3 ☐ DIFFERENTIATION RULES

3.1 Derivatives of Polynomials and Exponential Functions

1. (a) e is the number such that $\lim_{h \to 0} \frac{e^h - 1}{h} = 1$.

(b)

x	$(2.7^x - 1)/x$
-0.001	0.9928
-0.0001	0.9932
0.001	0.9937
0.0001	0.9933

x	$(2.8^x - 1)/x$
-0.001	1.0291
-0.0001	1.0296
0.001	1.0301
0.0001	1.0297

From the tables (to two decimal places), $\lim_{h \to 0} \frac{2.7^h - 1}{h} = 0.99$ and $\lim_{h \to 0} \frac{2.8^h - 1}{h} = 1.03$. Since $0.99 < 1 < 1.03$, $2.7 < e < 2.8$.

3. $f(x) = 186.5$ is a constant function, so its derivative is 0, that is, $f'(x) = 0$.

5. $f(x) = 5x - 1 \quad \Rightarrow \quad f'(x) = 5 - 0 = 5$

7. $f(x) = x^2 + 3x - 4 \quad \Rightarrow \quad f'(x) = 2x^{2-1} + 3 - 0 = 2x + 3$

9. $f(t) = \frac{1}{4}(t^4 + 8) \quad \Rightarrow \quad f'(t) = \frac{1}{4}(t^4 + 8)' = \frac{1}{4}(4t^{4-1} + 0) = t^3$

11. $y = x^{-2/5} \quad \Rightarrow \quad y' = -\frac{2}{5}x^{(-2/5)-1} = -\frac{2}{5}x^{-7/5} = -\frac{2}{5x^{7/5}}$

13. $V(r) = \frac{4}{3}\pi r^3 \quad \Rightarrow \quad V'(r) = \frac{4}{3}\pi(3r^2) = 4\pi r^2$

15. $Y(t) = 6t^{-9} \quad \Rightarrow \quad Y'(t) = 6(-9)t^{-10} = -54t^{-10}$

17. $G(x) = \sqrt{x} - 2e^x = x^{1/2} - 2e^x \quad \Rightarrow \quad G'(x) = \frac{1}{2}x^{-1/2} - 2e^x = \frac{1}{2\sqrt{x}} - 2e^x$

19. $F(x) = (\frac{1}{2}x)^5 = (\frac{1}{2})^5 x^5 = \frac{1}{32}x^5 \quad \Rightarrow \quad F'(x) = \frac{1}{32}(5x^4) = \frac{5}{32}x^4$

21. $g(x) = x^2 + \frac{1}{x^2} = x^2 + x^{-2} \quad \Rightarrow \quad g'(x) = 2x + (-2)x^{-3} = 2x - \frac{2}{x^3}$

23. $y = \frac{x^2 + 4x + 3}{\sqrt{x}} = x^{3/2} + 4x^{1/2} + 3x^{-1/2} \quad \Rightarrow$

$y' = \frac{3}{2}x^{1/2} + 4(\frac{1}{2})x^{-1/2} + 3(-\frac{1}{2})x^{-3/2} = \frac{3}{2}\sqrt{x} + \frac{2}{\sqrt{x}} - \frac{3}{2x\sqrt{x}}$

[note that $x^{3/2} = x^{2/2} \cdot x^{1/2} = x\sqrt{x}$]

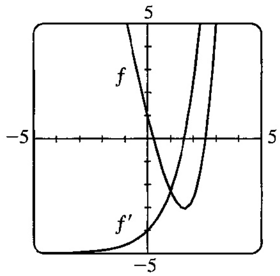
25. $y = 4\pi^2 \quad \Rightarrow \quad y' = 0$ since $4\pi^2$ is a constant.

27. $y = ax^2 + bx + c \quad \Rightarrow \quad y' = 2ax + b$

29. $v = t^2 - \frac{1}{\sqrt[4]{t^3}} = t^2 - t^{-3/4} \quad \Rightarrow \quad v' = 2t - (-\frac{3}{4})t^{-7/4} = 2t + \frac{3}{4t^{7/4}} = 2t + \frac{3}{4t\sqrt[4]{t^3}}$

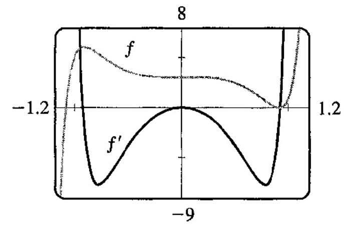
31. $z = \frac{A}{y^{10}} + Be^y = Ay^{-10} + Be^y \quad \Rightarrow \quad z' = -10Ay^{-11} + Be^y = -\frac{10A}{y^{11}} + Be^y$

33. $f(x) = e^x - 5x \quad \Rightarrow \quad f'(x) = e^x - 5.$



Notice that $f'(x) = 0$ when f has a horizontal tangent, f' is positive when f is increasing, and f' is negative when f is decreasing.

35. $f(x) = 3x^{15} - 5x^3 + 3 \quad \Rightarrow$
$f'(x) = 45x^{14} - 15x^2.$

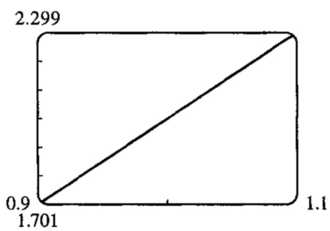


Notice that $f'(x) = 0$ when f has a horizontal tangent, f' is positive when f is increasing, and f' is negative when f is decreasing.

37. To graphically estimate the value of $f'(1)$ for $f(x) = 3x^2 - x^3$, we'll graph f in the viewing rectangle $[1 - 0.1, 1 + 0.1]$ by $[f(0.9), f(1.1)]$, as shown in the figure. [When assigning values to the window variables, it is convenient to use $\mathrm{Y}_1(0.9)$ for $\mathrm{Y}_{\min}$ and $\mathrm{Y}_1(1.1)$ for $\mathrm{Y}_{\max}$.] If we have sufficiently zoomed in on the graph of f, we should obtain a graph that looks like a diagonal line; if not, graph again with $1 - 0.01$ and $1 + 0.01$, etc.

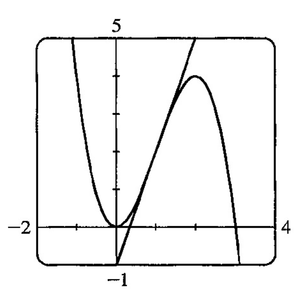
Estimated value:
$f'(1) \approx \dfrac{2.299 - 1.701}{1.1 - 0.9} = \dfrac{0.589}{0.2} = 2.99.$

Exact value: $f(x) = 3x^2 - x^3 \quad \Rightarrow \quad f'(x) = 6x - 3x^2,$
so $f'(1) = 6 - 3 = 3.$

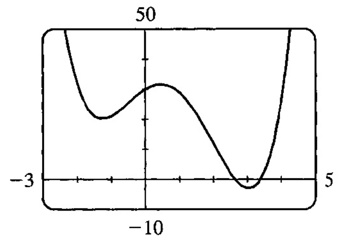


39. $y = x^4 + 2e^x \quad \Rightarrow \quad y' = 4x^3 + 2e^x$. At $(0, 2)$, $y' = 2$ and an equation of the tangent line is $y - 2 = 2(x - 0)$ or $y = 2x + 2$.

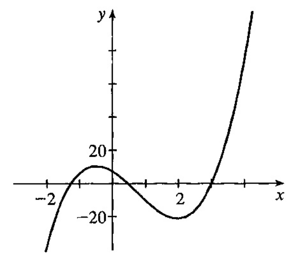
41. $y = 3x^2 - x^3 \quad \Rightarrow \quad y' = 6x - 3x^2$. At $(1, 2)$, $y' = 6 - 3 = 3$, so an equation of the tangent line is $y - 2 = 3(x - 1)$, or $y = 3x - 1$.



43. (a)

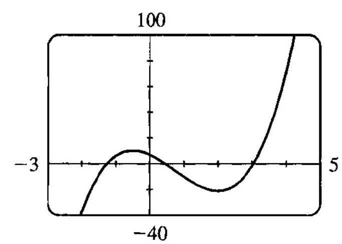


(b)



From the graph in part (a), it appears that f' is zero at $x_1 \approx -1.25$, $x_2 \approx 0.5$, and $x_3 \approx 3$. The slopes are negative (so f' is negative) on $(-\infty, x_1)$ and (x_2, x_3). The slopes are positive (so f' is positive) on (x_1, x_2) and (x_3, ∞).

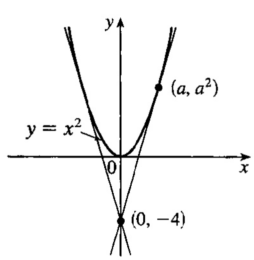
(c) $f(x) = x^4 - 3x^3 - 6x^2 + 7x + 30 \;\Rightarrow$
$f'(x) = 4x^3 - 9x^2 - 12x + 7$



45. The curve $y = 2x^3 + 3x^2 - 12x + 1$ has a horizontal tangent when $y' = 6x^2 + 6x - 12 = 0 \;\Leftrightarrow$ $6(x^2 + x - 2) = 0 \;\Leftrightarrow\; 6(x+2)(x-1) = 0 \;\Leftrightarrow\; x = -2$ or $x = 1$. The points on the curve are $(-2, 21)$ and $(1, -6)$.

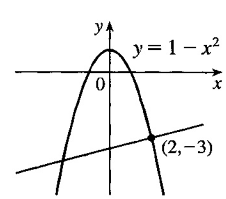
47. $y = 6x^3 + 5x - 3 \;\Rightarrow\; m = y' = 18x^2 + 5$, but $x^2 \geq 0$ for all x, so $m \geq 5$ for all x.

49.



Let (a, a^2) be a point on the parabola at which the tangent line passes through the point $(0, -4)$. The tangent line has slope $2a$ and equation $y - (-4) = 2a(x - 0) \;\Leftrightarrow\; y = 2ax - 4$. Since (a, a^2) also lies on the line, $a^2 = 2a(a) - 4$, or $a^2 = 4$. So $a = \pm 2$ and the points are $(2, 4)$ and $(-2, 4)$.

51. $y = f(x) = 1 - x^2 \;\Rightarrow\; f'(x) = -2x$, so the tangent line at $(2, -3)$ has slope $f'(2) = -4$. The normal line has slope $-\frac{1}{-4} = \frac{1}{4}$ and equation $y + 3 = \frac{1}{4}(x - 2)$ or $y = \frac{1}{4}x - \frac{7}{2}$.



53. $f'(x) = \lim_{h\to 0} \dfrac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \dfrac{\dfrac{1}{x+h} - \dfrac{1}{x}}{h} = \lim_{h\to 0} \dfrac{x-(x+h)}{hx(x+h)}$

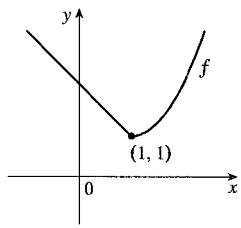
$= \lim_{h\to 0} \dfrac{-h}{hx(x+h)} = \lim_{h\to 0} \dfrac{-1}{x(x+h)} = -\dfrac{1}{x^2}$

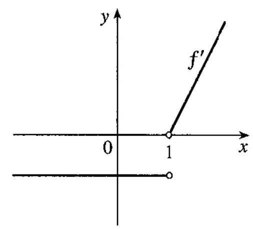
55. $f(x) = 2 - x$ if $x \le 1$ and $f(x) = x^2 - 2x + 2$ if $x > 1$. Now we compute the right- and left-hand derivatives defined in Exercise 2.9.46:

$f'_-(1) = \lim_{h\to 0^-} \dfrac{f(1+h)-f(1)}{h} = \lim_{h\to 0^-} \dfrac{2-(1+h)-1}{h} = \lim_{h\to 0^-} \dfrac{-h}{h} = \lim_{h\to 0^-} -1 = -1$ and

$f'_+(1) = \lim_{h\to 0+} \dfrac{f(1+h)-f(1)}{h} = \lim_{h\to 0+} \dfrac{(1+h)^2 - 2(1+h) + 2 - 1}{h} = \lim_{h\to 0+} \dfrac{h^2}{h} = \lim_{h\to 0+} h = 0.$

Thus, $f'(1)$ does not exist since $f'_-(1) \ne f'_+(1)$, so f is not differentiable at 1. But $f'(x) = -1$ for $x < 1$ and $f'(x) = 2x - 2$ if $x > 1$.





57. (a) Note that $x^2 - 9 < 0$ for $x^2 < 9 \iff |x| < 3 \iff -3 < x < 3$. So

$$f(x) = \begin{cases} x^2 - 9 & \text{if } x \le -3 \\ -x^2 + 9 & \text{if } -3 < x < 3 \\ x^2 - 9 & \text{if } x \ge 3 \end{cases} \Rightarrow$$

$$f'(x) = \begin{cases} 2x & \text{if } x < -3 \\ -2x & \text{if } -3 < x < 3 \\ 2x & \text{if } x > 3 \end{cases} = \begin{cases} 2x & \text{if } |x| > 3 \\ -2x & \text{if } |x| < 3 \end{cases}$$

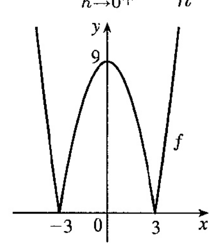
To show that $f'(3)$ does not exist we investigate $\lim_{h\to 0} \dfrac{f(3+h)-f(3)}{h}$ by computing the left- and right-hand derivatives defined in Exercise 2.9.46.

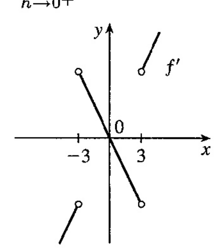
$f'_-(3) = \lim_{h\to 0^-} \dfrac{f(3+h)-f(3)}{h} = \lim_{h\to 0^-} \dfrac{[-(3+h)^2 + 9] - 0}{h} = \lim_{h\to 0^-} (-6-h) = -6$ and

$f'_+(3) = \lim_{h\to 0+} \dfrac{f(3+h)-f(3)}{h} = \lim_{h\to 0+} \dfrac{[(3+h)^2 - 9] - 0}{h} = \lim_{h\to 0+} \dfrac{6h + h^2}{h} = \lim_{h\to 0+} (6+h) = 6.$

Since the left and right limits are different, $\lim_{h\to 0} \dfrac{f(3+h)-f(3)}{h}$ does not exist, that is, $f'(3)$ does not exist. Similarly, $f'(-3)$ does not exist. Therefore, f is not differentiable at 3 or at -3.

(b)





The notations $\overset{\text{PR}}{\Rightarrow}$ and $\overset{\text{QR}}{\Rightarrow}$ indicate the use of the Product and Quotient Rules, respectively.

7. $g(x) = \dfrac{3x-1}{2x+1} \overset{\text{QR}}{\Rightarrow} g'(x) = \dfrac{(2x+1)(3)-(3x-1)(2)}{(2x+1)^2} = \dfrac{6x+3-6x+2}{(2x+1)^2} = \dfrac{5}{(2x+1)^2}$

9. $V(x) = (2x^3+3)(x^4-2x) \overset{\text{PR}}{\Rightarrow}$
$V'(x) = (2x^3+3)(4x^3-2) + (x^4-2x)(6x^2) = (8x^6+8x^3-6) + (6x^6-12x^3) = 14x^6-4x^3-6$

11. $F(y) = \left(\dfrac{1}{y^2} - \dfrac{3}{y^4}\right)(y+5y^3) = (y^{-2}-3y^{-4})(y+5y^3) \overset{\text{PR}}{\Rightarrow}$

$$\begin{aligned} F'(y) &= (y^{-2}-3y^{-4})(1+15y^2) + (y+5y^3)(-2y^{-3}+12y^{-5}) \\ &= (y^{-2}+15-3y^{-4}-45y^{-2}) + (-2y^{-2}+12y^{-4}-10+60y^{-2}) \\ &= 5+14y^{-2}+9y^{-4} \text{ or } 5+14/y^2+9/y^4 \end{aligned}$$

13. $y = \dfrac{t^2}{3t^2-2t+1} \overset{\text{QR}}{\Rightarrow}$

$$\begin{aligned} y' &= \frac{(3t^2-2t+1)(2t)-t^2(6t-2)}{(3t^2-2t+1)^2} = \frac{2t[3t^2-2t+1-t(3t-1)]}{(3t^2-2t+1)^2} \\ &= \frac{2t(3t^2-2t+1-3t^2+t)}{(3t^2-2t+1)^2} = \frac{2t(1-t)}{(3t^2-2t+1)^2} \end{aligned}$$

15. $y = (r^2-2r)e^r \overset{\text{PR}}{\Rightarrow} y' = (r^2-2r)(e^r) + e^r(2r-2) = e^r(r^2-2r+2r-2) = e^r(r^2-2)$

17. $y = \dfrac{v^3-2v\sqrt{v}}{v} = v^2-2\sqrt{v} = v^2-2v^{1/2} \Rightarrow y' = 2v-2\left(\tfrac{1}{2}\right)v^{-1/2} = 2v-v^{-1/2}$.

