \cos x$ and $a = \pi$.

Or: By Definition 2, $\lim_{h \rightarrow 0} \frac{\cos(\pi + h) + 1}{h} = f'(0)$, where $f(x) = \cos(\pi + x)$ and $a = 0$.

25. $v(2) = f'(2) = \lim_{h \rightarrow 0} \frac{f(2+h) - f(2)}{h} = \lim_{h \rightarrow 0} \frac{[(2+h)^2 - 6(2+h) - 5] - [2^2 - 6(2) - 5]}{h}$
 $= \lim_{h \rightarrow 0} \frac{(4 + 4h + h^2 - 12 - 6h - 5) - (-13)}{h} = \lim_{h \rightarrow 0} \frac{h^2 - 2h}{h} = \lim_{h \rightarrow 0} (h - 2) = -2 \text{ m/s}$

27. (a) $f'(x)$ is the rate of change of the production cost with respect to the number of ounces of gold produced. Its units are dollars per ounce.

(b) After 800 ounces of gold have been produced, the rate at which the production cost is increasing is \$17/ounce. So the cost of producing the 800th (or 801st) ounce is about \$17.

(c) In the short term, the values of $f'(x)$ will decrease because more efficient use is made of start-up costs as x increases. But eventually $f'(x)$ might increase due to large-scale operations.

29. (a) $f'(v)$ is the rate at which the fuel consumption is changing with respect to the speed. Its units are (gal/h)/(mi/h).

(b) The fuel consumption is decreasing by 0.05 (gal/h)/(mi/h) as the car's speed reaches 20 mi/h. So if you increase your speed to 21 mi/h, you could expect to decrease your fuel consumption by about 0.05 (gal/h)/(mi/h).

31. $T'(10)$ is the rate at which the temperature is changing at 10:00 A.M. To estimate the value of $T'(10)$, we will average the difference quotients obtained using the times $t = 8$ and $t = 12$. Let

$$A = \frac{T(8) - T(10)}{8 - 10} = \frac{72 - 81}{-2} = 4.5 \text{ and } B = \frac{T(12) - T(10)}{12 - 10} = \frac{88 - 81}{2} = 3.5. \text{ Then}$$

$$T'(10) = \lim_{t \rightarrow 10} \frac{T(t) - T(10)}{t - 10} \approx \frac{A + B}{2} = \frac{4.5 + 3.5}{2} = 4^\circ\text{F/h}.$$

33. (a) $S'(T)$ is the rate at which the oxygen solubility changes with respect to the water temperature. Its units are (mg/L)/°C.

(b) For $T = 16^\circ\text{C}$, it appears that the tangent line to the curve goes through the points (0, 14) and (32, 6). So

$$S'(16) \approx \frac{6 - 14}{32 - 0} = -\frac{8}{32} = -0.25 \text{ (mg/L)/}^\circ\text{C.}$$

This means that as the temperature increases past 16°C , the oxygen solubility is decreasing at a rate of 0.25 (mg/L)/°C.

35. Since $f(x) = x \sin(1/x)$ when $x \neq 0$ and $f(0) = 0$, we have

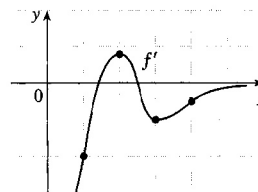
$$f'(0) = \lim_{h \rightarrow 0} \frac{f(0+h) - f(0)}{h} = \lim_{h \rightarrow 0} \frac{h \sin(1/h) - 0}{h} = \lim_{h \rightarrow 0} \sin(1/h).$$

This limit does not exist since $\sin(1/h)$ takes the values -1 and 1 on any interval containing 0 . (Compare with Example 4 in Section 2.2.)

2.9 The Derivative as a Function

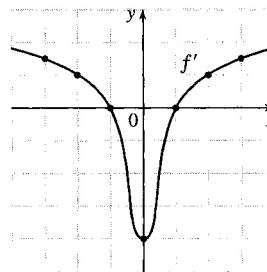
1. *Note:* Your answers may vary depending on your estimates. By estimating the slopes of tangent lines on the graph of f , it appears that

- (a) $f'(1) \approx -2$ (b) $f'(2) \approx 0.8$
 (c) $f'(3) \approx -1$ (d) $f'(4) \approx -0.5$



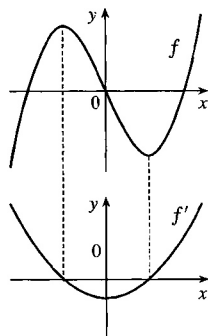
3. It appears that f is an odd function, so f' will be an even function—that is, $f'(-a) = f'(a)$.

- (a) $f'(-3) \approx 1.5$ (b) $f'(-2) \approx 1$
 (c) $f'(-1) \approx 0$ (d) $f'(0) \approx -4$
 (e) $f'(1) \approx 0$ (f) $f'(2) \approx 1$
 (g) $f'(3) \approx 1.5$

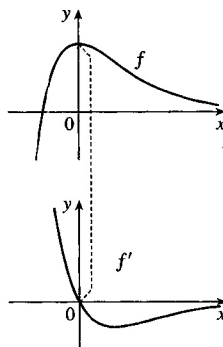


Hints for Exercises 5–13: First plot x -intercepts on the graph of f' for any horizontal tangents on the graph of f . Look for any corners on the graph of f —there will be a discontinuity on the graph of f' . On any interval where f has a tangent with positive (or negative) slope, the graph of f' will be positive (or negative). If the graph of the function is linear, the graph of f' will be a horizontal line.

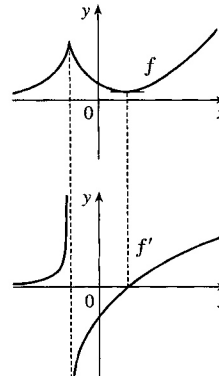
5.



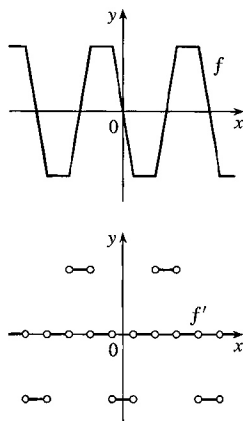
7.



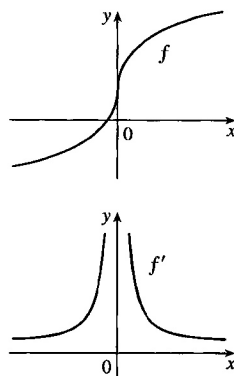
9.