We can change the form of the answer as follows: $2v-v^{-1/2} = 2v-\dfrac{1}{\sqrt{v}} = \dfrac{2v\sqrt{v}-1}{\sqrt{v}} = \dfrac{2v^{3/2}-1}{\sqrt{v}}$

19. $y = \dfrac{1}{x^4+x^2+1} \Rightarrow y' = \dfrac{(x^4+x^2+1)(0)-1(4x^3+2x)}{(x^4+x^2+1)^2} = -\dfrac{2x(2x^2+1)}{(x^4+x^2+1)^2}$

21. $f(x) = \dfrac{x}{x+c/x} \Rightarrow$

$$f'(x) = \frac{(x+c/x)(1)-x(1-c/x^2)}{\left(x+\frac{c}{x}\right)^2} = \frac{x+c/x-x+c/x}{\left(\frac{x^2+c}{x}\right)^2} = \frac{2c/x}{\frac{(x^2+c)^2}{x^2}} \cdot \frac{x^2}{x^2} = \frac{2cx}{(x^2+c)^2}$$

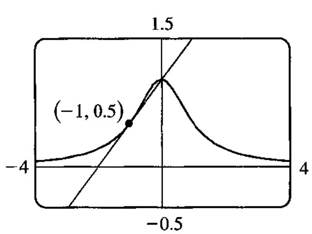
23. $y = \dfrac{2x}{x+1} \Rightarrow y' = \dfrac{(x+1)(2)-(2x)(1)}{(x+1)^2} = \dfrac{2}{(x+1)^2}$. At $(1,1)$, $y' = \frac{1}{2}$, and an equation of the tangent line is $y-1 = \frac{1}{2}(x-1)$, or $y = \frac{1}{2}x+\frac{1}{2}$.

25. $y = 2xe^x \Rightarrow y' = 2(x \cdot e^x + e^x \cdot 1) = 2e^x(x+1)$. At $(0,0)$, $y' = 2e^0(0+1) = 2 \cdot 1 \cdot 1 = 2$, and an equation of the tangent line is $y-0 = 2(x-0)$, or $y = 2x$.

27. (a) $y = f(x) = \dfrac{1}{1+x^2} \Rightarrow$

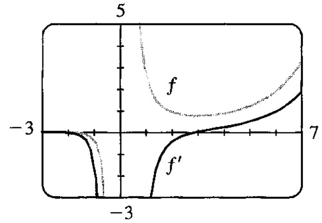
$f'(x) = \dfrac{(1+x^2)(0)-1(2x)}{(1+x^2)^2} = \dfrac{-2x}{(1+x^2)^2}$. So the slope of the tangent line at the point $\left(-1,\frac{1}{2}\right)$ is $f'(-1) = \dfrac{2}{2^2} = \frac{1}{2}$ and its equation is $y-\frac{1}{2} = \frac{1}{2}(x+1)$ or $y = \frac{1}{2}x+1$.

(b)



29. (a) $f(x) = \dfrac{e^x}{x^3} \quad \Rightarrow \quad f'(x) = \dfrac{x^3(e^x) - e^x(3x^2)}{(x^3)^2} = \dfrac{x^2 e^x(x-3)}{x^6} = \dfrac{e^x(x-3)}{x^4}$

(b)



$f' = 0$ when f has a horizontal tangent line, f' is negative when f is decreasing, and f' is positive when f is increasing.

31. We are given that $f(5) = 1$, $f'(5) = 6$, $g(5) = -3$, and $g'(5) = 2$.

(a) $(fg)'(5) = f(5)g'(5) + g(5)f'(5) = (1)(2) + (-3)(6) = 2 - 18 = -16$

(b) $\left(\dfrac{f}{g}\right)'(5) = \dfrac{g(5)f'(5) - f(5)g'(5)}{[g(5)]^2} = \dfrac{(-3)(6) - (1)(2)}{(-3)^2} = -\dfrac{20}{9}$

(c) $\left(\dfrac{g}{f}\right)'(5) = \dfrac{f(5)g'(5) - g(5)f'(5)}{[f(5)]^2} = \dfrac{(1)(2) - (-3)(6)}{(1)^2} = 20$

33. $f(x) = e^x g(x) \quad \Rightarrow \quad f'(x) = e^x g'(x) + g(x)e^x = e^x[g'(x) + g(x)]$.
$f'(0) = e^0[g'(0) + g(0)] = 1(5+2) = 7$

35. (a) From the graphs of f and g, we obtain the following values: $f(1) = 2$ since the point $(1,2)$ is on the graph of f; $g(1) = 1$ since the point $(1,1)$ is on the graph of g; $f'(1) = 2$ since the slope of the line segment between $(0,0)$ and $(2,4)$ is $\dfrac{4-0}{2-0} = 2$; $g'(1) = -1$ since the slope of the line segment between $(-2,4)$ and $(2,0)$ is $\dfrac{0-4}{2-(-2)} = -1$. Now $u(x) = f(x)g(x)$, so $u'(1) = f(1)g'(1) + g(1)\,f'(1) = 2\cdot(-1) + 1\cdot 2 = 0$.

(b) $v(x) = f(x)/g(x)$, so $v'(5) = \dfrac{g(5)f'(5) - f(5)g'(5)}{[g(5)]^2} = \dfrac{2\left(-\frac{1}{3}\right) - 3\cdot\frac{2}{3}}{2^2} = \dfrac{-\frac{8}{3}}{4} = -\dfrac{2}{3}$

37. (a) $y = xg(x) \quad \Rightarrow \quad y' = xg'(x) + g(x)\cdot 1 = xg'(x) + g(x)$

(b) $y = \dfrac{x}{g(x)} \quad \Rightarrow \quad y' = \dfrac{g(x)\cdot 1 - xg'(x)}{[g(x)]^2} = \dfrac{g(x) - xg'(x)}{[g(x)]^2}$

(c) $y = \dfrac{g(x)}{x} \quad \Rightarrow \quad y' = \dfrac{xg'(x) - g(x)\cdot 1}{(x)^2} = \dfrac{xg'(x) - g(x)}{x^2}$

39. If $P(t)$ denotes the population at time t and $A(t)$ the average annual income, then $T(t) = P(t)A(t)$ is the total personal income. The rate at which $T(t)$ is rising is given by $T'(t) = P(t)A'(t) + A(t)P'(t) \quad \Rightarrow$
$T'(1999) = P(1999)A'(1999) + A(1999)P'(1999) = (961{,}400)(\$1400/\text{yr}) + (\$30{,}593)(9200/\text{yr})$
$= \$1{,}345{,}960{,}000/\text{yr} + \$281{,}455{,}600/\text{yr} = \$1{,}627{,}415{,}600/\text{yr}$

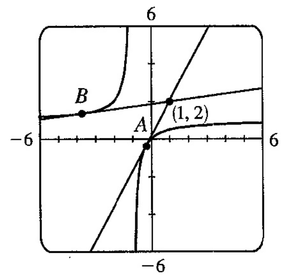
So the total personal income was rising by about \$1.627 billion per year in 1999.

The term $P(t)A'(t) \approx \$1.346$ billion represents the portion of the rate of change of total income due to the existing population's increasing income. The term $A(t)P'(t) \approx \$281$ million represents the portion of the rate of change of total income due to increasing population.

41. If $y = f(x) = \dfrac{x}{x+1}$, then $f'(x) = \dfrac{(x+1)(1) - x(1)}{(x+1)^2} = \dfrac{1}{(x+1)^2}$. When $x = a$, the equation of the tangent line is $y - \dfrac{a}{a+1} = \dfrac{1}{(a+1)^2}(x-a)$. This line passes through $(1,2)$ when $2 - \dfrac{a}{a+1} = \dfrac{1}{(a+1)^2}(1-a) \Leftrightarrow$ $2(a+1)^2 - a(a+1) = 1 - a \Leftrightarrow 2a^2 + 4a + 2 - a^2 - a - 1 + a = 0 \Leftrightarrow a^2 + 4a + 1 = 0$.

The quadratic formula gives the roots of this equation as $a = \dfrac{-4 \pm \sqrt{4^2 - 4(1)(1)}}{2(1)} = \dfrac{-4 \pm \sqrt{12}}{2} = -2 \pm \sqrt{3}$,

so there are two such tangent lines. Since



$$f\left(-2 \pm \sqrt{3}\right) = \frac{-2 \pm \sqrt{3}}{-2 \pm \sqrt{3} + 1} = \frac{-2 \pm \sqrt{3}}{-1 \pm \sqrt{3}} \cdot \frac{-1 \mp \sqrt{3}}{-1 \mp \sqrt{3}}$$

$$= \frac{2 \pm 2\sqrt{3} \mp \sqrt{3} - 3}{1 - 3} = \frac{-1 \pm \sqrt{3}}{-2} = \frac{1 \mp \sqrt{3}}{2},$$

the lines touch the curve at $A\left(-2 + \sqrt{3}, \frac{1-\sqrt{3}}{2}\right) \approx (-0.27, -0.37)$ and

$B\left(-2 - \sqrt{3}, \frac{1+\sqrt{3}}{2}\right) \approx (-3.73, 1.37)$.

We will sometimes use the form $f'g + fg'$ rather than the form $fg' + gf'$ for the Product Rule.

43. (a) $(fgh)' = [(fg)h]' = (fg)'h + (fg)h' = (f'g + fg')h + (fg)h' = f'gh + fg'h + fgh'$

(b) Putting $f = g = h$ in part (a), we have

$$\frac{d}{dx}[f(x)]^3 = (fff)' = f'ff + ff'f + fff' = 3fff' = 3[f(x)]^2 f'(x).$$

(c) $\dfrac{d}{dx}\left(e^{3x}\right) = \dfrac{d}{dx}\left(e^x\right)^3 = 3\left(e^x\right)^2 e^x = 3e^{2x}e^x = 3e^{3x}$

3.3 Rates of Change in the Natural and Social Sciences

1. (a) $s = f(t) = t^2 - 10t + 12 \Rightarrow v(t) = f'(t) = 2t - 10$

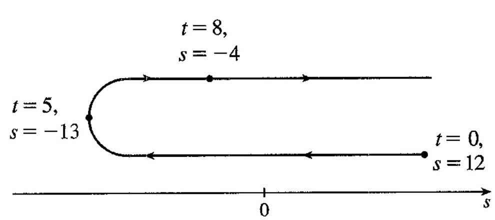
(b) $v(3) = 2(3) - 10 = -4$ ft/s

(c) The particle is at rest when $v(t) = 0 \Leftrightarrow 2t - 10 = 0 \Leftrightarrow t = 5$ s.

(d) The particle is moving in the positive direction when $v(t) > 0 \Leftrightarrow 2t - 10 > 0 \Leftrightarrow 2t > 10 \Leftrightarrow t > 5$.

(e) Since the particle is moving in the positive direction and in the negative direction, we need to calculate the distance traveled in the intervals $[0, 5]$ and $[5, 8]$ separately. $|f(5) - f(0)| = |-13 - 12| = 25$ ft and $|f(8) - f(5)| = |-4 - (-13)| = 9$ ft. The total distance traveled during the first 8 s is $25 + 9 = 34$ ft.

(f)



3. (a) $s = f(t) = t^3 - 12t^2 + 36t \Rightarrow v(t) = f'(t) = 3t^2 - 24t + 36$

(b) $v(3) = 27 - 72 + 36 = -9$ ft/s

(c) The particle is at rest when $v(t) = 0$. $3t^2 - 24t + 36 = 0 \Rightarrow 3(t-2)(t-6) = 0 \Rightarrow t = 2$ s or 6 s.

(d) The particle is moving in the positive direction when $v(t) > 0$. $3(t-2)(t-6) > 0 \Leftrightarrow 0 \le t < 2$ or $t > 6$.

(e) Since the particle is moving in the positive direction and in the negative direction, we need to calculate the distance traveled in the intervals $[0,2]$, $[2,6]$, and $[6,8]$ separately.

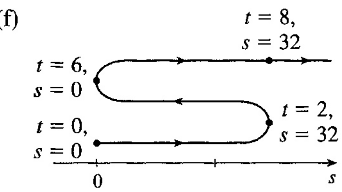
$|f(2)-f(0)| = |32-0| = 32.$

$|f(6)-f(2)| = |0-32| = 32.$

$|f(8)-f(6)| = |32-0| = 32.$

The total distance is $32+32+32 = 96$ ft.

(f)



5. (a) $s = \dfrac{t}{t^2+1} \;\Rightarrow\; v(t) = s'(t) = \dfrac{(t^2+1)(1)-t(2t)}{(t^2+1)^2} = \dfrac{1-t^2}{(t^2+1)^2}$

(b) $v(3) = \dfrac{1-(3)^2}{(3^2+1)^2} = \dfrac{1-9}{10^2} = \dfrac{-8}{100} = -\dfrac{2}{25}$ ft/s

(c) It is at rest when $v=0 \;\Leftrightarrow\; 1-t^2=0 \;\Leftrightarrow\; t=1$ s $\quad [t \neq -1$ since $t \geq 0]$.

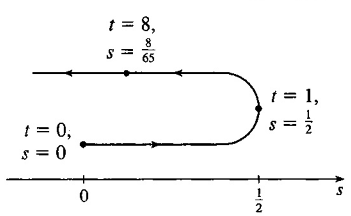
(d) It moves in the positive direction when $v>0 \;\Leftrightarrow\; 1-t^2>0 \;\Leftrightarrow\; t^2<1 \;\Leftrightarrow\; 0 \leq t < 1$.

(e) Distance in positive direction $= |s(1)-s(0)| = \left|\frac{1}{2}-0\right| = \frac{1}{2}$ ft

Distance in negative direction $= |s(8)-s(1)| = \left|\frac{8}{65}-\frac{1}{2}\right| = \frac{49}{130}$ ft

Total distance traveled $= \frac{1}{2}+\frac{49}{130} = \frac{57}{65}$ ft

(f)



7. $s(t) = t^3-4.5t^2-7t \;\Rightarrow\; v(t) = s'(t) = 3t^2-9t-7 = 5 \;\Leftrightarrow\; 3t^2-9t-12=0 \;\Leftrightarrow$ $3(t-4)(t+1)=0 \;\Leftrightarrow\; t=4$ or -1. Since $t \geq 0$, the particle reaches a velocity of 5 m/s at $t=4$ s.

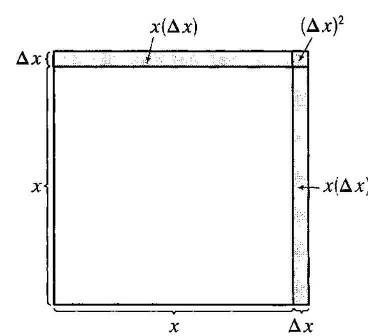
9. (a) $h = 10t-0.83t^2 \;\Rightarrow\; v(t) = \dfrac{dh}{dt} = 10-1.66t$, so $v(3) = 10-1.66(3) = 5.02$ m/s.

(b) $h=25 \;\Rightarrow\; 10t-0.83t^2=25 \;\Rightarrow\; 0.83t^2-10t+25=0 \;\Rightarrow\; t = \dfrac{10 \pm \sqrt{17}}{1.66} \approx 3.54$ or 8.51.

The value $t_1 = \left(10-\sqrt{17}\right)/1.66$ corresponds to the time it takes for the stone to rise 25 m and $t_2 = \left(10+\sqrt{17}\right)/1.66$ corresponds to the time when the stone is 25 m high on the way down. Thus, $v(t_1) = 10-1.66\left[\left(10-\sqrt{17}\right)/1.66\right] = \sqrt{17} \approx 4.12$ m/s.

11. (a) $A(x) = x^2 \;\Rightarrow\; A'(x) = 2x$. $A'(15) = 30\ \text{mm}^2/\text{mm}$ is the rate at which the area is increasing with respect to the side length as x reaches 15 mm.

(b) The perimeter is $P(x) = 4x$, so $A'(x) = 2x = \frac{1}{2}(4x) = \frac{1}{2}P(x)$. The figure suggests that if Δx is small, then the change in the area of the square is approximately half of its perimeter (2 of the 4 sides) times Δx. From the figure, $\Delta A = 2x(\Delta x)+(\Delta x)^2$. If Δx is small, then $\Delta A \approx 2x(\Delta x)$ and so $\Delta A/\Delta x \approx 2x$.



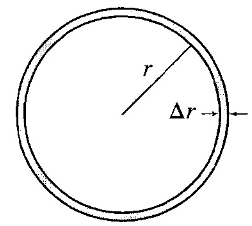
13. (a) Using $A(r) = \pi r^2$, we find that the average rate of change is:

(i) $\dfrac{A(3) - A(2)}{3 - 2} = \dfrac{9\pi - 4\pi}{1} = 5\pi$ (ii) $\dfrac{A(2.5) - A(2)}{2.5 - 2} = \dfrac{6.25\pi - 4\pi}{0.5} = 4.5\pi$

(iii) $\dfrac{A(2.1) - A(2)}{2.1 - 2} = \dfrac{4.41\pi - 4\pi}{0.1} = 4.1\pi$

(b) $A(r) = \pi r^2 \quad \Rightarrow \quad A'(r) = 2\pi r$, so $A'(2) = 4\pi$.

(c) The circumference is $C(r) = 2\pi r = A'(r)$. The figure suggests that if Δr is small, then the change in the area of the circle (a ring around the outside) is approximately equal to its circumference times Δr. Straightening out this ring gives us a shape that is approximately rectangular with length $2\pi r$ and width Δr, so $\Delta A \approx 2\pi r(\Delta r)$. Algebraically,



$\Delta A = A(r + \Delta r) - A(r) = \pi(r + \Delta r)^2 - \pi r^2 = 2\pi r(\Delta r) + \pi(\Delta r)^2$.

So we see that if Δr is small, then $\Delta A \approx 2\pi r(\Delta r)$ and therefore, $\Delta A/\Delta r \approx 2\pi r$.

15. $S(r) = 4\pi r^2 \quad \Rightarrow \quad S'(r) = 8\pi r \quad \Rightarrow$

(a) $S'(1) = 8\pi \text{ ft}^2/\text{ft}$ (b) $S'(2) = 16\pi \text{ ft}^2/\text{ft}$ (c) $S'(3) = 24\pi \text{ ft}^2/\text{ft}$

As the radius increases, the surface area grows at an increasing rate. In fact, the rate of change is linear with respect to the radius.

17. The mass is $f(x) = 3x^2$, so the linear density at x is $\rho(x) = f'(x) = 6x$.

(a) $\rho(1) = 6 \text{ kg/m}$ (b) $\rho(2) = 12 \text{ kg/m}$ (c) $\rho(3) = 18 \text{ kg/m}$

Since ρ is an increasing function, the density will be the highest at the right end of the rod and lowest at the left end.

19. The quantity of charge is $Q(t) = t^3 - 2t^2 + 6t - 2$, so the current is $Q'(t) = 3t^2 - 4t + 6$.

(a) $Q'(0.5) = 3(0.5)^2 - 4(0.5) + 6 = 4.75 \text{ A}$ (b) $Q'(1) = 3(1)^2 - 4(1) + 6 = 5 \text{ A}$

The current is lowest when Q' has a minimum. $Q''(t) = 6t - 4 < 0$ when $t < \frac{2}{3}$. So the current decreases when $t < \frac{2}{3}$ and increases when $t > \frac{2}{3}$. Thus, the current is lowest at $t = \frac{2}{3}$ s.

21. (a) To find the rate of change of volume with respect to pressure, we first solve for V in terms of P.

$$PV = C \quad \Rightarrow \quad V = \frac{C}{P} \quad \Rightarrow \quad \frac{dV}{dP} = -\frac{C}{P^2}.$$

(b) From the formula for dV/dP in part (a), we see that as P increases, the absolute value of dV/dP decreases. Thus, the volume is decreasing more rapidly at the beginning.

(c) $\beta = -\dfrac{1}{V}\dfrac{dV}{dP} = -\dfrac{1}{V}\left(-\dfrac{C}{P^2}\right) = \dfrac{C}{(PV)P} = \dfrac{C}{CP} = \dfrac{1}{P}$

23. (a) **1920:** $m_1 = \dfrac{1860 - 1750}{1920 - 1910} = \frac{110}{10} = 11$, $m_2 = \dfrac{2070 - 1860}{1930 - 1920} = \frac{210}{10} = 21$,

$(m_1 + m_2)/2 = (11 + 21)/2 = 16 \text{ million/year}$

1980: $m_1 = \dfrac{4450 - 3710}{1980 - 1970} = \frac{740}{10} = 74$, $m_2 = \dfrac{5280 - 4450}{1990 - 1980} = \frac{830}{10} = 83$,

$(m_1 + m_2)/2 = (74 + 83)/2 = 78.5 \text{ million/year}$

(b) $P(t) = at^3 + bt^2 + ct + d$ (in millions of people), where $a \approx 0.0012937063$, $b \approx -7.061421911$, $c \approx 12{,}822.97902$, and $d \approx -7{,}743{,}770.396$.

(c) $P(t) = at^3 + bt^2 + ct + d \Rightarrow P'(t) = 3at^2 + 2bt + c$ (in millions of people per year)

(d) $P'(1920) = 3(0.0012937063)(1920)^2 + 2(-7.061421911)(1920) + 12{,}822.97902$

≈ 14.48 million/year [smaller than the answer in part (a), but close to it]

$P'(1980) \approx 75.29$ million/year (smaller, but close)

(e) $P'(1985) \approx 81.62$ million/year, so the rate of growth in 1985 was about 81.62 million/year.

25. (a) $[\mathrm{C}] = \dfrac{a^2kt}{akt+1} \Rightarrow$

rate of reaction $= \dfrac{d\,[\mathrm{C}]}{dt} = \dfrac{(akt+1)(a^2k) - (a^2kt)(ak)}{(akt+1)^2} = \dfrac{a^2k(akt+1-akt)}{(akt+1)^2} = \dfrac{a^2k}{(akt+1)^2}$

(b) If $x = [\mathrm{C}]$, then $a - x = a - \dfrac{a^2kt}{akt+1} = \dfrac{a^2kt + a - a^2kt}{akt+1} = \dfrac{a}{akt+1}$.

So $k(a-x)^2 = k\left(\dfrac{a}{akt+1}\right)^2 = \dfrac{a^2k}{(akt+1)^2} = \dfrac{d\,[\mathrm{C}]}{dt}$ [from part (a)] $= \dfrac{dx}{dt}$.

(c) As $t \to \infty$, $[\mathrm{C}] = \dfrac{a^2kt}{akt+1} = \dfrac{(a^2kt)/t}{(akt+1)/t} = \dfrac{a^2k}{ak+(1/t)} \to \dfrac{a^2k}{ak} = a$ moles/L.

(d) As $t \to \infty$, $\dfrac{d[\mathrm{C}]}{dt} = \dfrac{a^2k}{(akt+1)^2} \to 0$.

(e) As t increases, nearly all of the reactants A and B are converted into product C. In practical terms, the reaction virtually stops.

27. (a) Using $v = \dfrac{P}{4\eta l}\left(R^2 - r^2\right)$ with $R = 0.01$, $l = 3$, $P = 3000$, and $\eta = 0.027$, we have v as a function of r:

$v(r) = \dfrac{3000}{4(0.027)3}\left(0.01^2 - r^2\right)$. $v(0) = 0.\overline{925}$ cm/s, $v(0.005) = 0.69\overline{4}$ cm/s, $v(0.01) = 0$.

(b) $v(r) = \dfrac{P}{4\eta l}\left(R^2 - r^2\right) \Rightarrow v'(r) = \dfrac{P}{4\eta l}(-2r) = -\dfrac{Pr}{2\eta l}$. When $l = 3$, $P = 3000$, and $\eta = 0.027$, we have

$v'(r) = -\dfrac{3000r}{2(0.027)3}$. $v'(0) = 0$, $v'(0.005) = -92.\overline{592}$ (cm/s)/cm, and $v'(0.01) = -185.\overline{185}$ (cm/s)/cm.

(c) The velocity is greatest where $r = 0$ (at the center) and the velocity is changing most where $r = R = 0.01$ cm (at the edge).

29. (a) $C(x) = 2000 + 3x + 0.01x^2 + 0.0002x^3 \Rightarrow C'(x) = 3 + 0.02x + 0.0006x^2$

(b) $C'(100) = 3 + 0.02(100) + 0.0006(10{,}000) = 3 + 2 + 6 = \$11$/pair. $C'(100)$ is the rate at which the cost is increasing as the 100th pair of jeans is produced. It predicts the cost of the 101st pair.

(c) The cost of manufacturing the 101st pair of jeans is

$$C(101) - C(100) = (2000 + 303 + 102.01 + 206.0602) - (2000 + 300 + 100 + 200)$$
$$= 11.0702 \approx \$11.07$$

31. (a) $A(x) = \dfrac{p(x)}{x} \Rightarrow A'(x) = \dfrac{xp'(x) - p(x)\cdot 1}{x^2} = \dfrac{xp'(x) - p(x)}{x^2}$. $A'(x) > 0 \Rightarrow A(x)$ is increasing; that is, the average productivity increases as the size of the workforce increases.

(b) $p'(x)$ is greater than the average productivity $\Rightarrow p'(x) > A(x) \Rightarrow p'(x) > \dfrac{p(x)}{x} \Rightarrow$

$xp'(x) > p(x) \Rightarrow xp'(x) - p(x) > 0 \Rightarrow \dfrac{xp'(x) - p(x)}{x^2} > 0 \Rightarrow A'(x) > 0$.

33. $PV = nRT \Rightarrow T = \dfrac{PV}{nR} = \dfrac{PV}{(10)(0.0821)} = \dfrac{1}{0.821}(PV)$. Using the Product Rule, we have

$\dfrac{dT}{dt} = \dfrac{1}{0.821}[P(t)V'(t) + V(t)P'(t)] = \dfrac{1}{0.821}[(8)(-0.15) + (10)(0.10)] \approx -0.2436$ K/min.

35. (a) If the populations are stable, then the growth rates are neither positive nor negative; that is, $\dfrac{dC}{dt} = 0$ and $\dfrac{dW}{dt} = 0$.

(b) "The caribou go extinct" means that the population is zero, or mathematically, $C = 0$.

(c) We have the equations $\dfrac{dC}{dt} = aC - bCW$ and $\dfrac{dW}{dt} = -cW + dCW$. Let $dC/dt = dW/dt = 0$, $a = 0.05$, $b = 0.001$, $c = 0.05$, and $d = 0.0001$ to obtain $0.05C - 0.001CW = 0$ **(1)** and $-0.05W + 0.0001CW = 0$ **(2)**. Adding 10 times **(2)** to **(1)** eliminates the CW-terms and gives us $0.05C - 0.5W = 0 \Rightarrow C = 10W$. Substituting $C = 10W$ into **(1)** results in $0.05(10W) - 0.001(10W)W = 0 \Leftrightarrow 0.5W - 0.01W^2 = 0 \Leftrightarrow 50W - W^2 = 0 \Leftrightarrow W(50 - W) = 0 \Leftrightarrow W = 0$ or 50. Since $C = 10W$, $C = 0$ or 500. Thus, the population pairs (C, W) that lead to stable populations are $(0, 0)$ and $(500, 50)$. So it is possible for the two species to live in harmony.

3.4 Derivatives of Trigonometric Functions

1. $f(x) = x - 3\sin x \Rightarrow f'(x) = 1 - 3\cos x$

3. $y = \sin x + 10\tan x \Rightarrow y' = \cos x + 10\sec^2 x$

5. $g(t) = t^3\cos t \Rightarrow g'(t) = t^3(-\sin t) + (\cos t)\cdot 3t^2 = 3t^2\cos t - t^3\sin t$ or $t^2(3\cos t - t\sin t)$

7. $h(\theta) = \csc\theta + e^\theta\cot\theta \Rightarrow$

$h'(\theta) = -\csc\theta\cot\theta + e^\theta(-\csc^2\theta) + (\cot\theta)e^\theta = -\csc\theta\cot\theta + e^\theta(\cot\theta - \csc^2\theta)$

9. $y = \dfrac{x}{\cos x} \Rightarrow y' = \dfrac{(\cos x)(1) - (x)(-\sin x)}{(\cos x)^2} = \dfrac{\cos x + x\sin x}{\cos^2 x}$

11. $f(\theta) = \dfrac{\sec\theta}{1 + \sec\theta} \Rightarrow$

$f'(\theta) = \dfrac{(1 + \sec\theta)(\sec\theta\tan\theta) - (\sec\theta)(\sec\theta\tan\theta)}{(1 + \sec\theta)^2} = \dfrac{(\sec\theta\tan\theta)[(1 + \sec\theta) - \sec\theta]}{(1 + \sec\theta)^2} = \dfrac{\sec\theta\tan\theta}{(1 + \sec\theta)^2}$

13. $y = \dfrac{\sin x}{x^2} \Rightarrow y' = \dfrac{x^2\cos x - (\sin x)(2x)}{(x^2)^2} = \dfrac{x(x\cos x - 2\sin x)}{x^4} = \dfrac{x\cos x - 2\sin x}{x^3}$

15. $y = \sec\theta\tan\theta \Rightarrow y' = \sec\theta(\sec^2\theta) + \tan\theta(\sec\theta\tan\theta) = \sec\theta(\sec^2\theta + \tan^2\theta)$

Using the identity $1 + \tan^2\theta = \sec^2\theta$, we can write alternative forms of the answer as

$$\sec\theta\,(1 + 2\tan^2\theta) \qquad \text{or} \qquad \sec\theta\,(2\sec^2\theta - 1)$$

17. $\dfrac{d}{dx}(\csc x) = \dfrac{d}{dx}\left(\dfrac{1}{\sin x}\right) = \dfrac{(\sin x)(0) - 1(\cos x)}{\sin^2 x} = \dfrac{-\cos x}{\sin^2 x} = -\dfrac{1}{\sin x}\cdot\dfrac{\cos x}{\sin x} = -\csc x\cot x$

19. $\dfrac{d}{dx}(\cot x) = \dfrac{d}{dx}\left(\dfrac{\cos x}{\sin x}\right) = \dfrac{(\sin x)(-\sin x) - (\cos x)(\cos x)}{\sin^2 x} = -\dfrac{\sin^2 x + \cos^2 x}{\sin^2 x} = -\dfrac{1}{\sin^2 x} = -\csc^2 x$

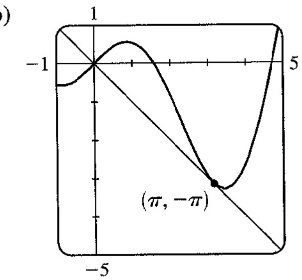
21. $y = \tan x \Rightarrow y' = \sec^2 x \Rightarrow$ the slope of the tangent line at $\left(\frac{\pi}{4}, 1\right)$ is $\sec^2\frac{\pi}{4} = \left(\sqrt{2}\right)^2 = 2$ and an equation of the tangent line is $y - 1 = 2\left(x - \frac{\pi}{4}\right)$ or $y = 2x + 1 - \frac{\pi}{2}$.

23. $y = x + \cos x \quad \Rightarrow \quad y' = 1 - \sin x$. At $(0, 1)$, $y' = 1$, and an equation of the tangent line is $y - 1 = 1(x - 0)$, or $y = x + 1$.

25. (a) $y = x\cos x \quad \Rightarrow \quad y' = x(-\sin x) + \cos x(1) = \cos x - x\sin x$.

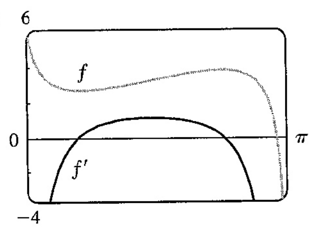
So the slope of the tangent at the point $(\pi, -\pi)$ is $\cos\pi - \pi\sin\pi = -1 - \pi(0) = -1$, and an equation is $y + \pi = -(x - \pi)$ or $y = -x$.

(b)



27. (a) $f(x) = 2x + \cot x \quad \Rightarrow \quad f'(x) = 2 - \csc^2 x$

(b)



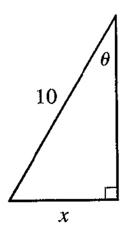
Notice that $f'(x) = 0$ when f has a horizontal tangent. f' is positive when f is increasing and f' is negative when f is decreasing. Also, $f'(x)$ is large negative when the graph of f is steep.

29. $f(x) = x + 2\sin x$ has a horizontal tangent when $f'(x) = 0 \quad \Leftrightarrow \quad 1 + 2\cos x = 0 \quad \Leftrightarrow \quad \cos x = -\frac{1}{2} \quad \Leftrightarrow$ $x = \frac{2\pi}{3} + 2\pi n$ or $\frac{4\pi}{3} + 2\pi n$, where n is an integer. Note that $\frac{4\pi}{3}$ and $\frac{2\pi}{3}$ are $\pm\frac{\pi}{3}$ units from π. This allows us to write the solutions in the more compact equivalent form $(2n + 1)\pi \pm \frac{\pi}{3}$, n an integer.