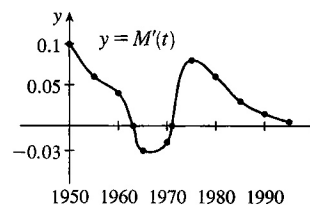
11.



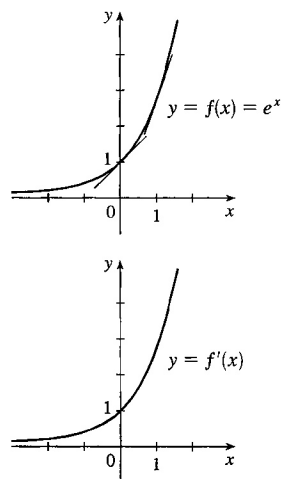
13.



15. It appears that there are horizontal tangents on the graph of M for $t = 1963$ and $t = 1971$. Thus, there are zeros for those values of t on the graph of M' . The derivative is negative for the years 1963 to 1971.



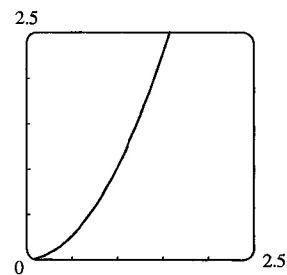
17.



The slope at 0 appears to be 1 and the slope at 1 appears to be 2.7. As x decreases, the slope gets closer to 0. Since the graphs are so similar, we might guess that $f'(x) = e^x$.

19. (a) By zooming in, we estimate that $f'(0) = 0$, $f'(\frac{1}{2}) = 1$, $f'(1) = 2$, and $f'(2) = 4$.
 (b) By symmetry, $f'(-x) = -f'(x)$. So $f'(-\frac{1}{2}) = -1$, $f'(-1) = -2$, and $f'(-2) = -4$.
 (c) It appears that $f'(x)$ is twice the value of x , so we guess that $f'(x) = 2x$.

$$\begin{aligned} \text{(d) } f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \rightarrow 0} \frac{(x+h)^2 - x^2}{h} \\ &= \lim_{h \rightarrow 0} \frac{(x^2 + 2hx + h^2) - x^2}{h} = \lim_{h \rightarrow 0} \frac{2hx + h^2}{h} = \lim_{h \rightarrow 0} \frac{h(2x + h)}{h} = \lim_{h \rightarrow 0} (2x + h) = 2x \end{aligned}$$



$$21. f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \rightarrow 0} \frac{37 - 37}{h} = \lim_{h \rightarrow 0} \frac{0}{h} = \lim_{h \rightarrow 0} 0 = 0$$

Domain of f = domain of $f' = \mathbb{R}$.

$$\begin{aligned} 23. f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \rightarrow 0} \frac{[1 - 3(x+h)^2] - (1 - 3x^2)}{h} \\ &= \lim_{h \rightarrow 0} \frac{[1 - 3(x^2 + 2xh + h^2)] - (1 - 3x^2)}{h} = \lim_{h \rightarrow 0} \frac{1 - 3x^2 - 6xh - 3h^2 - 1 + 3x^2}{h} \\ &= \lim_{h \rightarrow 0} \frac{-6xh - 3h^2}{h} = \lim_{h \rightarrow 0} \frac{h(-6x - 3h)}{h} = \lim_{h \rightarrow 0} (-6x - 3h) = -6x \end{aligned}$$

Domain of f = domain of $f' = \mathbb{R}$.

$$\begin{aligned} 25. f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \rightarrow 0} \frac{[(x+h)^3 - 3(x+h) + 5] - (x^3 - 3x + 5)}{h} \\ &= \lim_{h \rightarrow 0} \frac{(x^3 + 3x^2h + 3xh^2 + h^3 - 3x - 3h + 5) - (x^3 - 3x + 5)}{h} \\ &= \lim_{h \rightarrow 0} \frac{3x^2h + 3xh^2 + h^3 - 3h}{h} = \lim_{h \rightarrow 0} \frac{h(3x^2 + 3xh + h^2 - 3)}{h} \\ &= \lim_{h \rightarrow 0} (3x^2 + 3xh + h^2 - 3) = 3x^2 - 3 \end{aligned}$$

Domain of f = domain of $f' = \mathbb{R}$.

$$\begin{aligned} 27. g'(x) &= \lim_{h \rightarrow 0} \frac{g(x+h) - g(x)}{h} = \lim_{h \rightarrow 0} \frac{\sqrt{1+2(x+h)} - \sqrt{1+2x}}{h} \left[\frac{\sqrt{1+2(x+h)} + \sqrt{1+2x}}{\sqrt{1+2(x+h)} + \sqrt{1+2x}} \right] \\ &= \lim_{h \rightarrow 0} \frac{(1+2x+2h) - (1+2x)}{h \left[\sqrt{1+2(x+h)} + \sqrt{1+2x} \right]} = \lim_{h \rightarrow 0} \frac{2}{\sqrt{1+2x+2h} + \sqrt{1+2x}} = \frac{2}{2\sqrt{1+2x}} = \frac{1}{\sqrt{1+2x}} \end{aligned}$$

Domain of $g = [-\frac{1}{2}, \infty)$, domain of $g' = (-\frac{1}{2}, \infty)$.

$$\begin{aligned} 29. G'(t) &= \lim_{h \rightarrow 0} \frac{G(t+h) - G(t)}{h} = \lim_{h \rightarrow 0} \frac{\frac{4(t+h)}{(t+h)+1} - \frac{4t}{t+1}}{h} = \lim_{h \rightarrow 0} \frac{\frac{4(t+h)(t+1) - 4t(t+h+1)}{(t+h+1)(t+1)}}{h} \\ &= \lim_{h \rightarrow 0} \frac{(4t^2 + 4ht + 4t + 4h) - (4t^2 + 4ht + 4t)}{h(t+h+1)(t+1)} \\ &= \lim_{h \rightarrow 0} \frac{4h}{h(t+h+1)(t+1)} = \lim_{h \rightarrow 0} \frac{4}{(t+h+1)(t+1)} = \frac{4}{(t+1)^2} \end{aligned}$$

Domain of G = domain of $G' = (-\infty, -1) \cup (-1, \infty)$.