31. (a) $x(t) = 8\sin t \quad \Rightarrow \quad v(t) = x'(t) = 8\cos t$

(b) The mass at time $t = \frac{2\pi}{3}$ has position $x\left(\frac{2\pi}{3}\right) = 8\sin\frac{2\pi}{3} = 8\left(\frac{\sqrt{3}}{2}\right) = 4\sqrt{3}$ and velocity $v\left(\frac{2\pi}{3}\right) = 8\cos\frac{2\pi}{3} = 8\left(-\frac{1}{2}\right) = -4$. Since $v\left(\frac{2\pi}{3}\right) < 0$, the particle is moving to the left.

33.



From the diagram we can see that $\sin\theta = x/10 \quad \Leftrightarrow \quad x = 10\sin\theta$. We want to find the rate of change of x with respect to θ; that is, $dx/d\theta$. Taking the derivative of the above expression, $dx/d\theta = 10(\cos\theta)$. So when $\theta = \frac{\pi}{3}$,

$$dx/d\theta = 10\cos\frac{\pi}{3} = 10\left(\tfrac{1}{2}\right) = 5 \text{ ft/rad}$$

35.
$$\begin{aligned}
\lim_{x\to 0}\frac{\sin 3x}{x} &= \lim_{x\to 0}\frac{3\sin 3x}{3x} && \text{[multiply numerator and denominator by 3]}\\
&= 3\lim_{3x\to 0}\frac{\sin 3x}{3x} && \text{[as } x \to 0,\ 3x \to 0\text{]}\\
&= 3\lim_{\theta\to 0}\frac{\sin\theta}{\theta} && \text{[let } \theta = 3x\text{]}\\
&= 3(1) && \text{[Equation 2]}\\
&= 3
\end{aligned}$$

37.
$$\begin{aligned}
\lim_{t\to 0}\frac{\tan 6t}{\sin 2t} &= \lim_{t\to 0}\left(\frac{\sin 6t}{t}\cdot\frac{1}{\cos 6t}\cdot\frac{t}{\sin 2t}\right) = \lim_{t\to 0}\frac{6\sin 6t}{6t}\cdot\lim_{t\to 0}\frac{1}{\cos 6t}\cdot\lim_{t\to 0}\frac{2t}{2\sin 2t}\\
&= 6\lim_{t\to 0}\frac{\sin 6t}{6t}\cdot\lim_{t\to 0}\frac{1}{\cos 6t}\cdot\frac{1}{2}\lim_{t\to 0}\frac{2t}{\sin 2t} = 6(1)\cdot\frac{1}{1}\cdot\frac{1}{2}(1) = 3
\end{aligned}$$

39. $\displaystyle\lim_{\theta\to 0}\frac{\sin(\cos\theta)}{\sec\theta}=\frac{\sin\left(\lim_{\theta\to 0}\cos\theta\right)}{\lim_{\theta\to 0}\sec\theta}=\frac{\sin 1}{1}=\sin 1$

41. $\displaystyle\lim_{x\to 0}\frac{\cot 2x}{\csc x}=\lim_{x\to 0}\frac{\cos 2x\sin x}{\sin 2x}=\lim_{x\to 0}\cos 2x\left[\frac{(\sin x)/x}{(\sin 2x)/x}\right]=\lim_{x\to 0}\cos 2x\left[\frac{\lim_{x\to 0}[(\sin x)/x]}{2\lim_{x\to 0}[(\sin 2x)/2x]}\right]$

$\displaystyle=1\cdot\frac{1}{2\cdot 1}=\frac{1}{2}$

43. Divide numerator and denominator by θ. ($\sin\theta$ also works.)

$$\lim_{\theta\to 0}\frac{\sin\theta}{\theta+\tan\theta}=\lim_{\theta\to 0}\frac{\frac{\sin\theta}{\theta}}{1+\frac{\sin\theta}{\theta}\cdot\frac{1}{\cos\theta}}=\frac{\lim_{\theta\to 0}\frac{\sin\theta}{\theta}}{1+\lim_{\theta\to 0}\frac{\sin\theta}{\theta}\lim_{\theta\to 0}\frac{1}{\cos\theta}}=\frac{1}{1+1\cdot 1}=\frac{1}{2}$$

45. (a) $\displaystyle\frac{d}{dx}\tan x=\frac{d}{dx}\frac{\sin x}{\cos x}\quad\Rightarrow\quad\sec^2 x=\frac{\cos x\cos x-\sin x\,(-\sin x)}{\cos^2 x}=\frac{\cos^2 x+\sin^2 x}{\cos^2 x}$. So $\displaystyle\sec^2 x=\frac{1}{\cos^2 x}$.

(b) $\displaystyle\frac{d}{dx}\sec x=\frac{d}{dx}\frac{1}{\cos x}\quad\Rightarrow\quad\sec x\tan x=\frac{(\cos x)(0)-1(-\sin x)}{\cos^2 x}$. So $\displaystyle\sec x\tan x=\frac{\sin x}{\cos^2 x}$.

(c) $\displaystyle\frac{d}{dx}(\sin x+\cos x)=\frac{d}{dx}\frac{1+\cot x}{\csc x}\quad\Rightarrow$

$\displaystyle\cos x-\sin x=\frac{\csc x\,(-\csc^2 x)-(1+\cot x)(-\csc x\cot x)}{\csc^2 x}=\frac{\csc x\left[-\csc^2 x+(1+\cot x)\cot x\right]}{\csc^2 x}$

$\displaystyle=\frac{-\csc^2 x+\cot^2 x+\cot x}{\csc x}=\frac{-1+\cot x}{\csc x}$

So $\displaystyle\cos x-\sin x=\frac{\cot x-1}{\csc x}$.

47. By the definition of radian measure, $s=r\theta$, where r is the radius of the circle.

By drawing the bisector of the angle θ, we can see that $\displaystyle\sin\frac{\theta}{2}=\frac{d/2}{r}\quad\Rightarrow\quad d=2r\sin\frac{\theta}{2}$.

So $\displaystyle\lim_{\theta\to 0^+}\frac{s}{d}=\lim_{\theta\to 0^+}\frac{r\theta}{2r\sin(\theta/2)}=\lim_{\theta\to 0^+}\frac{2\cdot(\theta/2)}{2\sin(\theta/2)}=\lim_{\theta\to 0}\frac{\theta/2}{\sin(\theta/2)}=1$. [This is just the reciprocal of the limit $\lim_{x\to 0}\frac{\sin x}{x}=1$ combined with the fact that as $\theta\to 0$, $\frac{\theta}{2}\to 0$ also.]

3.5 The Chain Rule

1. Let $u=g(x)=4x$ and $y=f(u)=\sin u$. Then $\displaystyle\frac{dy}{dx}=\frac{dy}{du}\frac{du}{dx}=(\cos u)(4)=4\cos 4x$.

3. Let $u=g(x)=1-x^2$ and $y=f(u)=u^{10}$. Then $\displaystyle\frac{dy}{dx}=\frac{dy}{du}\frac{du}{dx}=(10u^9)(-2x)=-20x(1-x^2)^9$.

5. Let $u=g(x)=\sqrt{x}$ and $y=f(u)=e^u$.

Then $\displaystyle\frac{dy}{dx}=\frac{dy}{du}\frac{du}{dx}=(e^u)\left(\tfrac{1}{2}x^{-1/2}\right)=e^{\sqrt{x}}\cdot\frac{1}{2\sqrt{x}}=\frac{e^{\sqrt{x}}}{2\sqrt{x}}$.

7. $F(x) = (x^3 + 4x)^7 \quad\Rightarrow\quad F'(x) = 7(x^3 + 4x)^6(3x^2 + 4) \quad [\text{or } 7x^6(x^2 + 4)^6(3x^2 + 4)]$

9. $F(x) = \sqrt[4]{1 + 2x + x^3} = (1 + 2x + x^3)^{1/4} \quad\Rightarrow$

$$F'(x) = \tfrac{1}{4}(1 + 2x + x^3)^{-3/4} \cdot \frac{d}{dx}(1 + 2x + x^3) = \frac{1}{4(1 + 2x + x^3)^{3/4}} \cdot (2 + 3x^2)$$
$$= \frac{2 + 3x^2}{4(1 + 2x + x^3)^{3/4}} = \frac{2 + 3x^2}{4\sqrt[4]{(1 + 2x + x^3)^3}}$$

11. $g(t) = \dfrac{1}{(t^4 + 1)^3} = (t^4 + 1)^{-3} \quad\Rightarrow\quad g'(t) = -3(t^4 + 1)^{-4}(4t^3) = -12t^3(t^4 + 1)^{-4} = \dfrac{-12t^3}{(t^4 + 1)^4}$

13. $y = \cos(a^3 + x^3) \quad\Rightarrow\quad y' = -\sin(a^3 + x^3) \cdot 3x^2$ [a^3 is just a constant] $= -3x^2 \sin(a^3 + x^3)$

15. $y = e^{-mx} \quad\Rightarrow\quad y' = e^{-mx}\dfrac{d}{dx}(-mx) = e^{-mx}(-m) = -me^{-mx}$

17. $g(x) = (1 + 4x)^5(3 + x - x^2)^8 \quad\Rightarrow$

$$g'(x) = (1 + 4x)^5 \cdot 8(3 + x - x^2)^7(1 - 2x) + (3 + x - x^2)^8 \cdot 5(1 + 4x)^4 \cdot 4$$
$$= 4(1 + 4x)^4(3 + x - x^2)^7\,[2(1 + 4x)(1 - 2x) + 5(3 + x - x^2)]$$
$$= 4(1 + 4x)^4(3 + x - x^2)^7\,[(2 + 4x - 16x^2) + (15 + 5x - 5x^2)]$$
$$= 4(1 + 4x)^4(3 + x - x^2)^7\,(17 + 9x - 21x^2)$$

19. $y = (2x - 5)^4(8x^2 - 5)^{-3} \quad\Rightarrow$

$$y' = 4(2x - 5)^3(2)(8x^2 - 5)^{-3} + (2x - 5)^4(-3)(8x^2 - 5)^{-4}(16x)$$
$$= 8(2x - 5)^3(8x^2 - 5)^{-3} - 48x(2x - 5)^4(8x^2 - 5)^{-4}$$

[This simplifies to $8(2x - 5)^3(8x^2 - 5)^{-4}(-4x^2 + 30x - 5)$.]

21. $y = xe^{-x^2} \quad\Rightarrow\quad y' = xe^{-x^2}(-2x) + e^{-x^2} \cdot 1 = e^{-x^2}(-2x^2 + 1) = e^{-x^2}(1 - 2x^2)$

23. $y = e^{x\cos x} \quad\Rightarrow\quad y' = e^{x\cos x} \cdot \dfrac{d}{dx}(x\cos x) = e^{x\cos x}[x(-\sin x) + (\cos x) \cdot 1] = e^{x\cos x}(\cos x - x\sin x)$

25. $F(z) = \sqrt{\dfrac{z - 1}{z + 1}} = \left(\dfrac{z - 1}{z + 1}\right)^{1/2} \quad\Rightarrow$

$$F'(z) = \frac{1}{2}\left(\frac{z - 1}{z + 1}\right)^{-1/2} \cdot \frac{d}{dz}\left(\frac{z - 1}{z + 1}\right) = \frac{1}{2}\left(\frac{z + 1}{z - 1}\right)^{1/2} \cdot \frac{(z + 1)(1) - (z - 1)(1)}{(z + 1)^2}$$
$$= \frac{1}{2}\frac{(z + 1)^{1/2}}{(z - 1)^{1/2}} \cdot \frac{z + 1 - z + 1}{(z + 1)^2} = \frac{1}{2}\frac{(z + 1)^{1/2}}{(z - 1)^{1/2}} \cdot \frac{2}{(z + 1)^2} = \frac{1}{(z - 1)^{1/2}(z + 1)^{3/2}}$$

27. $y = \dfrac{r}{\sqrt{r^2+1}} \quad \Rightarrow$

$$y' = \frac{\sqrt{r^2+1}\,(1) - r\cdot\frac{1}{2}\left(r^2+1\right)^{-1/2}(2r)}{\left(\sqrt{r^2+1}\right)^2} = \frac{\sqrt{r^2+1} - \dfrac{r^2}{\sqrt{r^2+1}}}{\left(\sqrt{r^2+1}\right)^2} = \frac{\dfrac{\sqrt{r^2+1}\,\sqrt{r^2+1} - r^2}{\sqrt{r^2+1}}}{\left(\sqrt{r^2+1}\right)^2}$$

$$= \frac{\left(r^2+1\right) - r^2}{\left(\sqrt{r^2+1}\right)^3} = \frac{1}{(r^2+1)^{3/2}} \text{ or } \left(r^2+1\right)^{-3/2}$$

Another solution: Write y as a product and make use of the Product Rule. $y = r\left(r^2+1\right)^{-1/2} \quad \Rightarrow$

$$y' = r\cdot -\tfrac{1}{2}\left(r^2+1\right)^{-3/2}(2r) + \left(r^2+1\right)^{-1/2}\cdot 1$$

$$= \left(r^2+1\right)^{-3/2}\left[-r^2 + \left(r^2+1\right)^1\right] = \left(r^2+1\right)^{-3/2}(1) = \left(r^2+1\right)^{-3/2}$$

The step that students usually have trouble with is factoring out $\left(r^2+1\right)^{-3/2}$. But this is no different than factoring out x^2 from x^2+x^5; that is, we are just factoring out a factor with the *smallest* exponent that appears on it. In this case, $-\frac{3}{2}$ is smaller than $-\frac{1}{2}$.

29. $y = \tan(\cos x) \quad \Rightarrow \quad y' = \sec^2(\cos x)\cdot(-\sin x) = -\sin x\sec^2(\cos x)$

31. Using Formula 5 and the Chain Rule, $y = 2^{\sin \pi x} \quad \Rightarrow$

$$y' = 2^{\sin\pi x}(\ln 2)\cdot\frac{d}{dx}(\sin\pi x) = 2^{\sin\pi x}(\ln 2)\cdot\cos\pi x\cdot\pi = 2^{\sin\pi x}(\pi\ln 2)\cos\pi x$$

33. $y = (1+\cos^2 x)^6 \quad \Rightarrow \quad y' = 6(1+\cos^2 x)^5\cdot 2\cos x\,(-\sin x) = -12\cos x\,\sin x\,(1+\cos^2 x)^5$

35. $y = \sec^2 x + \tan^2 x = (\sec x)^2 + (\tan x)^2 \quad \Rightarrow$

$$y' = 2(\sec x)(\sec x\tan x) + 2(\tan x)(\sec^2 x) = 2\sec^2 x\tan x + 2\sec^2 x\tan x = 4\sec^2 x\tan x$$

37. $y = \cot^2(\sin\theta) = [\cot(\sin\theta)]^2 \quad \Rightarrow$

$$y' = 2\,[\cot(\sin\theta)]\cdot\frac{d}{d\theta}[\cot(\sin\theta)] = 2\cot(\sin\theta)\cdot\left[-\csc^2(\sin\theta)\cdot\cos\theta\right] = -2\cos\theta\,\cot(\sin\theta)\,\csc^2(\sin\theta)$$

39. $y = \sqrt{x+\sqrt{x}} \quad \Rightarrow \quad y' = \frac{1}{2}\left(x+\sqrt{x}\right)^{-1/2}\left(1+\frac{1}{2}x^{-1/2}\right) = \dfrac{1}{2\sqrt{x+\sqrt{x}}}\left(1+\dfrac{1}{2\sqrt{x}}\right)$

41. $y = \sin\left(\tan\sqrt{\sin x}\right) \quad \Rightarrow$

$$y' = \cos\left(\tan\sqrt{\sin x}\right)\cdot\frac{d}{dx}\left(\tan\sqrt{\sin x}\right) = \cos\left(\tan\sqrt{\sin x}\right)\sec^2\sqrt{\sin x}\cdot\frac{d}{dx}(\sin x)^{1/2}$$

$$= \cos\left(\tan\sqrt{\sin x}\right)\sec^2\sqrt{\sin x}\cdot\tfrac{1}{2}(\sin x)^{-1/2}\cdot\cos x$$

$$= \cos\left(\tan\sqrt{\sin x}\right)\left(\sec^2\sqrt{\sin x}\right)\left(\frac{1}{2\sqrt{\sin x}}\right)(\cos x)$$

43. $y = (1+2x)^{10} \quad \Rightarrow \quad y' = 10(1+2x)^9\cdot 2 = 20(1+2x)^9$. At $(0,1)$, $y' = 20(1+0)^9 = 20$, and an equation of the tangent line is $y - 1 = 20(x-0)$, or $y = 20x+1$.

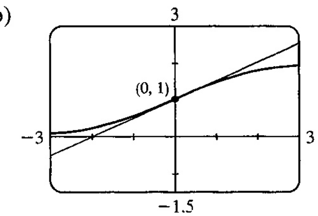
45. $y = \sin(\sin x) \quad \Rightarrow \quad y' = \cos(\sin x)\cdot\cos x$. At $(\pi, 0)$, $y' = \cos(\sin\pi)\cdot\cos\pi = \cos(0)\cdot(-1) = 1(-1) = -1$, and an equation of the tangent line is $y - 0 = -1(x-\pi)$, or $y = -x+\pi$.

47. (a) $y = \dfrac{2}{1+e^{-x}} \Rightarrow y' = \dfrac{(1+e^{-x})(0) - 2(-e^{-x})}{(1+e^{-x})^2} = \dfrac{2e^{-x}}{(1+e^{-x})^2}$.

At $(0,1)$, $y' = \dfrac{2e^0}{(1+e^0)^2} = \dfrac{2(1)}{(1+1)^2} = \dfrac{2}{2^2} = \dfrac{1}{2}$.

So an equation of the tangent line is $y - 1 = \frac{1}{2}(x-0)$ or $y = \frac{1}{2}x + 1$.

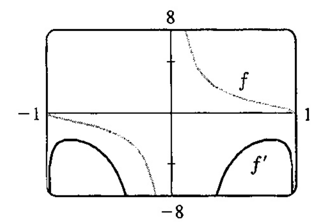
(b)



49. (a) $f(x) = \dfrac{\sqrt{1-x^2}}{x} \Rightarrow$

$$f'(x) = \frac{x \cdot \frac{1}{2}(1-x^2)^{-1/2}(-2x) - \sqrt{1-x^2}\,(1)}{x^2} \cdot \frac{\sqrt{1-x^2}}{\sqrt{1-x^2}}$$
$$= \frac{-x^2 - (1-x^2)}{x^2\sqrt{1-x^2}} = \frac{-1}{x^2\sqrt{1-x^2}}$$

(b)



Notice that all tangents to the graph of f have negative slopes and $f'(x) < 0$ always.

51. For the tangent line to be horizontal, $f'(x) = 0$. $f(x) = 2\sin x + \sin^2 x \Rightarrow$ $f'(x) = 2\cos x + 2\sin x \cos x = 0 \Leftrightarrow 2\cos x\,(1 + \sin x) = 0 \Leftrightarrow \cos x = 0$ or $\sin x = -1$, so $x = \frac{\pi}{2} + 2n\pi$ or $\frac{3\pi}{2} + 2n\pi$, where n is any integer. Now $f\left(\frac{\pi}{2}\right) = 3$ and $f\left(\frac{3\pi}{2}\right) = -1$, so the points on the curve with a horizontal tangent are $\left(\frac{\pi}{2} + 2n\pi, 3\right)$ and $\left(\frac{3\pi}{2} + 2n\pi, -1\right)$, where n is any integer.

53. $F(x) = f(g(x)) \Rightarrow F'(x) = f'(g(x)) \cdot g'(x)$, so $F'(3) = f'(g(3)) \cdot g'(3) = f'(6) \cdot g'(3) = 7 \cdot 4 = 28$. Notice that we did not use $f'(3) = 2$.

55. (a) $h(x) = f(g(x)) \Rightarrow h'(x) = f'(g(x)) \cdot g'(x)$, so $h'(1) = f'(g(1)) \cdot g'(1) = f'(2) \cdot 6 = 5 \cdot 6 = 30$.

(b) $H(x) = g(f(x)) \Rightarrow H'(x) = g'(f(x)) \cdot f'(x)$, so $H'(1) = g'(f(1)) \cdot f'(1) = g'(3) \cdot 4 = 9 \cdot 4 = 36$.

57. (a) $u(x) = f(g(x)) \Rightarrow u'(x) = f'(g(x))g'(x)$. So $u'(1) = f'(g(1))g'(1) = f'(3)g'(1)$. To find $f'(3)$, note that f is linear from $(2,4)$ to $(6,3)$, so its slope is $\dfrac{3-4}{6-2} = -\dfrac{1}{4}$. To find $g'(1)$, note that g is linear from $(0,6)$ to $(2,0)$, so its slope is $\dfrac{0-6}{2-0} = -3$. Thus, $f'(3)g'(1) = \left(-\frac{1}{4}\right)(-3) = \frac{3}{4}$.

(b) $v(x) = g(f(x)) \Rightarrow v'(x) = g'(f(x))f'(x)$. So $v'(1) = g'(f(1))f'(1) = g'(2)f'(1)$, which does not exist since $g'(2)$ does not exist.

(c) $w(x) = g(g(x)) \Rightarrow w'(x) = g'(g(x))g'(x)$. So $w'(1) = g'(g(1))g'(1) = g'(3)g'(1)$. To find $g'(3)$, note that g is linear from $(2,0)$ to $(5,2)$, so its slope is $\dfrac{2-0}{5-2} = \dfrac{2}{3}$. Thus, $g'(3)g'(1) = \left(\frac{2}{3}\right)(-3) = -2$.

59. $h(x) = f(g(x)) \Rightarrow h'(x) = f'(g(x))g'(x)$. So $h'(0.5) = f'(g(0.5))g'(0.5) = f'(0.1)g'(0.5)$. We can estimate the derivatives by taking the average of two secant slopes.

For $f'(0.1)$: $m_1 = \dfrac{14.8 - 12.6}{0.1 - 0} = 22$, $m_2 = \dfrac{18.4 - 14.8}{0.2 - 0.1} = 36$. So $f'(0.1) \approx \dfrac{m_1 + m_2}{2} = \dfrac{22 + 36}{2} = 29$.

For $g'(0.5)$: $m_1 = \dfrac{0.10 - 0.17}{0.5 - 0.4} = -0.7$, $m_2 = \dfrac{0.05 - 0.10}{0.6 - 0.5} = -0.5$. So $g'(0.5) \approx \dfrac{m_1 + m_2}{2} = -0.6$.

Hence, $h'(0.5) = f'(0.1)g'(0.5) \approx (29)(-0.6) = -17.4$.

61. (a) $F(x) = f(e^x) \Rightarrow F'(x) = f'(e^x)\dfrac{d}{dx}(e^x) = f'(e^x)\,e^x$

(b) $G(x) = e^{f(x)} \Rightarrow G'(x) = e^{f(x)}\dfrac{d}{dx}f(x) = e^{f(x)}f'(x)$

63. (a) $f(x) = L(x^4) \Rightarrow f'(x) = L'(x^4)\cdot 4x^3 = (1/x^4)\cdot 4x^3 = 4/x$ for $x > 0$.

(b) $g(x) = L(4x) \Rightarrow g'(x) = L'(4x)\cdot 4 = (1/(4x))\cdot 4 = 1/x$ for $x > 0$.

(c) $F(x) = [L(x)]^4 \Rightarrow F'(x) = 4[L(x)]^3\cdot L'(x) = 4[L(x)]^3\cdot(1/x) = 4[L(x)]^3/x$

(d) $G(x) = L(1/x) \Rightarrow G'(x) = L'(1/x)\cdot(-1/x^2) = (1/(1/x))\cdot(-1/x^2) = x\cdot(-1/x^2) = -1/x$ for $x > 0$.

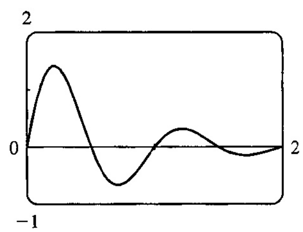
65. $s(t) = 10 + \frac{1}{4}\sin(10\pi t) \Rightarrow$ the velocity after t seconds is
$v(t) = s'(t) = \frac{1}{4}\cos(10\pi t)(10\pi) = \frac{5\pi}{2}\cos(10\pi t)$ cm/s.

67. (a) $B(t) = 4.0 + 0.35\sin\dfrac{2\pi t}{5.4} \Rightarrow \dfrac{dB}{dt} = \left(0.35\cos\dfrac{2\pi t}{5.4}\right)\left(\dfrac{2\pi}{5.4}\right) = \dfrac{0.7\pi}{5.4}\cos\dfrac{2\pi t}{5.4} = \dfrac{7\pi}{54}\cos\dfrac{2\pi t}{5.4}$

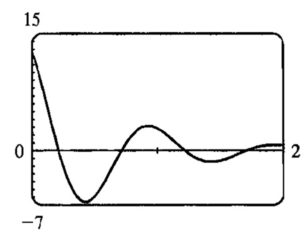
(b) At $t = 1$, $\dfrac{dB}{dt} = \dfrac{7\pi}{54}\cos\dfrac{2\pi}{5.4} \approx 0.16$.

69. $s(t) = 2e^{-1.5t}\sin 2\pi t \Rightarrow$
$v(t) = s'(t) = 2\left[e^{-1.5t}(\cos 2\pi t)(2\pi) + (\sin 2\pi t)e^{-1.5t}(-1.5)\right] = 2e^{-1.5t}(2\pi\cos 2\pi t - 1.5\sin 2\pi t)$

Graph of position



Graph of velocity



71. (a) Using a calculator or CAS, we obtain the model $Q = ab^t$ with $a = 100.0124369$ and $b = 0.000045145933$. We can change this model to one with base e and exponent $\ln b$ [$b^t = e^{t\ln b}$ from precalculus mathematics or from Section 7.3]: $Q = ae^{t\ln b} = 100.012437e^{-10.005531t}$.

(b) Use $Q'(t) = ab^t\ln b$ or the calculator command `nDeriv(Y₁,X,.04)` with `Y₁=abˣ` to get $Q'(0.04) \approx -670.63\ \mu$A. The result of Example 2 in Section 2.1 was $-670\ \mu$A.

73. (a) Derive gives $g'(t) = \dfrac{45(t-2)^8}{(2t+1)^{10}}$ without simplifying. With either Maple or Mathematica, we first get $g'(t) = 9\dfrac{(t-2)^8}{(2t+1)^9} - 18\dfrac{(t-2)^9}{(2t+1)^{10}}$, and the simplification command results in the above expression.

(b) Derive gives $y' = 2(x^3 - x + 1)^3(2x+1)^4(17x^3 + 6x^2 - 9x + 3)$ without simplifying. With either Maple or Mathematica, we first get $y' = 10(2x+1)^4(x^3 - x + 1)^4 + 4(2x+1)^5(x^3 - x + 1)^3(3x^2 - 1)$. If we use Mathematica's `Factor` or `Simplify`, or Maple's `factor`, we get the above expression, but Maple's `simplify` gives the polynomial expansion instead. For locating horizontal tangents, the factored form is the most helpful.

75. (a) If f is even, then $f(x) = f(-x)$. Using the Chain Rule to differentiate this equation, we get $f'(x) = f'(-x)\dfrac{d}{dx}(-x) = -f'(-x)$. Thus, $f'(-x) = -f'(x)$, so f' is odd.

(b) If f is odd, then $f(x) = -f(-x)$. Differentiating this equation, we get $f'(x) = -f'(-x)(-1) = f'(-x)$, so f' is even.

77. (a)
$$\begin{aligned}\frac{d}{dx}(\sin^n x \cos nx) &= n\sin^{n-1} x \cos x \cos nx + \sin^n x(-n \sin nx) && \text{[Product Rule]}\\ &= n\sin^{n-1} x(\cos nx \cos x - \sin nx \sin x) && [\text{factor out } n\sin^{n-1} x]\\ &= n\sin^{n-1} x \cos(nx + x) && \text{[Addition Formula for cosine]}\\ &= n\sin^{n-1} x \cos[(n+1)x] && [\text{factor out } x]\end{aligned}$$

(b)
$$\begin{aligned}\frac{d}{dx}(\cos^n x \cos nx) &= n\cos^{n-1} x(-\sin x)\cos nx + \cos^n x(-n \sin nx) && \text{[Product Rule]}\\ &= -n\cos^{n-1} x(\cos nx \sin x + \sin nx \cos x) && [\text{factor out } -n\cos^{n-1} x]\\ &= -n\cos^{n-1} x \sin(nx + x) && \text{[Addition Formula for sine]}\\ &= -n\cos^{n-1} x \sin[(n+1)x] && [\text{factor out } x]\end{aligned}$$

79. Since $\theta^\circ = \left(\frac{\pi}{180}\right)\theta$ rad, we have $\frac{d}{d\theta}(\sin\theta^\circ) = \frac{d}{d\theta}\left(\sin\frac{\pi}{180}\theta\right) = \frac{\pi}{180}\cos\frac{\pi}{180}\theta = \frac{\pi}{180}\cos\theta^\circ$.

81. First note that products and differences of polynomials are polynomials and that the derivative of a polynomial is also a polynomial. When $n = 1$, $f^{(1)}(x) = \left(\dfrac{P(x)}{Q(x)}\right)' = \dfrac{Q(x)P'(x) - P(x)Q'(x)}{[Q(x)]^2} = \dfrac{A_1(x)}{[Q(x)]^{1+1}}$, where $A_1(x) = Q(x)P'(x) - P(x)Q'(x)$. Suppose the result is true for $n = k$, where $k \geq 1$. Then $f^{(k)}(x) = \dfrac{A_k(x)}{[Q(x)]^{k+1}}$, so

$$\begin{aligned}f^{(k+1)}(x) &= \left(\frac{A_k(x)}{[Q(x)]^{k+1}}\right)' = \frac{[Q(x)]^{k+1}A_k'(x) - A_k(x)\cdot(k+1)[Q(x)]^k\cdot Q'(x)}{\{[Q(x)]^{k+1}\}^2}\\ &= \frac{[Q(x)]^{k+1}A_k'(x) - (k+1)A_k(x)[Q(x)]^kQ'(x)}{[Q(x)]^{2k+2}}\\ &= \frac{[Q(x)]^k\{[Q(x)]^1A_k'(x) - (k+1)A_k(x)Q'(x)\}}{[Q(x)]^k[Q(x)]^{k+2}} = \frac{Q(x)A_k'(x) - (k+1)A_k(x)Q'(x)}{[Q(x)]^{k+2}}\\ &= A_{k+1}(x)/[Q(x)]^{k+2}, \text{ where } A_{k+1}(x) = Q(x)A_k'(x) - (k+1)A_k(x)Q'(x).\end{aligned}$$

We have shown that the formula holds for $n = 1$, and that when it holds for $n = k$ it also holds for $n = k + 1$. Thus, by mathematical induction, the formula holds for all positive integers n.

3.6 Implicit Differentiation

1. (a) $\frac{d}{dx}(xy + 2x + 3x^2) = \frac{d}{dx}(4) \Rightarrow (x\cdot y' + y\cdot 1) + 2 + 6x = 0 \Rightarrow xy' = -y - 2 - 6x \Rightarrow$
$y' = \dfrac{-y - 2 - 6x}{x}$ or $y' = -6 - \dfrac{y+2}{x}$.

(b) $xy + 2x + 3x^2 = 4 \Rightarrow xy = 4 - 2x - 3x^2 \Rightarrow y = \dfrac{4 - 2x - 3x^2}{x} = \dfrac{4}{x} - 2 - 3x$, so $y' = -\dfrac{4}{x^2} - 3$.

(c) From part (a), $y' = \dfrac{-y - 2 - 6x}{x} = \dfrac{-(4/x - 2 - 3x) - 2 - 6x}{x} = \dfrac{-4/x - 3x}{x} = -\dfrac{4}{x^2} - 3$.

3. (a) $\frac{d}{dx}\left(\frac{1}{x}+\frac{1}{y}\right)=\frac{d}{dx}(1) \Rightarrow -\frac{1}{x^2}-\frac{1}{y^2}y'=0 \Rightarrow -\frac{1}{y^2}y'=\frac{1}{x^2} \Rightarrow y'=-\frac{y^2}{x^2}$

(b) $\frac{1}{x}+\frac{1}{y}=1 \Rightarrow \frac{1}{y}=1-\frac{1}{x}=\frac{x-1}{x} \Rightarrow y=\frac{x}{x-1}$, so $y'=\frac{(x-1)(1)-(x)(1)}{(x-1)^2}=\frac{-1}{(x-1)^2}$.