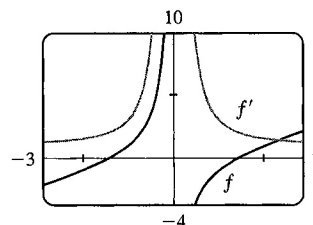
$$\begin{aligned} 31. f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \rightarrow 0} \frac{(x+h)^4 - x^4}{h} = \lim_{h \rightarrow 0} \frac{(x^4 + 4x^3h + 6x^2h^2 + 4xh^3 + h^4) - x^4}{h} \\ &= \lim_{h \rightarrow 0} \frac{4x^3h + 6x^2h^2 + 4xh^3 + h^4}{h} = \lim_{h \rightarrow 0} (4x^3 + 6x^2h + 4xh^2 + h^3) = 4x^3 \end{aligned}$$

Domain of f = domain of $f' = \mathbb{R}$.

$$\begin{aligned}
 33. \text{ (a) } f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \rightarrow 0} \frac{\left[x+h - \left(\frac{2}{x+h} \right) \right] - \left[x - \left(\frac{2}{x} \right) \right]}{h} \\
 &= \lim_{h \rightarrow 0} \left[\frac{h - \frac{2}{(x+h)} + \frac{2}{x}}{h} \right] = \lim_{h \rightarrow 0} \left[1 + \frac{-2x + 2(x+h)}{hx(x+h)} \right] = \lim_{h \rightarrow 0} \left[1 + \frac{2h}{hx(x+h)} \right] \\
 &= \lim_{h \rightarrow 0} \left[1 + \frac{2}{x(x+h)} \right] = 1 + \frac{2}{x^2}
 \end{aligned}$$

(b) Notice that when f has steep tangent lines, $f'(x)$ is very large.

When f is flatter, $f'(x)$ is smaller.



35. (a) $U'(t)$ is the rate at which the unemployment rate is changing with respect to time. Its units are percent per year.

(b) To find $U'(t)$, we use $\lim_{h \rightarrow 0} \frac{U(t+h) - U(t)}{h} \approx \frac{U(t+h) - U(t)}{h}$ for small values of h .

$$\text{For 1991: } U'(1991) = \frac{U(1992) - U(1991)}{1992 - 1991} = \frac{7.5 - 6.8}{1} = 0.70$$

For 1992: We estimate $U'(1992)$ by using $h = -1$ and $h = 1$, and then average the two results to obtain a final estimate.

$$h = -1 \Rightarrow U'(1992) \approx \frac{U(1991) - U(1992)}{1991 - 1992} = \frac{6.8 - 7.5}{-1} = 0.70;$$

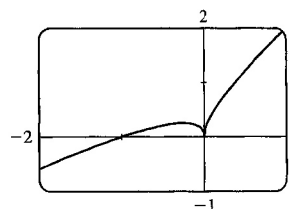
$$h = 1 \Rightarrow U'(1992) \approx \frac{U(1993) - U(1992)}{1993 - 1992} = \frac{6.9 - 7.5}{1} = -0.60.$$

So we estimate that $U'(1992) \approx \frac{1}{2}[0.70 + (-0.60)] = 0.05$.

t	1991	1992	1993	1994	1995	1996	1997	1998	1999	2000
$U'(t)$	0.70	0.05	-0.70	-0.65	-0.35	-0.35	-0.45	-0.35	-0.25	-0.20

37. f is not differentiable at $x = -1$ or at $x = 11$ because the graph has vertical tangents at those points; at $x = 4$, because there is a discontinuity there; and at $x = 8$, because the graph has a corner there.

39. As we zoom in toward $(-1, 0)$, the curve appears more and more like a straight line, so $f(x) = x + \sqrt{|x|}$ is differentiable at $x = -1$. But no matter how much we zoom in toward the origin, the curve doesn't straighten out—we can't eliminate the sharp point (a cusp). So f is not differentiable at $x = 0$.



41. (a) Note that we have factored
- $x - a$
- as the difference of two cubes in the third step.

$$\begin{aligned} f'(a) &= \lim_{x \rightarrow a} \frac{f(x) - f(a)}{x - a} = \lim_{x \rightarrow a} \frac{x^{1/3} - a^{1/3}}{x - a} = \lim_{x \rightarrow a} \frac{x^{1/3} - a^{1/3}}{(x^{1/3} - a^{1/3})(x^{2/3} + x^{1/3}a^{1/3} + a^{2/3})} \\ &= \lim_{x \rightarrow a} \frac{1}{x^{2/3} + x^{1/3}a^{1/3} + a^{2/3}} = \frac{1}{3a^{2/3}} \text{ or } \frac{1}{3}a^{-2/3} \end{aligned}$$

- (b) $f'(0) = \lim_{h \rightarrow 0} \frac{f(0+h) - f(0)}{h} = \lim_{h \rightarrow 0} \frac{\sqrt[3]{h} - 0}{h} = \lim_{h \rightarrow 0} \frac{1}{h^{2/3}}$. This function increases without bound, so the limit does not exist, and therefore $f'(0)$ does not exist.

- (c) $\lim_{x \rightarrow 0} |f'(x)| = \lim_{x \rightarrow 0} \frac{1}{3x^{2/3}} = \infty$ and f is continuous at $x = 0$ (root function), so f has a vertical tangent at $x = 0$.

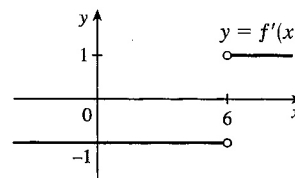
$$43. f(x) = |x - 6| = \begin{cases} -(x - 6) & \text{if } x < 6 \\ x - 6 & \text{if } x \geq 6 \end{cases} = \begin{cases} 6 - x & \text{if } x < 6 \\ x - 6 & \text{if } x \geq 6 \end{cases}$$

$$\lim_{x \rightarrow 6^+} \frac{f(x) - f(6)}{x - 6} = \lim_{x \rightarrow 6^+} \frac{|x - 6| - 0}{x - 6} = \lim_{x \rightarrow 6^+} \frac{x - 6}{x - 6} = \lim_{x \rightarrow 6^+} 1 = 1.$$

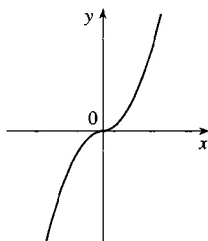
$$\begin{aligned} \text{But } \lim_{x \rightarrow 6^-} \frac{f(x) - f(6)}{x - 6} &= \lim_{x \rightarrow 6^-} \frac{|x - 6| - 0}{x - 6} = \lim_{x \rightarrow 6^-} \frac{6 - x}{x - 6} \\ &= \lim_{x \rightarrow 6^-} (-1) = -1 \end{aligned}$$

$$\text{So } f'(6) = \lim_{x \rightarrow 6} \frac{f(x) - f(6)}{x - 6} \text{ does not exist. However, } f'(x) = \begin{cases} -1 & \text{if } x < 6 \\ 1 & \text{if } x > 6 \end{cases}$$

Another way of writing the answer is $f'(x) = \frac{x - 6}{|x - 6|}$.



$$45. (a) f(x) = x|x| = \begin{cases} x^2 & \text{if } x \geq 0 \\ -x^2 & \text{if } x < 0 \end{cases}$$



- (b) Since $f(x) = x^2$ for $x \geq 0$, we have $f'(x) = 2x$ for $x > 0$.