(c) $y'=-\frac{y^2}{x^2}=-\frac{[x/(x-1)]^2}{x^2}=-\frac{x^2}{x^2(x-1)^2}=-\frac{1}{(x-1)^2}$

5. $\frac{d}{dx}\left(x^2+y^2\right)=\frac{d}{dx}(1) \Rightarrow 2x+2yy'=0 \Rightarrow 2yy'=-2x \Rightarrow y'=-\frac{x}{y}$

7. $\frac{d}{dx}\left(x^3+x^2y+4y^2\right)=\frac{d}{dx}(6) \Rightarrow 3x^2+\left(x^2y'+y\cdot 2x\right)+8yy'=0 \Rightarrow x^2y'+8yy'=-3x^2-2xy$
$\Rightarrow \left(x^2+8y\right)y'=-3x^2-2xy \Rightarrow y'=-\frac{3x^2+2xy}{x^2+8y}=-\frac{x(3x+2y)}{x^2+8y}$

9. $\frac{d}{dx}\left(x^2y+xy^2\right)=\frac{d}{dx}(3x) \Rightarrow \left(x^2y'+y\cdot 2x\right)+\left(x\cdot 2yy'+y^2\cdot 1\right)=3 \Rightarrow$
$x^2y'+2xyy'=3-2xy-y^2 \Rightarrow y'\left(x^2+2xy\right)=3-2xy-y^2 \Rightarrow y'=\frac{3-2xy-y^2}{x^2+2xy}$

11. $\frac{d}{dx}\left(x^2y^2+x\sin y\right)=\frac{d}{dx}(4) \Rightarrow x^2\cdot 2yy'+y^2\cdot 2x+x\cos y\cdot y'+\sin y\cdot 1=0 \Rightarrow$
$2x^2yy'+x\cos y\cdot y'=-2xy^2-\sin y \Rightarrow (2x^2y+x\cos y)y'=-2xy^2-\sin y \Rightarrow y'=\frac{-2xy^2-\sin y}{2x^2y+x\cos y}$

13. $\frac{d}{dx}(4\cos x\sin y)=\frac{d}{dx}(1) \Rightarrow 4\left[\cos x\cdot\cos y\cdot y'+\sin y\cdot(-\sin x)\right]=0 \Rightarrow$
$y'(4\cos x\cos y)=4\sin x\sin y \Rightarrow y'=\frac{4\sin x\sin y}{4\cos x\cos y}=\tan x\tan y$

15. $\frac{d}{dx}\left(e^{x^2y}\right)=\frac{d}{dx}(x+y) \Rightarrow e^{x^2y}(x^2y'+y\cdot 2x)=1+y' \Rightarrow x^2e^{x^2y}y'+2xye^{x^2y}=1+y' \Rightarrow$
$x^2e^{x^2y}y'-y'=1-2xye^{x^2y} \Rightarrow y'(x^2e^{x^2y}-1)=1-2xye^{x^2y} \Rightarrow y'=\frac{1-2xye^{x^2y}}{x^2e^{x^2y}-1}$

17. $\sqrt{xy}=1+x^2y \Rightarrow \frac{1}{2}(xy)^{-1/2}(xy'+y\cdot 1)=0+x^2y'+y\cdot 2x \Rightarrow \frac{x}{2\sqrt{xy}}y'+\frac{y}{2\sqrt{xy}}=x^2y'+2xy$
$\Rightarrow y'\left(\frac{x}{2\sqrt{xy}}-x^2\right)=2xy-\frac{y}{2\sqrt{xy}} \Rightarrow y'\left(\frac{x-2x^2\sqrt{xy}}{2\sqrt{xy}}\right)=\frac{4xy\sqrt{xy}-y}{2\sqrt{xy}} \Rightarrow y'=\frac{4xy\sqrt{xy}-y}{x-2x^2\sqrt{xy}}$

19. $xy=\cot(xy) \Rightarrow y+xy'=-\csc^2(xy)(y+xy') \Rightarrow (y+xy')\left[1+\csc^2(xy)\right]=0 \Rightarrow$
$y+xy'=0$ [since $1+\csc^2(xy)>0$] $\Rightarrow y'=-y/x$

21. $\frac{d}{dx}\left\{1+f(x)+x^2[f(x)]^3\right\}=\frac{d}{dx}(0) \Rightarrow f'(x)+x^2\cdot 3[f(x)]^2\cdot f'(x)+[f(x)]^3\cdot 2x=0$. If $x=1$, we have
$f'(1)+1^2\cdot 3[f(1)]^2\cdot f'(1)+[f(1)]^3\cdot 2(1)=0 \Rightarrow f'(1)+1\cdot 3\cdot 2^2\cdot f'(1)+2^3\cdot 2=0 \Rightarrow$
$f'(1)+12f'(1)=-16 \Rightarrow 13f'(1)=-16 \Rightarrow f'(1)=-\frac{16}{13}$.

23. $y^4 + x^2y^2 + yx^4 = y + 1 \;\Rightarrow\; 4y^3 + \left(x^2 \cdot 2y + y^2 \cdot 2x\,\dfrac{dx}{dy}\right) + \left(y \cdot 4x^3\,\dfrac{dx}{dy} + x^4 \cdot 1\right) = 1 \;\Rightarrow$

$2xy^2\,\dfrac{dx}{dy} + 4x^3y\,\dfrac{dx}{dy} = 1 - 4y^3 - 2x^2y - x^4 \;\Rightarrow\; \dfrac{dx}{dy} = \dfrac{1 - 4y^3 - 2x^2y - x^4}{2xy^2 + 4x^3y}$

25. $x^2 + xy + y^2 = 3 \;\Rightarrow\; 2x + xy' + y \cdot 1 + 2yy' = 0 \;\Rightarrow\; xy' + 2yy' = -2x - y \;\Rightarrow$

$y'(x + 2y) = -2x - y \;\Rightarrow\; y' = \dfrac{-2x - y}{x + 2y}$. When $x = 1$ and $y = 1$, we have $y' = \dfrac{-2 - 1}{1 + 2 \cdot 1} = \dfrac{-3}{3} = -1$, so an equation of the tangent line is $y - 1 = -1(x - 1)$ or $y = -x + 2$.

27. $x^2 + y^2 = (2x^2 + 2y^2 - x)^2 \;\Rightarrow\; 2x + 2yy' = 2(2x^2 + 2y^2 - x)(4x + 4yy' - 1)$. When $x = 0$ and $y = \frac{1}{2}$, we have $0 + y' = 2(\frac{1}{2})(2y' - 1) \;\Rightarrow\; y' = 2y' - 1 \;\Rightarrow\; y' = 1$, so an equation of the tangent line is $y - \frac{1}{2} = 1(x - 0)$ or $y = x + \frac{1}{2}$.

29. $2(x^2 + y^2)^2 = 25(x^2 - y^2) \;\Rightarrow\; 4(x^2 + y^2)(2x + 2yy') = 25(2x - 2yy') \;\Rightarrow$

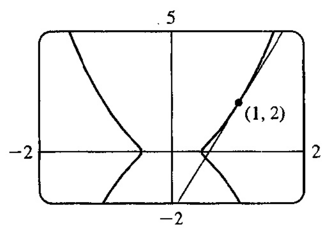
$4(x + yy')(x^2 + y^2) = 25(x - yy') \;\Rightarrow\; 4yy'(x^2 + y^2) + 25yy' = 25x - 4x(x^2 + y^2) \;\Rightarrow$

$y' = \dfrac{25x - 4x(x^2 + y^2)}{25y + 4y(x^2 + y^2)}$. When $x = 3$ and $y = 1$, we have $y' = \frac{75 - 120}{25 + 40} = -\frac{45}{65} = -\frac{9}{13}$, so an equation of the tangent line is $y - 1 = -\frac{9}{13}(x - 3)$ or $y = -\frac{9}{13}x + \frac{40}{13}$.

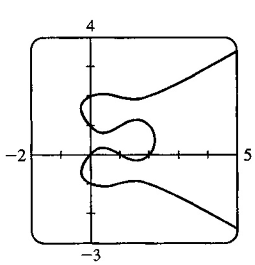
31. (a) $y^2 = 5x^4 - x^2 \;\Rightarrow\; 2yy' = 5(4x^3) - 2x \;\Rightarrow\; y' = \dfrac{10x^3 - x}{y}$.

So at the point $(1, 2)$ we have $y' = \dfrac{10(1)^3 - 1}{2} = \dfrac{9}{2}$, and an equation of the tangent line is $y - 2 = \frac{9}{2}(x - 1)$ or $y = \frac{9}{2}x - \frac{5}{2}$.

(b)



33. (a)



There are eight points with horizontal tangents: four at $x \approx 1.57735$ and four at $x \approx 0.42265$.

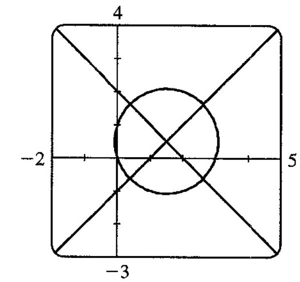
(b) $y' = \dfrac{3x^2 - 6x + 2}{2(2y^3 - 3y^2 - y + 1)} \;\Rightarrow\; y' = -1$ at $(0, 1)$ and $y' = \frac{1}{3}$ at $(0, 2)$.

Equations of the tangent lines are $y = -x + 1$ and $y = \frac{1}{3}x + 2$.

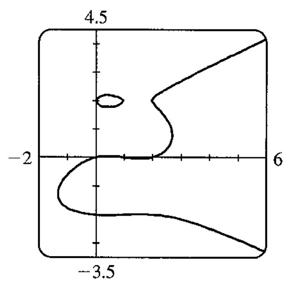
(c) $y' = 0 \;\Rightarrow\; 3x^2 - 6x + 2 = 0 \;\Rightarrow$ $x = 1 \pm \frac{1}{3}\sqrt{3}$

(d) By multiplying the right side of the equation by $x - 3$, we obtain the first graph.

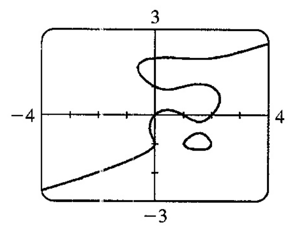
By modifying the equation in other ways, we can generate the other graphs.



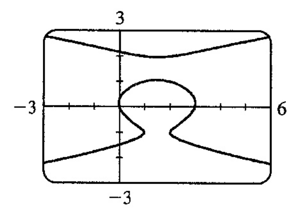
$y(y^2 - 1)(y - 2)$
$= x(x - 1)(x - 2)(x - 3)$



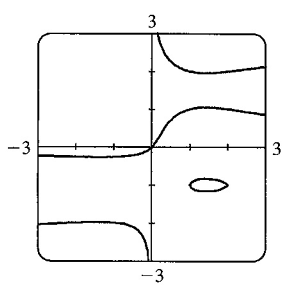
$y(y^2 - 4)(y - 2)$
$= x(x - 1)(x - 2)$



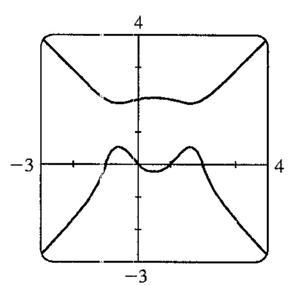
$y(y + 1)(y^2 - 1)(y - 2)$
$= x(x - 1)(x - 2)$



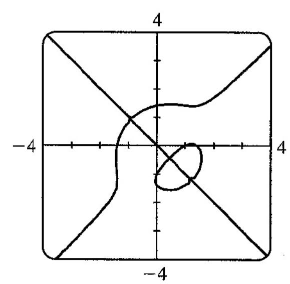
$(y + 1)(y^2 - 1)(y - 2)$
$= (x - 1)(x - 2)$



$x(y + 1)(y^2 - 1)(y - 2)$
$= y(x - 1)(x - 2)$



$y(y^2 + 1)(y - 2)$
$= x(x^2 - 1)(x - 2)$



$y(y + 1)(y^2 - 2)$
$= x(x - 1)(x^2 - 2)$

35. From Exercise 29, a tangent to the lemniscate will be horizontal if $y' = 0 \Rightarrow 25x - 4x(x^2 + y^2) = 0 \Rightarrow x[25 - 4(x^2 + y^2)] = 0 \Rightarrow x^2 + y^2 = \frac{25}{4}$ **(1)**. (Note that when x is 0, y is also 0, and there is no horizontal tangent at the origin.) Substituting $\frac{25}{4}$ for $x^2 + y^2$ in the equation of the lemniscate, $2(x^2 + y^2)^2 = 25(x^2 - y^2)$, we get $x^2 - y^2 = \frac{25}{8}$ **(2)**. Solving **(1)** and **(2)**, we have $x^2 = \frac{75}{16}$ and $y^2 = \frac{25}{16}$, so the four points are $\left(\pm\frac{5\sqrt{3}}{4}, \pm\frac{5}{4}\right)$.

37. $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1 \Rightarrow \dfrac{2x}{a^2} - \dfrac{2yy'}{b^2} = 0 \Rightarrow y' = \dfrac{b^2x}{a^2y} \Rightarrow$ an equation of the tangent line at (x_0, y_0) is $y - y_0 = \dfrac{b^2x_0}{a^2y_0}(x - x_0)$. Multiplying both sides by $\frac{y_0}{b^2}$ gives $\dfrac{y_0y}{b^2} - \dfrac{y_0^2}{b^2} = \dfrac{x_0x}{a^2} - \dfrac{x_0^2}{a^2}$. Since (x_0, y_0) lies on the hyperbola, we have $\dfrac{x_0x}{a^2} - \dfrac{y_0y}{b^2} = \dfrac{x_0^2}{a^2} - \dfrac{y_0^2}{b^2} = 1$.

39. If the circle has radius r, its equation is $x^2 + y^2 = r^2 \Rightarrow 2x + 2yy' = 0 \Rightarrow y' = -\frac{x}{y}$, so the slope of the tangent line at $P(x_0, y_0)$ is $-\frac{x_0}{y_0}$. The negative reciprocal of that slope is $\frac{-1}{-x_0/y_0} = \frac{y_0}{x_0}$, which is the slope of OP, so the tangent line at P is perpendicular to the radius OP.

41. $y = \tan^{-1}\sqrt{x} \Rightarrow y' = \frac{1}{1+(\sqrt{x})^2} \cdot \frac{d}{dx}(\sqrt{x}) = \frac{1}{1+x}\left(\frac{1}{2}x^{-1/2}\right) = \frac{1}{2\sqrt{x}(1+x)}$

43. $y = \sin^{-1}(2x+1) \Rightarrow$

$$y' = \frac{1}{\sqrt{1-(2x+1)^2}} \cdot \frac{d}{dx}(2x+1) = \frac{1}{\sqrt{1-(4x^2+4x+1)}} \cdot 2 = \frac{2}{\sqrt{-4x^2-4x}} = \frac{1}{\sqrt{-x^2-x}}$$

45. $H(x) = (1+x^2)\arctan x \Rightarrow H'(x) = (1+x^2)\frac{1}{1+x^2} + (\arctan x)(2x) = 1 + 2x\arctan x$

47. $h(t) = \cot^{-1}(t) + \cot^{-1}(1/t) \Rightarrow$

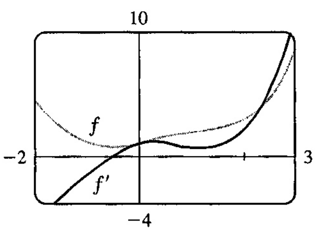
$$h'(t) = -\frac{1}{1+t^2} - \frac{1}{1+(1/t)^2} \cdot \frac{d}{dt}\frac{1}{t} = -\frac{1}{1+t^2} - \frac{t^2}{t^2+1} \cdot \left(-\frac{1}{t^2}\right) = -\frac{1}{1+t^2} + \frac{1}{t^2+1} = 0.$$

Note that this makes sense because $h(t) = \frac{\pi}{2}$ for $t > 0$ and $h(t) = -\frac{\pi}{2}$ for $t < 0$.

49. $y = \cos^{-1}(e^{2x}) \Rightarrow y' = -\frac{1}{\sqrt{1-(e^{2x})^2}} \cdot \frac{d}{dx}(e^{2x}) = -\frac{2e^{2x}}{\sqrt{1-e^{4x}}}$

51. $f(x) = e^x - x^2\arctan x \Rightarrow$

$$f'(x) = e^x - \left[x^2\left(\frac{1}{1+x^2}\right) + (\arctan x)(2x)\right]$$
$$= e^x - \frac{x^2}{1+x^2} - 2x\arctan x$$



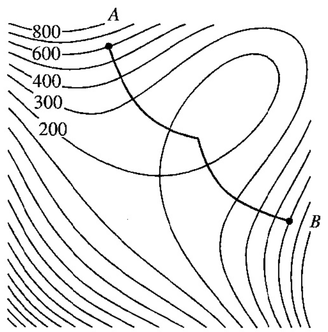
This is reasonable because the graphs show that f is increasing when f' is positive, and f' is zero when f has a minimum.

53. Let $y = \cos^{-1} x$. Then $\cos y = x$ and $0 \le y \le \pi \Rightarrow -\sin y\frac{dy}{dx} = 1 \Rightarrow$

$\frac{dy}{dx} = -\frac{1}{\sin y} = -\frac{1}{\sqrt{1-\cos^2 y}} = -\frac{1}{\sqrt{1-x^2}}$. (Note that $\sin y \ge 0$ for $0 \le y \le \pi$.)