[See Exercise 2.9.19(d).] Similarly, since $f(x) = -x^2$ for

$x < 0$, we have $f'(x) = -2x$ for $x < 0$. At $x = 0$, we have

$$f'(0) = \lim_{x \rightarrow 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x \rightarrow 0} \frac{x|x|}{x} = \lim_{x \rightarrow 0} |x| = 0.$$

So f is differentiable at 0. Thus, f is differentiable for all x .

$$(c) \text{ From part (b), we have } f'(x) = \begin{cases} 2x & \text{if } x \geq 0 \\ -2x & \text{if } x < 0 \end{cases} = 2|x|.$$

47. (a) If
- f
- is even, then

$$\begin{aligned} f'(-x) &= \lim_{h \rightarrow 0} \frac{f(-x+h) - f(-x)}{h} = \lim_{h \rightarrow 0} \frac{f[-(x-h)] - f(-x)}{h} = \lim_{h \rightarrow 0} \frac{f(x-h) - f(x)}{h} \\ &= -\lim_{h \rightarrow 0} \frac{f(x-h) - f(x)}{-h} \quad [\text{let } \Delta x = -h] = -\lim_{\Delta x \rightarrow 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} = -f'(x) \end{aligned}$$

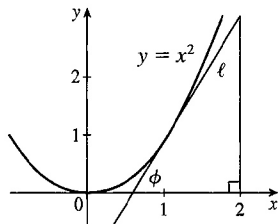
Therefore, f' is odd.

(b) If f is odd, then

$$\begin{aligned} f'(-x) &= \lim_{h \rightarrow 0} \frac{f(-x+h) - f(-x)}{h} = \lim_{h \rightarrow 0} \frac{f[-(x-h)] - f(-x)}{h} = \lim_{h \rightarrow 0} \frac{-f(x-h) + f(x)}{h} \\ &= \lim_{h \rightarrow 0} \frac{f(x-h) - f(x)}{-h} \quad [\text{let } \Delta x = -h] = \lim_{\Delta x \rightarrow 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} = f'(x) \end{aligned}$$

Therefore, f' is even.

49.

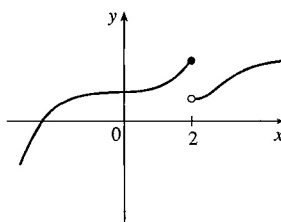


In the right triangle in the diagram, let Δy be the side opposite angle ϕ and Δx the side adjacent angle ϕ . Then the slope of the tangent line ℓ is $m = \Delta y / \Delta x = \tan \phi$. Note that $0 < \phi < \frac{\pi}{2}$. We know (see Exercise 19) that the derivative of $f(x) = x^2$ is $f'(x) = 2x$. So the slope of the tangent to the curve at the point $(1, 1)$ is 2. Thus, ϕ is the angle between 0 and $\frac{\pi}{2}$ whose tangent is 2; that is, $\phi = \tan^{-1} 2 \approx 63^\circ$.

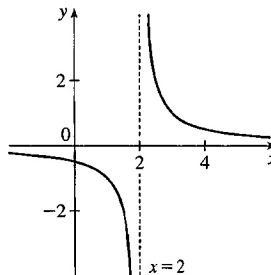
2 Review

CONCEPT CHECK

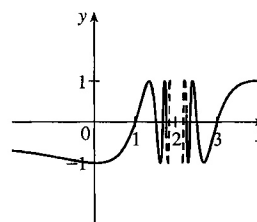
- $\lim_{x \rightarrow a} f(x) = L$: See Definition 2.2.1 and Figures 1 and 2 in Section 2.2.
 - $\lim_{x \rightarrow a^+} f(x) = L$: See the paragraph after Definition 2.2.2 and Figure 9(b) in Section 2.2.
 - $\lim_{x \rightarrow a^-} f(x) = L$: See Definition 2.2.2 and Figure 9(a) in Section 2.2.
 - $\lim_{x \rightarrow a} f(x) = \infty$: See Definition 2.2.4 and Figure 12 in Section 2.2.
 - $\lim_{x \rightarrow \infty} f(x) = L$: See Definition 2.6.1 and Figure 2 in Section 2.6.
- In general, the limit of a function fails to exist when the function does not approach a fixed number. For each of the following functions, the limit fails to exist at $x = 2$.



The left- and right-hand limits are not equal.



There is an infinite discontinuity.

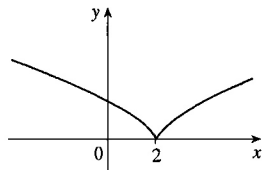


There are an infinite number of oscillations.

- (a)–(g) See the statements of Limit Laws 1–6 and 11 in Section 2.3.
- See Theorem 3 in Section 2.3.

5. (a) See Definition 2.2.6 and Figures 12–14 in Section 2.2.
 (b) See Definition 2.6.3 and Figures 3 and 4 in Section 2.6.
6. (a) $y = x^4$: No asymptote
 (b) $y = \sin x$: No asymptote
 (c) $y = \tan x$: Vertical asymptotes $x = \frac{\pi}{2} + \pi n$,
 n an integer
 (d) $y = \tan^{-1} x$: Horizontal asymptotes $y = \pm \frac{\pi}{2}$
 (e) $y = e^x$: Horizontal asymptote $y = 0$
 (f) $y = \ln x$: Vertical asymptote $x = 0$
 $(\lim_{x \rightarrow -\infty} e^x = 0)$
 $(\lim_{x \rightarrow 0^+} \ln x = -\infty)$
 (g) $y = 1/x$: Vertical asymptote $x = 0$,
 horizontal asymptote $y = 0$
 (h) $y = \sqrt{x}$: No asymptote
7. (a) A function f is continuous at a number a if $f(x)$ approaches $f(a)$ as x approaches a ; that is,
 $\lim_{x \rightarrow a} f(x) = f(a)$.
 (b) A function f is continuous on the interval $(-\infty, \infty)$ if f is continuous at every real number a . The graph of such a function has no breaks and every vertical line crosses it.
8. See Theorem 2.5.10.
9. See Definition 2.7.1.
10. See the paragraph containing Formula 3 in Section 2.7.
11. (a) The average rate of change of y with respect to x over the interval $[x_1, x_2]$ is $\frac{f(x_2) - f(x_1)}{x_2 - x_1}$.
 (b) The instantaneous rate of change of y with respect to x at $x = x_1$ is $\lim_{x_2 \rightarrow x_1} \frac{f(x_2) - f(x_1)}{x_2 - x_1}$.
12. See Definition 2.8.2. The pages following the definition discuss interpretations of $f'(a)$ as the slope of a tangent line to the graph of f at $x = a$ and as an instantaneous rate of change of $f(x)$ with respect to x when $x = a$.
13. (a) A function f is differentiable at a number a if its derivative f' exists at $x = a$; that is, if $f'(a)$ exists.
 (b) See Theorem 2.9.4. This theorem also tells us that if f is *not* continuous at a , then f is *not* differentiable at a .

(c)



14. See the discussion and Figure 8 on page 172.

 TRUE-FALSE QUIZ

1. False. Limit Law 2 applies only if the individual limits exist (these don't).
3. True. Limit Law 5 applies.
5. False. Consider $\lim_{x \rightarrow 5} \frac{x(x-5)}{x-5}$ or $\lim_{x \rightarrow 5} \frac{\sin(x-5)}{x-5}$. The first limit exists and is equal to 5. By Example 3 in Section 2.2, we know that the latter limit exists (and it is equal to 1).

7. True. A polynomial is continuous everywhere, so $\lim_{x \rightarrow b} p(x)$ exists and is equal to $p(b)$.
9. True. See Figure 4 in Section 2.6.
11. False. Consider $f(x) = \begin{cases} 1/(x-1) & \text{if } x \neq 1 \\ 2 & \text{if } x = 1 \end{cases}$
13. True. Use Theorem 2.5.8 with $a = 2$, $b = 5$, and $g(x) = 4x^2 - 11$. Note that $f(4) = 3$ is not needed.
15. True, by the definition of a limit with $\varepsilon = 1$.
17. False. See the note after Theorem 4 in Section 2.9.

EXERCISES

1. (a) (i) $\lim_{x \rightarrow 2^+} f(x) = 3$ (ii) $\lim_{x \rightarrow -3^+} f(x) = 0$
 (iii) $\lim_{x \rightarrow -3} f(x)$ does not exist since the left and right limits are not equal. (The left limit is -2 .) (iv) $\lim_{x \rightarrow 4} f(x) = 2$
 (v) $\lim_{x \rightarrow 0} f(x) = \infty$ (vi) $\lim_{x \rightarrow 2^-} f(x) = -\infty$
 (vii) $\lim_{x \rightarrow \infty} f(x) = 4$ (viii) $\lim_{x \rightarrow -\infty} f(x) = -1$
- (b) The equations of the horizontal asymptotes are $y = -1$ and $y = 4$.
- (c) The equations of the vertical asymptotes are $x = 0$ and $x = 2$.
- (d) f is discontinuous at $x = -3, 0, 2$, and 4 . The discontinuities are jump, infinite, infinite, and removable, respectively.

3. Since the exponential function is continuous, $\lim_{x \rightarrow 1} e^{x^3 - x} = e^{1-1} = e^0 = 1$.

$$5. \lim_{x \rightarrow -3} \frac{x^2 - 9}{x^2 + 2x - 3} = \lim_{x \rightarrow -3} \frac{(x+3)(x-3)}{(x+3)(x-1)} = \lim_{x \rightarrow -3} \frac{x-3}{x-1} = \frac{-3-3}{-3-1} = \frac{-6}{-4} = \frac{3}{2}$$

$$7. \lim_{h \rightarrow 0} \frac{(h-1)^3 + 1}{h} = \lim_{h \rightarrow 0} \frac{(h^3 - 3h^2 + 3h - 1) + 1}{h} = \lim_{h \rightarrow 0} \frac{h^3 - 3h^2 + 3h}{h} = \lim_{h \rightarrow 0} (h^2 - 3h + 3) = 3$$

Another solution: Factor the numerator as a sum of two cubes and then simplify.

$$\begin{aligned} \lim_{h \rightarrow 0} \frac{(h-1)^3 + 1}{h} &= \lim_{h \rightarrow 0} \frac{(h-1)^3 + 1^3}{h} = \lim_{h \rightarrow 0} \frac{[(h-1) + 1][(h-1)^2 - 1(h-1) + 1^2]}{h} \\ &= \lim_{h \rightarrow 0} [(h-1)^2 - h + 2] = 1 - 0 + 2 = 3 \end{aligned}$$

$$9. \lim_{r \rightarrow 9} \frac{\sqrt{r}}{(r-9)^4} = \infty \text{ since } (r-9)^4 \rightarrow 0 \text{ as } r \rightarrow 9 \text{ and } \frac{\sqrt{r}}{(r-9)^4} > 0 \text{ for } r \neq 9.$$

$$11. \lim_{s \rightarrow 16} \frac{4 - \sqrt{s}}{s - 16} = \lim_{s \rightarrow 16} \frac{4 - \sqrt{s}}{(\sqrt{s} + 4)(\sqrt{s} - 4)} = \lim_{s \rightarrow 16} \frac{-1}{\sqrt{s} + 4} = \frac{-1}{\sqrt{16} + 4} = -\frac{1}{8}$$

$$13. \frac{|x-8|}{x-8} = \begin{cases} \frac{x-8}{x-8} & \text{if } x-8 > 0 \\ \frac{-(x-8)}{x-8} & \text{if } x-8 < 0 \end{cases} = \begin{cases} 1 & \text{if } x > 8 \\ -1 & \text{if } x < 8 \end{cases}$$

$$\text{Thus, } \lim_{x \rightarrow 8^-} \frac{|x-8|}{x-8} = \lim_{x \rightarrow 8^-} (-1) = -1.$$

$$15. \lim_{x \rightarrow 0} \frac{1 - \sqrt{1 - x^2}}{x} \cdot \frac{1 + \sqrt{1 - x^2}}{1 + \sqrt{1 - x^2}} = \lim_{x \rightarrow 0} \frac{1 - (1 - x^2)}{x(1 + \sqrt{1 - x^2})} = \lim_{x \rightarrow 0} \frac{x^2}{x(1 + \sqrt{1 - x^2})} = \lim_{x \rightarrow 0} \frac{x}{1 + \sqrt{1 - x^2}} = 0$$

$$17. \lim_{x \rightarrow \infty} \frac{1 + 2x - x^2}{1 - x + 2x^2} = \lim_{x \rightarrow \infty} \frac{(1 + 2x - x^2)/x^2}{(1 - x + 2x^2)/x^2} = \lim_{x \rightarrow \infty} \frac{1/x^2 + 2/x - 1}{1/x^2 - 1/x + 2} = \frac{0 + 0 - 1}{0 - 0 + 2} = -\frac{1}{2}$$