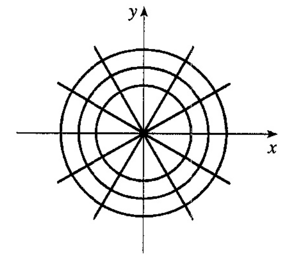
55. $2x^2 + y^2 = 3$ and $x = y^2$ intersect when $2x^2 + x - 3 = 0 \Leftrightarrow (2x+3)(x-1) = 0 \Leftrightarrow x = -\frac{3}{2}$ or 1, but $-\frac{3}{2}$ is extraneous since $x = y^2$ is nonnegative. When $x = 1$, $1 = y^2 \Rightarrow y = \pm 1$, so there are two points of intersection: $(1, \pm 1)$. $2x^2 + y^2 = 3 \Rightarrow 4x + 2yy' = 0 \Rightarrow y' = -2x/y$, and $x = y^2 \Rightarrow 1 = 2yy' \Rightarrow y' = 1/(2y)$. At $(1, 1)$, the slopes are $m_1 = -2(1)/1 = -2$ and $m_2 = 1/(2 \cdot 1) = \frac{1}{2}$, so the curves are orthogonal (since m_1 and m_2 are negative reciprocals of each other). By symmetry, the curves are also orthogonal at $(1, -1)$.

57.



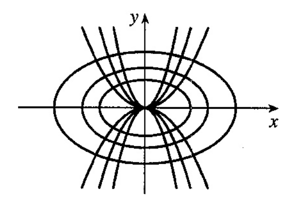
59. $x^2 + y^2 = r^2$ is a circle with center O and $ax + by = 0$ is a line through O.

$x^2 + y^2 = r^2 \;\Rightarrow\; 2x + 2yy' = 0 \;\Rightarrow\; y' = -x/y$, so the slope of the tangent line at $P_0\,(x_0, y_0)$ is $-x_0/y_0$. The slope of the line OP_0 is y_0/x_0, which is the negative reciprocal of $-x_0/y_0$. Hence, the curves are orthogonal, and the families of curves are orthogonal trajectories of each other.



61. $y = cx^2 \;\Rightarrow\; y' = 2cx$ and $x^2 + 2y^2 = k \;\Rightarrow\; 2x + 4yy' = 0 \;\Rightarrow$

$2yy' = -x \;\Rightarrow\; y' = -\dfrac{x}{2(y)} = -\dfrac{x}{2(cx^2)} = -\dfrac{1}{2cx}$, so the curves are orthogonal.



63. To find the points at which the ellipse $x^2 - xy + y^2 = 3$ crosses the x-axis, let $y = 0$ and solve for x.

$y = 0 \;\Rightarrow\; x^2 - x(0) + 0^2 = 3 \;\Leftrightarrow\; x = \pm\sqrt{3}$. So the graph of the ellipse crosses the x-axis at the points $\left(\pm\sqrt{3}, 0\right)$. Using implicit differentiation to find y', we get $2x - xy' - y + 2yy' = 0 \;\Rightarrow\; y'(2y - x) = y - 2x$

$\Leftrightarrow\; y' = \dfrac{y - 2x}{2y - x}$. So y' at $\left(\sqrt{3}, 0\right)$ is $\dfrac{0 - 2\sqrt{3}}{2(0) - \sqrt{3}} = 2$ and y' at $\left(-\sqrt{3}, 0\right)$ is $\dfrac{0 + 2\sqrt{3}}{2(0) + \sqrt{3}} = 2$. Thus, the tangent lines at these points are parallel.

65. $x^2y^2 + xy = 2 \;\Rightarrow\; x^2 \cdot 2yy' + y^2 \cdot 2x + x \cdot y' + y \cdot 1 = 0 \;\Leftrightarrow\; y'\left(2x^2y + x\right) = -2xy^2 - y \;\Leftrightarrow$

$y' = -\dfrac{2xy^2 + y}{2x^2y + x}$. So $-\dfrac{2xy^2 + y}{2x^2y + x} = -1 \;\Leftrightarrow\; 2xy^2 + y = 2x^2y + x \;\Leftrightarrow\; y(2xy + 1) = x(2xy + 1) \;\Leftrightarrow$

$y(2xy + 1) - x(2xy + 1) = 0 \;\Leftrightarrow\; (2xy + 1)(y - x) = 0 \;\Leftrightarrow\; xy = -\frac{1}{2}$ or $y = x$. But $xy = -\frac{1}{2} \;\Rightarrow$

$x^2y^2 + xy = \frac{1}{4} - \frac{1}{2} \neq 2$, so we must have $x = y$. Then $x^2y^2 + xy = 2 \;\Rightarrow\; x^4 + x^2 = 2 \;\Leftrightarrow$

$x^4 + x^2 - 2 = 0 \;\Leftrightarrow\; \left(x^2 + 2\right)\left(x^2 - 1\right) = 0$. So $x^2 = -2$, which is impossible, or $x^2 = 1 \;\Leftrightarrow\; x = \pm 1$.

Since $x = y$, the points on the curve where the tangent line has a slope of -1 are $(-1, -1)$ and $(1, 1)$.

67. (a) If $y = f^{-1}(x)$, then $f(y) = x$. Differentiating implicitly with respect to x and remembering that y is a function of x, we get $f'(y)\,\dfrac{dy}{dx} = 1$, so $\dfrac{dy}{dx} = \dfrac{1}{f'(y)} \;\Rightarrow\; \left(f^{-1}\right)'(x) = \dfrac{1}{f'(f^{-1}(x))}$.

(b) $f(4) = 5 \Rightarrow f^{-1}(5) = 4$. By part (a), $(f^{-1})'(5) = 1/f'(f^{-1}(5)) = 1/f'(4) = 1/\left(\frac{2}{3}\right) = \frac{3}{2}$.

69. $x^2 + 4y^2 = 5 \Rightarrow 2x + 4(2yy') = 0 \Rightarrow y' = -\dfrac{x}{4y}$. Now let h be the height of the lamp, and let (a, b) be the point of tangency of the line passing through the points $(3, h)$ and $(-5, 0)$. This line has slope $(h - 0)/[3 - (-5)] = \frac{1}{8}h$. But the slope of the tangent line through the point (a, b) can be expressed as $y' = -\dfrac{a}{4b}$, or as $\dfrac{b - 0}{a - (-5)} = \dfrac{b}{a + 5}$ [since the line passes through $(-5, 0)$ and (a, b)], so $-\dfrac{a}{4b} = \dfrac{b}{a + 5} \Leftrightarrow$ $4b^2 = -a^2 - 5a \Leftrightarrow a^2 + 4b^2 = -5a$. But $a^2 + 4b^2 = 5$ [since (a, b) is on the ellipse], so $5 = -5a \Leftrightarrow$ $a = -1$. Then $4b^2 = -a^2 - 5a = -1 - 5(-1) = 4 \Rightarrow b = 1$, since the point is on the top half of the ellipse. So $\dfrac{h}{8} = \dfrac{b}{a + 5} = \dfrac{1}{-1 + 5} = \dfrac{1}{4} \Rightarrow h = 2$. So the lamp is located 2 units above the x-axis.

3.7 Higher Derivatives

1. $a = f$, $b = f'$, $c = f''$. We can see this because where a has a horizontal tangent, $b = 0$, and where b has a horizontal tangent, $c = 0$. We can immediately see that c can be neither f nor f', since at the points where c has a horizontal tangent, neither a nor b is equal to 0.

3. We can immediately see that a is the graph of the acceleration function, since at the points where a has a horizontal tangent, neither c nor b is equal to 0. Next, we note that $a = 0$ at the point where b has a horizontal tangent, so b must be the graph of the velocity function, and hence, $b' = a$. We conclude that c is the graph of the position function.

5. $f(x) = x^5 + 6x^2 - 7x \Rightarrow f'(x) = 5x^4 + 12x - 7 \Rightarrow f''(x) = 20x^3 + 12$

7. $y = \cos 2\theta \Rightarrow y' = -2\sin 2\theta \Rightarrow y'' = -4\cos 2\theta$

9. $F(t) = (1 - 7t)^6 \Rightarrow F'(t) = 6(1 - 7t)^5(-7) = -42(1 - 7t)^5 \Rightarrow$
$F''(t) = -42 \cdot 5(1 - 7t)^4(-7) = 1470(1 - 7t)^4$

11. $h(u) = \dfrac{1 - 4u}{1 + 3u} \Rightarrow h'(u) = \dfrac{(1 + 3u)(-4) - (1 - 4u)(3)}{(1 + 3u)^2} = \dfrac{-4 - 12u - 3 + 12u}{(1 + 3u)^2} = \dfrac{-7}{(1 + 3u)^2}$ or
$-7(1 + 3u)^{-2} \Rightarrow h''(u) = -7(-2)(1 + 3u)^{-3}(3) = 42(1 + 3u)^{-3}$ or $\dfrac{42}{(1 + 3u)^3}$

13. $h(x) = \sqrt{x^2 + 1} \Rightarrow h'(x) = \frac{1}{2}(x^2 + 1)^{-1/2}(2x) = \dfrac{x}{\sqrt{x^2 + 1}} \Rightarrow$

$$h''(x) = \frac{\sqrt{x^2 + 1} \cdot 1 - x\left[\frac{1}{2}(x^2 + 1)^{-1/2}(2x)\right]}{\left(\sqrt{x^2 + 1}\right)^2} = \frac{(x^2 + 1)^{-1/2}\left[(x^2 + 1) - x^2\right]}{(x^2 + 1)^1} = \frac{1}{(x^2 + 1)^{3/2}}$$

15. $y = (x^3 + 1)^{2/3} \Rightarrow y' = \frac{2}{3}(x^3 + 1)^{-1/3}(3x^2) = 2x^2(x^3 + 1)^{-1/3} \Rightarrow$
$y'' = 2x^2\left(-\frac{1}{3}\right)(x^3 + 1)^{-4/3}(3x^2) + (x^3 + 1)^{-1/3}(4x) = 4x(x^3 + 1)^{-1/3} - 2x^4(x^3 + 1)^{-4/3}$

17. $H(t) = \tan 3t \Rightarrow H'(t) = 3\sec^2 3t \Rightarrow$
$H''(t) = 2 \cdot 3\sec 3t \dfrac{d}{dt}(\sec 3t) = 6\sec 3t\,(3\sec 3t\tan 3t) = 18\sec^2 3t\tan 3t$

19. $g(t) = t^3e^{5t} \Rightarrow g'(t) = t^3e^{5t} \cdot 5 + e^{5t} \cdot 3t^2 = t^2e^{5t}(5t+3) \Rightarrow$

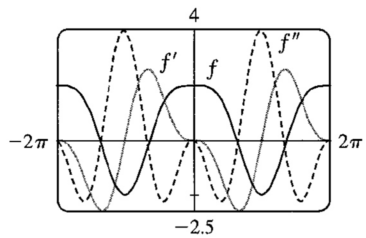
$g''(t) = (2t)e^{5t}(5t+3) + t^2(5e^{5t})(5t+3) + t^2e^{5t}(5)$

$= te^{5t}\,[2(5t+3) + 5t(5t+3) + 5t] = te^{5t}(25t^2 + 30t + 6)$

21. (a) $f(x) = 2\cos x + \sin^2 x \Rightarrow f'(x) = 2(-\sin x) + 2\sin x\,(\cos x) = \sin 2x - 2\sin x \Rightarrow$

$f''(x) = 2\cos 2x - 2\cos x = 2(\cos 2x - \cos x)$

(b)



We can see that our answers are plausible, since f has horizontal tangents where $f'(x) = 0$, and f' has horizontal tangents where $f''(x) = 0$.

23. $y = \sqrt{2x+3} = (2x+3)^{1/2} \Rightarrow y' = \frac{1}{2}(2x+3)^{-1/2} \cdot 2 = (2x+3)^{-1/2} \Rightarrow$

$y'' = -\frac{1}{2}(2x+3)^{-3/2} \cdot 2 = -(2x+3)^{-3/2} \Rightarrow y''' = \frac{3}{2}(2x+3)^{-5/2} \cdot 2 = 3(2x+3)^{-5/2}$

25. $f(t) = t\cos t \Rightarrow f'(t) = t(-\sin t) + \cos t \cdot 1 \Rightarrow f''(t) = t(-\cos t) - \sin t \cdot 1 - \sin t \Rightarrow$

$f'''(t) = t\sin t - \cos t \cdot 1 - \cos t - \cos t = t\sin t - 3\cos t$, so $f'''(0) = 0 - 3 = -3$.

27. $f(\theta) = \cot\theta \Rightarrow f'(\theta) = -\csc^2\theta \Rightarrow f''(\theta) = -2\csc\theta\,(-\csc\theta\cot\theta) = 2\csc^2\theta\cot\theta \Rightarrow$

$f'''(\theta) = 2\left(-2\csc^2\theta\cot\theta\right)\cot\theta + 2\csc^2\theta\left(-\csc^2\theta\right) = -2\csc^2\theta\left(2\cot^2\theta + \csc^2\theta\right) \Rightarrow$

$f'''\left(\frac{\pi}{6}\right) = -2(2)^2\left[2\left(\sqrt{3}\right)^2 + (2)^2\right] = -80$

29. $9x^2 + y^2 = 9 \Rightarrow 18x + 2yy' = 0 \Rightarrow 2yy' = -18x \Rightarrow y' = -9x/y \Rightarrow$

$y'' = -9\left(\dfrac{y \cdot 1 - x \cdot y'}{y^2}\right) = -9\left(\dfrac{y - x(-9x/y)}{y^2}\right) = -9 \cdot \dfrac{y^2 + 9x^2}{y^3} = -9 \cdot \dfrac{9}{y^3}$ [since x and y must satisfy the original equation, $9x^2 + y^2 = 9$]. Thus, $y'' = -81/y^3$.

31. $x^3 + y^3 = 1 \Rightarrow 3x^2 + 3y^2y' = 0 \Rightarrow y' = -\dfrac{x^2}{y^2} \Rightarrow$

$y'' = -\dfrac{y^2(2x) - x^2 \cdot 2yy'}{(y^2)^2} = -\dfrac{2xy^2 - 2x^2y(-x^2/y^2)}{y^4} = -\dfrac{2xy^4 + 2x^4y}{y^6} = -\dfrac{2xy(y^3 + x^3)}{y^6} = -\dfrac{2x}{y^5}$,

since x and y must satisfy the original equation, $x^3 + y^3 = 1$.

33. $f(x) = x^n \Rightarrow f'(x) = nx^{n-1} \Rightarrow f''(x) = n(n-1)x^{n-2} \Rightarrow \cdots \Rightarrow$

$f^{(n)}(x) = n(n-1)(n-2)\cdots 2 \cdot 1x^{n-n} = n!$

35. $f(x) = e^{2x} \Rightarrow f'(x) = 2e^{2x} \Rightarrow f''(x) = 2 \cdot 2e^{2x} = 2^2e^{2x} \Rightarrow$

$f'''(x) = 2^2 \cdot 2e^{2x} = 2^3e^{2x} \Rightarrow \cdots \Rightarrow f^{(n)}(x) = 2^ne^{2x}$

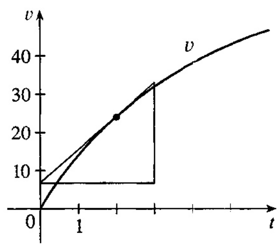
37. $f(x) = 1/(3x^3) = \frac{1}{3}x^{-3} \Rightarrow f'(x) = \frac{1}{3}(-3)x^{-4} \Rightarrow f''(x) = \frac{1}{3}(-3)(-4)x^{-5} \Rightarrow$

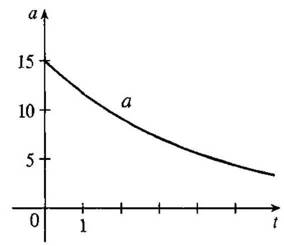
$f'''(x) = \frac{1}{3}(-3)(-4)(-5)x^{-6} \Rightarrow \cdots \Rightarrow$

$f^{(n)}(x) = \frac{1}{3}(-3)(-4)\cdots[-(n+2)]\,x^{-(n+3)} = \dfrac{(-1)^n \cdot 3 \cdot 4 \cdot 5 \cdot \cdots \cdot (n+2)}{3x^{n+3}} \cdot \dfrac{2}{2} = \dfrac{(-1)^n\,(n+2)!}{6x^{n+3}}$

39. Let $f(x) = \cos x$. Then $Df(2x) = 2f'(2x)$, $D^2f(2x) = 2^2f''(2x)$, $D^3f(2x) = 2^3f'''(2x), \ldots,$ $D^{(n)}f(2x) = 2^nf^{(n)}(2x)$. Since the derivatives of $\cos x$ occur in a cycle of four, and since $103 = 4(25) + 3$, we have $f^{(103)}(x) = f^{(3)}(x) = \sin x$ and $D^{103}\cos 2x = 2^{103}f^{(103)}(2x) = 2^{103}\sin 2x$.