19. Since x is positive, $\sqrt{x^2} = |x| = x$.

$$\lim_{x \rightarrow \infty} \frac{\sqrt{x^2 - 9}}{2x - 6} = \lim_{x \rightarrow \infty} \frac{\sqrt{x^2 - 9}/\sqrt{x^2}}{(2x - 6)/x} = \lim_{x \rightarrow \infty} \frac{\sqrt{1 - 9/x^2}}{2 - 6/x} = \frac{\sqrt{1 - 0}}{2 - 0} = \frac{1}{2}$$

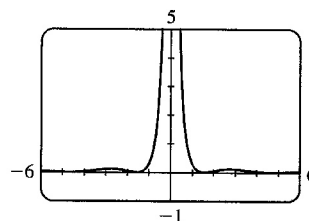
21. $\lim_{x \rightarrow \infty} e^{-3x} = 0$ since $-3x \rightarrow -\infty$ as $x \rightarrow \infty$ and $\lim_{t \rightarrow -\infty} e^t = 0$.

23. From the graph of $y = (\cos^2 x)/x^2$, it appears that $y = 0$ is the horizontal asymptote and $x = 0$ is the vertical asymptote. Now $0 \leq (\cos x)^2 \leq 1$

$$\Rightarrow \frac{0}{x^2} \leq \frac{\cos^2 x}{x^2} \leq \frac{1}{x^2} \Rightarrow 0 \leq \frac{\cos^2 x}{x^2} \leq \frac{1}{x^2}. \text{ But } \lim_{x \rightarrow \pm\infty} 0 = 0 \text{ and}$$

$$\lim_{x \rightarrow \pm\infty} \frac{1}{x^2} = 0, \text{ so by the Squeeze Theorem,}$$

$\lim_{x \rightarrow \pm\infty} \frac{\cos^2 x}{x^2} = 0$. Thus, $y = 0$ is the horizontal asymptote. $\lim_{x \rightarrow 0} \frac{\cos^2 x}{x^2} = \infty$ because $\cos^2 x \rightarrow 1$ and $x^2 \rightarrow 0$ as $x \rightarrow 0$, so $x = 0$ is the vertical asymptote.



25. Since $2x - 1 \leq f(x) \leq x^2$ for $0 < x < 3$ and $\lim_{x \rightarrow 1} (2x - 1) = 1 = \lim_{x \rightarrow 1} x^2$, we have $\lim_{x \rightarrow 1} f(x) = 1$ by the Squeeze Theorem.

27. Given $\varepsilon > 0$, we need $\delta > 0$ so that if $0 < |x - 5| < \delta$, then $|(7x - 27) - 8| < \varepsilon \Leftrightarrow |7x - 35| < \varepsilon \Leftrightarrow |x - 5| < \varepsilon/7$. So take $\delta = \varepsilon/7$. Then $0 < |x - 5| < \delta \Rightarrow |(7x - 27) - 8| < \varepsilon$. Thus, $\lim_{x \rightarrow 5} (7x - 27) = 8$ by the definition of a limit.

29. Given $\varepsilon > 0$, we need $\delta > 0$ so that if $0 < |x - 2| < \delta$, then $|x^2 - 3x - (-2)| < \varepsilon$. First, note that if $|x - 2| < 1$, then $-1 < x - 2 < 1$, so $0 < x - 1 < 2 \Rightarrow |x - 1| < 2$. Now let $\delta = \min\{\varepsilon/2, 1\}$. Then $0 < |x - 2| < \delta \Rightarrow |x^2 - 3x - (-2)| = |(x - 2)(x - 1)| = |x - 2||x - 1| < (\varepsilon/2)(2) = \varepsilon$.

Thus, $\lim_{x \rightarrow 2} (x^2 - 3x) = -2$ by the definition of a limit.

31. (a) $f(x) = \sqrt{-x}$ if $x < 0$, $f(x) = 3 - x$ if $0 \leq x < 3$, $f(x) = (x - 3)^2$ if $x > 3$.

$$(i) \lim_{x \rightarrow 0^+} f(x) = \lim_{x \rightarrow 0^+} (3 - x) = 3$$

$$(ii) \lim_{x \rightarrow 0^-} f(x) = \lim_{x \rightarrow 0^-} \sqrt{-x} = 0$$

(iii) Because of (i) and (ii), $\lim_{x \rightarrow 0} f(x)$ does not exist.

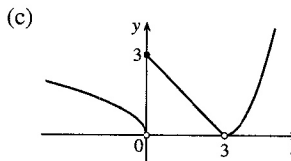
$$(iv) \lim_{x \rightarrow 3^-} f(x) = \lim_{x \rightarrow 3^-} (3 - x) = 0$$

$$(v) \lim_{x \rightarrow 3^+} f(x) = \lim_{x \rightarrow 3^+} (x - 3)^2 = 0$$

(vi) Because of (iv) and (v), $\lim_{x \rightarrow 3} f(x) = 0$.

(b) f is discontinuous at 0 since $\lim_{x \rightarrow 0} f(x)$ does not exist.

f is discontinuous at 3 since $f(3)$ does not exist.



33. $\sin x$ is continuous on \mathbb{R} by Theorem 7 in Section 2.5. Since e^x is continuous on \mathbb{R} , $e^{\sin x}$ is continuous on \mathbb{R} by Theorem 9 in Section 2.5. Lastly, x is continuous on \mathbb{R} since it's a polynomial and the product $xe^{\sin x}$ is continuous on its domain \mathbb{R} by Theorem 4 in Section 2.5.

35. $f(x) = 2x^3 + x^2 + 2$ is a polynomial, so it is continuous on $[-2, -1]$ and $f(-2) = -10 < 0 < 1 = f(-1)$. So by the Intermediate Value Theorem there is a number c in $(-2, -1)$ such that $f(c) = 0$, that is, the equation $2x^3 + x^2 + 2 = 0$ has a root in $(-2, -1)$.

37. (a) The slope of the tangent line at $(2, 1)$ is

$$\begin{aligned}\lim_{x \rightarrow 2} \frac{f(x) - f(2)}{x - 2} &= \lim_{x \rightarrow 2} \frac{9 - 2x^2 - 1}{x - 2} = \lim_{x \rightarrow 2} \frac{8 - 2x^2}{x - 2} = \lim_{x \rightarrow 2} \frac{-2(x^2 - 4)}{x - 2} = \lim_{x \rightarrow 2} \frac{-2(x - 2)(x + 2)}{x - 2} \\ &= \lim_{x \rightarrow 2} [-2(x + 2)] = -2 \cdot 4 = -8\end{aligned}$$

(b) An equation of this tangent line is $y - 1 = -8(x - 2)$ or $y = -8x + 17$.

39. (a) $s = s(t) = 1 + 2t + t^2/4$. The average velocity over the time interval $[1, 1 + h]$ is

$$v_{\text{ave}} = \frac{s(1+h) - s(1)}{(1+h) - 1} = \frac{1 + 2(1+h) + (1+h)^2/4 - 13/4}{h} = \frac{10h + h^2}{4h} = \frac{10 + h}{4}.$$

So for the following intervals the average velocities are:

(i) $[1, 3]$: $h = 2$, $v_{\text{ave}} = (10 + 2)/4 = 3$ m/s

(ii) $[1, 2]$: $h = 1$, $v_{\text{ave}} = (10 + 1)/4 = 2.75$ m/s

(iii) $[1, 1.5]$: $h = 0.5$, $v_{\text{ave}} = (10 + 0.5)/4 = 2.625$ m/s

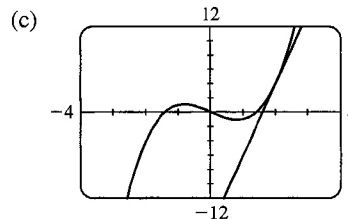
(iv) $[1, 1.1]$: $h = 0.1$, $v_{\text{ave}} = (10 + 0.1)/4 = 2.525$ m/s

(b) When $t = 1$, the instantaneous velocity is $\lim_{h \rightarrow 0} \frac{s(1+h) - s(1)}{h} = \lim_{h \rightarrow 0} \frac{10 + h}{4} = \frac{10}{4} = 2.5$ m/s.

41. (a) $f'(2) = \lim_{x \rightarrow 2} \frac{f(x) - f(2)}{x - 2} = \lim_{x \rightarrow 2} \frac{x^3 - 2x - 4}{x - 2}$

$$= \lim_{x \rightarrow 2} \frac{(x - 2)(x^2 + 2x + 2)}{x - 2} = \lim_{x \rightarrow 2} (x^2 + 2x + 2) = 10$$

(b) $y - 4 = 10(x - 2)$ or $y = 10x - 16$



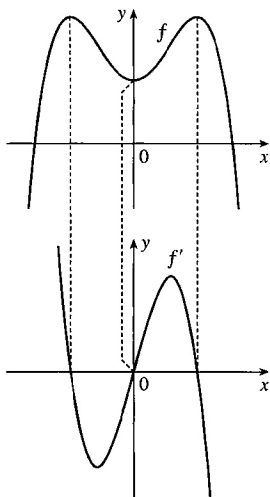
43. (a) $f'(r)$ is the rate at which the total cost changes with respect to the interest rate. Its units are dollars/(percent per year).

(b) The total cost of paying off the loan is increasing by \$1200/(percent per year) as the interest rate reaches 10%. So if the interest rate goes up from 10% to 11%, the cost goes up approximately \$1200.

(c) As r increases, C increases. So $f'(r)$ will always be positive.

For Exercises 44–46, see the hints before Exercise 5 in Section 2.9.

45.

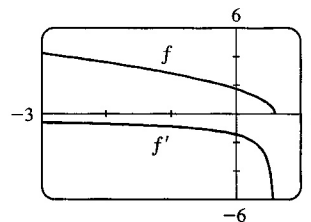


$$\begin{aligned}
 49. (a) f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \rightarrow 0} \frac{\sqrt{3-5(x+h)} - \sqrt{3-5x}}{h} \cdot \frac{\sqrt{3-5(x+h)} + \sqrt{3-5x}}{\sqrt{3-5(x+h)} + \sqrt{3-5x}} \\
 &= \lim_{h \rightarrow 0} \frac{[3-5(x+h)] - (3-5x)}{h(\sqrt{3-5(x+h)} + \sqrt{3-5x})} = \lim_{h \rightarrow 0} \frac{-5}{\sqrt{3-5(x+h)} + \sqrt{3-5x}} = \frac{-5}{2\sqrt{3-5x}}
 \end{aligned}$$

(b) Domain of f : (the radicand must be nonnegative) $3-5x \geq 0 \Rightarrow 5x \leq 3 \Rightarrow x \in (-\infty, \frac{3}{5}]$

Domain of f' : exclude $\frac{3}{5}$ because it makes the denominator zero; $x \in (-\infty, \frac{3}{5})$

(c) Our answer to part (a) is reasonable because $f'(x)$ is always negative and f is always decreasing.



49. f is not differentiable: at $x = -4$ because f is not continuous, at $x = -1$ because f has a corner, at $x = 2$ because f is not continuous, and at $x = 5$ because f has a vertical tangent.

51. $B'(1990)$ is the rate at which the total value of U.S. banknotes in circulation is changing in billions of dollars per year. To estimate the value of $B'(1990)$, we will average the difference quotients obtained using the times $t = 1985$

and $t = 1995$. Let $A = \frac{B(1985) - B(1990)}{1985 - 1990} = \frac{182.0 - 268.2}{-5} = 17.24$ and

$C = \frac{B(1995) - B(1990)}{1995 - 1990} = \frac{401.5 - 268.2}{5} = 26.66$. Then

$$B'(1990) = \lim_{t \rightarrow 1990} \frac{B(t) - B(1990)}{t - 1990} \approx \frac{A + C}{2} = \frac{17.24 + 26.66}{2} = 21.95 \text{ billions of dollars/year.}$$

53. $|f(x)| \leq g(x) \Leftrightarrow -g(x) \leq f(x) \leq g(x)$ and $\lim_{x \rightarrow a} g(x) = 0 = \lim_{x \rightarrow a} -g(x)$. Thus, by the Squeeze Theorem,

$$\lim_{x \rightarrow a} f(x) = 0.$$

□ PROBLEMS PLUS

1. Let $t = \sqrt[6]{x}$, so $x = t^6$. Then $t \rightarrow 1$ as $x \rightarrow 1$, so

$$\lim_{x \rightarrow 1} \frac{\sqrt[3]{x} - 1}{\sqrt{x} - 1} = \lim_{t \rightarrow 1} \frac{t^2 - 1}{t^3 - 1} = \lim_{t \rightarrow 1} \frac{(t-1)(t+1)}{(t-1)(t^2+t+1)} = \lim_{t \rightarrow 1} \frac{t+1}{t^2+t+1} = \frac{1+1}{1^2+1+1} = \frac{2}{3}.$$

Another method: Multiply both the numerator and the denominator by $(\sqrt{x} + 1)(\sqrt[3]{x^2} + \sqrt[3]{x} + 1)$.

3. For $-\frac{1}{2} < x < \frac{1}{2}$, we have $2x - 1 < 0$ and $2x + 1 > 0$, so $|2x - 1| = -(2x - 1)$ and $|2x + 1| = 2x + 1$.

$$\text{Therefore, } \lim_{x \rightarrow 0} \frac{|2x - 1| - |2x + 1|}{x} = \lim_{x \rightarrow 0} \frac{-(2x - 1) - (2x + 1)}{x} = \lim_{x \rightarrow 0} \frac{-4x}{x} = \lim_{x \rightarrow 0} (-4) = -4.$$

5. Since $\lfloor x \rfloor \leq x < \lfloor x \rfloor + 1$, we have $\frac{\lfloor x \rfloor}{\lfloor x \rfloor} \leq \frac{x}{\lfloor x \rfloor} < \frac{\lfloor x \rfloor + 1}{\lfloor x \rfloor} \Rightarrow 1 \leq \frac{x}{\lfloor x \rfloor} < 1 + \frac{1}{\lfloor x \rfloor}$ for $x \geq 1$. As $x \rightarrow \infty$,

$$\lfloor x \rfloor \rightarrow \infty, \text{ so } \frac{1}{\lfloor x \rfloor} \rightarrow 0 \text{ and } 1 + \frac{1}{\lfloor x \rfloor} \rightarrow 1. \text{ Thus, } \lim_{x \rightarrow \infty} \frac{x}{\lfloor x \rfloor} = 1 \text{ by the Squeeze Theorem.}$$

7. f is continuous on $(-\infty, a)$ and (a, ∞) . To make f continuous on \mathbb{R} , we must have continuity at a . Thus,

$$\lim_{x \rightarrow a^+} f(x) = \lim_{x \rightarrow a^-} f(x) \Rightarrow \lim_{x \rightarrow a^+} x^2 = \lim_{x \rightarrow a^-} (x + 1) \Rightarrow a^2 = a + 1 \Rightarrow a^2 - a - 1 = 0 \Rightarrow$$

$$[\text{by the quadratic formula}] \quad a = (1 \pm \sqrt{5})/2 \approx 1.618 \text{ or } -0.618.$$

9. $\lim_{x \rightarrow a} f(x) = \lim_{x \rightarrow a} \left(\frac{1}{2} [f(x) + g(x)] + \frac{1}{2} [f(x) - g(x)] \right)$

$$= \frac{1}{2} \lim_{x \rightarrow a} [f(x) + g(x)] + \frac{1}{2} \lim_{x \rightarrow a} [f(x) - g(x)]$$

$$= \frac{1}{2} \cdot 2 + \frac{1}{2} \cdot 1 = \frac{3}{2}, \text{ and}$$

$$\lim_{x \rightarrow a} g(x) = \lim_{x \rightarrow a} \left([f(x) + g(x)] - f(x) \right) = \lim_{x \rightarrow a} [f(x) + g(x)] - \lim_{x \rightarrow a} f(x) = 2 - \frac{3}{2} = \frac{1}{2}.$$

$$\text{So } \lim_{x \rightarrow a} [f(x)g(x)] = \left[\lim_{x \rightarrow a} f(x) \right] \left[\lim_{x \rightarrow a} g(x) \right] = \frac{3}{2} \cdot \frac{1}{2} = \frac{3}{4}.$$

Another solution: Since $\lim_{x \rightarrow a} [f(x) + g(x)]$ and $\lim_{x \rightarrow a} [f(x) - g(x)]$ exist, we must have

$$\lim_{x \rightarrow a} [f(x) + g(x)]^2 = \left(\lim_{x \rightarrow a} [f(x) + g(x)] \right)^2 \text{ and } \lim_{x \rightarrow a} [f(x) - g(x)]^2 = \left(\lim_{x \rightarrow a} [f(x) - g(x)] \right)^2, \text{ so}$$

$$\lim_{x \rightarrow a} [f(x)g(x)] = \lim_{x \rightarrow a} \frac{1}{4} ([f(x) + g(x)]^2 - [f(x) - g(x)]^2) \quad [\text{because all of the } f^2 \text{ and } g^2 \text{ cancel}]$$

$$= \frac{1}{4} \left(\lim_{x \rightarrow a} [f(x) + g(x)]^2 - \lim_{x \rightarrow a} [f(x) - g(x)]^2 \right) = \frac{1}{4} (2^2 - 1^2) = \frac{3}{4}.$$

11. (a) Consider $G(x) = T(x + 180^\circ) - T(x)$. Fix any number a . If $G(a) = 0$, we are done:

Temperature at a = Temperature at $a + 180^\circ$. If $G(a) > 0$, then $G(a + 180^\circ) = T(a + 360^\circ) - T(a + 180^\circ) = T(a) - T(a + 180^\circ) = -G(a) < 0$. Also, G is continuous since temperature varies continuously. So, by the Intermediate Value Theorem, G has a zero on the interval $[a, a + 180^\circ]$. If $G(a) < 0$, then a similar argument applies.

(b) Yes. The same argument applies.

(c) The same argument applies for quantities that vary continuously, such as barometric pressure. But one could argue that altitude above sea level is sometimes discontinuous, so the result might not always hold for that quantity.

13. (a) Put $x = 0$ and $y = 0$ in the equation: $f(0 + 0) = f(0) + f(0) + 0^2 \cdot 0 + 0 \cdot 0^2 \Rightarrow f(0) = 2f(0)$.

Subtracting $f(0)$ from each side of this equation gives $f(0) = 0$.

$$\begin{aligned} \text{(b) } f'(0) &= \lim_{h \rightarrow 0} \frac{f(0+h) - f(0)}{h} = \lim_{h \rightarrow 0} \frac{[f(0) + f(h) + 0^2h + 0h^2] - f(0)}{h} \\ &= \lim_{h \rightarrow 0} \frac{f(h)}{h} = \lim_{x \rightarrow 0} \frac{f(x)}{x} = 1 \end{aligned}$$

$$\begin{aligned} \text{(c) } f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \rightarrow 0} \frac{[f(x) + f(h) + x^2h + xh^2] - f(x)}{h} \\ &= \lim_{h \rightarrow 0} \frac{f(h) + x^2h + xh^2}{h} = \lim_{h \rightarrow 0} \left[\frac{f(h)}{h} + x^2 + xh \right] = 1 + x^2 \end{aligned}$$