41. By measuring the slope of the graph of $s = f(t)$ at $t = 0, 1, 2, 3, 4$, and 5, and using the method of Example 1 in Section 2.9, we plot the graph of the velocity function $v = f'(t)$ in the first figure. The acceleration when $t = 2$ s is $a = f''(2)$, the slope of the tangent line to the graph of f' when $t = 2$. We estimate the slope of this tangent line to be $a(2) = f''(2) = v'(2) \approx \frac{27}{3} = 9\text{ ft/s}^2$. Similar measurements enable us to graph the acceleration function in the second figure.





43. (a) $s = 2t^3 - 15t^2 + 36t + 2 \quad\Rightarrow\quad v(t) = s'(t) = 6t^2 - 30t + 36 \quad\Rightarrow\quad a(t) = v'(t) = 12t - 30$

(b) $a(1) = 12 \cdot 1 - 30 = -18\text{ m/s}^2$

(c) $v(t) = 6(t^2 - 5t + 6) = 6(t - 2)(t - 3) = 0$ when $t = 2$ or 3 and $a(2) = 24 - 30 = -6\text{ m/s}^2$, $a(3) = 36 - 30 = 6\text{ m/s}^2$.

45. (a) $s = \sin\left(\frac{\pi}{6}t\right) + \cos\left(\frac{\pi}{6}t\right), 0 \le t \le 2.$ $v(t) = s'(t) = \cos\left(\frac{\pi}{6}t\right) \cdot \frac{\pi}{6} - \sin\left(\frac{\pi}{6}t\right) \cdot \frac{\pi}{6} = \frac{\pi}{6}\left[\cos\left(\frac{\pi}{6}t\right) - \sin\left(\frac{\pi}{6}t\right)\right]$

$\Rightarrow\quad a(t) = v'(t) = \frac{\pi}{6}\left[-\sin\left(\frac{\pi}{6}t\right) \cdot \frac{\pi}{6} - \cos\left(\frac{\pi}{6}t\right) \cdot \frac{\pi}{6}\right] = -\frac{\pi^2}{36}\left[\sin\left(\frac{\pi}{6}t\right) + \cos\left(\frac{\pi}{6}t\right)\right]$

(b) $a(1) = -\frac{\pi^2}{36}\left[\sin\left(\frac{\pi}{6} \cdot 1\right) + \cos\left(\frac{\pi}{6} \cdot 1\right)\right] = -\frac{\pi^2}{36}\left[\frac{1}{2} + \frac{\sqrt{3}}{2}\right] = -\frac{\pi^2}{72}\left(1 + \sqrt{3}\right) \approx -0.3745\text{ m/s}^2$

(c) $v(t) = 0$ for $0 \le t \le 2 \quad\Rightarrow\quad \cos\left(\frac{\pi}{6}t\right) = \sin\left(\frac{\pi}{6}t\right) \quad\Rightarrow\quad 1 = \dfrac{\sin\left(\frac{\pi}{6}t\right)}{\cos\left(\frac{\pi}{6}t\right)} \quad\Rightarrow$

$\tan\left(\frac{\pi}{6}t\right) = 1 \quad\Rightarrow\quad \frac{\pi}{6}t = \tan^{-1} 1 \quad\Rightarrow\quad t = \frac{6}{\pi} \cdot \frac{\pi}{4} = \frac{3}{2} = 1.5$ s. Thus,

$a\left(\frac{3}{2}\right) = -\frac{\pi^2}{36}\left[\sin\left(\frac{\pi}{6} \cdot \frac{3}{2}\right) + \cos\left(\frac{\pi}{6} \cdot \frac{3}{2}\right)\right] = -\frac{\pi^2}{36}\left[\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\right] = -\frac{\pi^2}{36}\sqrt{2} \approx -0.3877\text{ m/s}^2.$

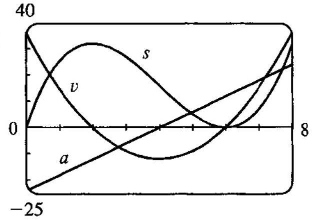
47. (a) $s(t) = t^4 - 4t^3 + 2 \quad\Rightarrow\quad v(t) = s'(t) = 4t^3 - 12t^2 \quad\Rightarrow\quad a(t) = v'(t) = 12t^2 - 24t = 12t(t - 2) = 0$ when $t = 0$ or 2.

(b) $s(0) = 2$ m, $v(0) = 0$ m/s, $s(2) = -14$ m, $v(2) = -16$ m/s

49. (a) $s = f(t) = t^3 - 12t^2 + 36t, t \ge 0 \quad\Rightarrow\quad v(t) = f'(t) = 3t^2 - 24t + 36.$

$a(t) = v'(t) = 6t - 24.$ $a(3) = 6(3) - 24 = -6\text{ (m/s)/s or m/s}^2.$

(b)



(c) The particle is speeding up when v and a have the same sign. This occurs when $2 < t < 4$ and when $t > 6$. It is slowing down when v and a have opposite signs; that is, when $0 \le t < 2$ and when $4 < t < 6$.

51. (a) $y(t) = A\sin\omega t \;\Rightarrow\; v(t) = y'(t) = A\omega\cos\omega t \;\Rightarrow\; a(t) = v'(t) = -A\omega^2\sin\omega t$

(b) $a(t) = -A\omega^2\sin\omega t = -\omega^2(A\sin\omega t) = -\omega^2 y(t)$, so $a(t)$ is proportional to $y(t)$.

(c) speed $= |v(t)| = A\omega\,|\cos\omega t|$ is a maximum when $\cos\omega t = \pm 1$. But when $\cos\omega t = \pm 1$, we have $\sin\omega t = 0$, and $a(t) = -A\omega^2\sin\omega t = -A\omega^2(0) = 0$.

53. Let $P(x) = ax^2 + bx + c$. Then $P'(x) = 2ax + b$ and $P''(x) = 2a$. $P''(2) = 2 \;\Rightarrow\; 2a = 2 \;\Rightarrow\; a = 1$.
$P'(2) = 3 \;\Rightarrow\; 2(1)(2) + b = 3 \;\Rightarrow\; 4 + b = 3 \;\Rightarrow\; b = -1$.
$P(2) = 5 \;\Rightarrow\; 1(2)^2 + (-1)(2) + c = 5 \;\Rightarrow\; 2 + c = 5 \;\Rightarrow\; c = 3$. So $P(x) = x^2 - x + 3$.

55. $y = A\sin x + B\cos x \;\Rightarrow\; y' = A\cos x - B\sin x \;\Rightarrow\; y'' = -A\sin x - B\cos x$. Substituting into $y'' + y' - 2y = \sin x$ gives us $(-3A - B)\sin x + (A - 3B)\cos x = 1\sin x$, so we must have $-3A - B = 1$ and $A - 3B = 0$. Solving for A and B, we add the first equation to three times the second to get $B = -\frac{1}{10}$ and $A = -\frac{3}{10}$.

57. $y = e^{rx} \;\Rightarrow\; y' = re^{rx} \;\Rightarrow\; y'' = r^2e^{rx}$, so
$y'' + 5y' - 6y = r^2e^{rx} + 5re^{rx} - 6e^{rx} = e^{rx}(r^2 + 5r - 6) = e^{rx}(r+6)(r-1) = 0 \;\Rightarrow$
$(r+6)(r-1) = 0 \;\Rightarrow\; r = 1$ or -6.

59. $f(x) = xg(x^2) \;\Rightarrow\; f'(x) = x \cdot g'(x^2) \cdot 2x + g(x^2) \cdot 1 = g(x^2) + 2x^2g'(x^2) \;\Rightarrow$
$f''(x) = g'(x^2) \cdot 2x + 2x^2 \cdot g''(x^2) \cdot 2x + g'(x^2) \cdot 4x = 6xg'(x^2) + 4x^3g''(x^2)$

61. $f(x) = g(\sqrt{x}\,) \;\Rightarrow\; f'(x) = g'(\sqrt{x}\,) \cdot \frac{1}{2}x^{-1/2} = \dfrac{g'(\sqrt{x}\,)}{2\sqrt{x}} \;\Rightarrow$

$$f''(x) = \frac{2\sqrt{x} \cdot g''(\sqrt{x}\,) \cdot \frac{1}{2}x^{-1/2} - g'(\sqrt{x}\,) \cdot 2 \cdot \frac{1}{2}x^{-1/2}}{(2\sqrt{x}\,)^2} = \frac{x^{-1/2}\left[\sqrt{x}\,g''(\sqrt{x}\,) - g'(\sqrt{x}\,)\right]}{4x}$$
$$= \frac{\sqrt{x}\,g''(\sqrt{x}\,) - g'(\sqrt{x}\,)}{4x\sqrt{x}}$$

63. (a) $f(x) = \dfrac{1}{x^2 + x} \;\Rightarrow\; f'(x) = \dfrac{-(2x+1)}{(x^2+x)^2} \;\Rightarrow$

$$f''(x) = \frac{(x^2+x)^2(-2) + (2x+1)(2)(x^2+x)(2x+1)}{(x^2+x)^4} = \frac{2(3x^2+3x+1)}{(x^2+x)^3} \;\Rightarrow$$
$$f'''(x) = \frac{(x^2+x)^3(2)(6x+3) - 2(3x^2+3x+1)(3)(x^2+x)^2(2x+1)}{(x^2+x)^6}$$
$$= \frac{-6(4x^3+6x^2+4x+1)}{(x^2+x)^4} \;\Rightarrow$$
$$f^{(4)}(x) = \frac{(x^2+x)^4(-6)(12x^2+12x+4) + 6(4x^3+6x^2+4x+1)(4)(x^2+x)^3(2x+1)}{(x^2+x)^8}$$
$$= \frac{24(5x^4+10x^3+10x^2+5x+1)}{(x^2+x)^5}$$

$f^{(5)}(x) = ?$

(b) $f(x) = \dfrac{1}{x(x+1)} = \dfrac{1}{x} - \dfrac{1}{x+1} \;\Rightarrow\; f'(x) = -x^{-2} + (x+1)^{-2} \;\Rightarrow\; f''(x) = 2x^{-3} - 2(x+1)^{-3} \;\Rightarrow$
$f'''(x) = (-3)(2)x^{-4} + (3)(2)(x+1)^{-4} \;\Rightarrow\; \cdots \;\Rightarrow\; f^{(n)}(x) = (-1)^n n!\left[x^{-(n+1)} - (x+1)^{-(n+1)}\right]$

65. For $f(x) = x^2 e^x$, $f'(x) = x^2 e^x + e^x(2x) = e^x(x^2 + 2x)$. Similarly, we have

$$\begin{aligned} f''(x) &= e^x(x^2 + 4x + 2) \\ f'''(x) &= e^x(x^2 + 6x + 6) \\ f^{(4)}(x) &= e^x(x^2 + 8x + 12) \\ f^{(5)}(x) &= e^x(x^2 + 10x + 20) \end{aligned}$$

It appears that the coefficient of x in the quadratic term increases by 2 with each differentiation. The pattern for the constant terms seems to be $0 = 1 \cdot 0$, $2 = 2 \cdot 1$, $6 = 3 \cdot 2$, $12 = 4 \cdot 3$, $20 = 5 \cdot 4$. So a reasonable guess is that $f^{(n)}(x) = e^x[x^2 + 2nx + n(n-1)]$.

Proof: Let S_n be the statement that $f^{(n)}(x) = e^x[x^2 + 2nx + n(n-1)]$.

1. S_1 is true because $f'(x) = e^x(x^2 + 2x)$.
2. Assume that S_k is true; that is, $f^{(k)}(x) = e^x[x^2 + 2kx + k(k-1)]$. Then

$$\begin{aligned} f^{(k+1)}(x) = \frac{d}{dx}\left[f^{(k)}(x)\right] &= e^x(2x + 2k) + \left[x^2 + 2kx + k(k-1)\right]e^x \\ &= e^x\left[x^2 + (2k+2)x + (k^2 + k)\right] = e^x\left[x^2 + 2(k+1)x + (k+1)k\right] \end{aligned}$$

   This shows that S_{k+1} is true.
3. Therefore, by mathematical induction, S_n is true for all n; that is, $f^{(n)}(x) = e^x[x^2 + 2nx + n(n-1)]$ for every positive integer n.

67. The Chain Rule says that $\dfrac{dy}{dx} = \dfrac{dy}{du}\dfrac{du}{dx}$, so

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(\frac{dy}{du}\frac{du}{dx}\right) = \left[\frac{d}{dx}\left(\frac{dy}{du}\right)\right]\frac{du}{dx} + \frac{dy}{du}\frac{d}{dx}\left(\frac{du}{dx}\right) \quad \text{[Product Rule]}$$

$$= \left[\frac{d}{du}\left(\frac{dy}{du}\right)\frac{du}{dx}\right]\frac{du}{dx} + \frac{dy}{du}\frac{d^2u}{dx^2} = \frac{d^2y}{du^2}\left(\frac{du}{dx}\right)^2 + \frac{dy}{du}\frac{d^2u}{dx^2}$$

69. We will show that for each positive integer n, the nth derivative $f^{(n)}$ exists and equals one of $f, f', f'', f''', \ldots, f^{(p-1)}$. Since $f^{(p)} = f$, the first p derivatives of f are $f', f'', f''', \ldots, f^{(p-1)}$, and f. In particular, our statement is true for $n = 1$. Suppose that k is an integer, $k \geq 1$, for which f is k-times differentiable with $f^{(k)}$ in the set $S = \{f, f', f'', \ldots, f^{(p-1)}\}$. Since f is p-times differentiable, every member of S [including $f^{(k)}$] is differentiable, so $f^{(k+1)}$ exists and equals the derivative of some member of S. Thus, $f^{(k+1)}$ is in the set $\{f', f'', f''', \ldots, f^{(p)}\}$, which equals S since $f^{(p)} = f$. We have shown that the statement is true for $n = 1$ and that its truth for $n = k$ implies its truth for $n = k + 1$. By mathematical induction, the statement is true for all positive integers n.

3.8 Derivatives of Logarithmic Functions

1. The differentiation formula for logarithmic functions, $\dfrac{d}{dx}(\log_a x) = \dfrac{1}{x \ln a}$, is simplest when $a = e$ because $\ln e = 1$.

3. $f(\theta) = \ln(\cos\theta) \quad\Rightarrow\quad f'(\theta) = \dfrac{1}{\cos\theta}\cdot\dfrac{d}{d\theta}(\cos\theta) = \dfrac{-\sin\theta}{\cos\theta} = -\tan\theta$

5. $f(x) = \log_2(1-3x) \quad \Rightarrow \quad f'(x) = \dfrac{1}{(1-3x)\ln 2}\dfrac{d}{dx}(1-3x) = \dfrac{-3}{(1-3x)\ln 2}$ or $\dfrac{3}{(3x-1)\ln 2}$

7. $f(x) = \sqrt[5]{\ln x} = (\ln x)^{1/5} \quad \Rightarrow \quad f'(x) = \frac{1}{5}(\ln x)^{-4/5}\dfrac{d}{dx}(\ln x) = \dfrac{1}{5(\ln x)^{4/5}} \cdot \dfrac{1}{x} = \dfrac{1}{5x\sqrt[5]{(\ln x)^4}}$

9. $f(x) = \sqrt{x}\ln x \quad \Rightarrow \quad f'(x) = \sqrt{x}\left(\dfrac{1}{x}\right) + (\ln x)\cdot\dfrac{1}{2\sqrt{x}} = \dfrac{1}{\sqrt{x}} + \dfrac{\ln x}{2\sqrt{x}} = \dfrac{2+\ln x}{2\sqrt{x}}$

11. $F(t) = \ln\dfrac{(2t+1)^3}{(3t-1)^4} = \ln(2t+1)^3 - \ln(3t-1)^4 = 3\ln(2t+1) - 4\ln(3t-1) \quad \Rightarrow$

$F'(t) = 3\cdot\dfrac{1}{2t+1}\cdot 2 - 4\cdot\dfrac{1}{3t-1}\cdot 3 = \dfrac{6}{2t+1} - \dfrac{12}{3t-1}$, or combined, $\dfrac{-6(t+3)}{(2t+1)(3t-1)}$.

13. $g(x) = \ln\dfrac{a-x}{a+x} = \ln(a-x) - \ln(a+x) \quad \Rightarrow$

$g'(x) = \dfrac{1}{a-x}(-1) - \dfrac{1}{a+x} = \dfrac{-(a+x)-(a-x)}{(a-x)(a+x)} = \dfrac{-2a}{a^2-x^2}$

15. $f(u) = \dfrac{\ln u}{1+\ln(2u)} \quad \Rightarrow$

$$f'(u) = \frac{[1+\ln(2u)]\cdot\frac{1}{u} - \ln u\cdot\frac{1}{2u}\cdot 2}{[1+\ln(2u)]^2} = \frac{\frac{1}{u}[1+\ln(2u)-\ln u]}{[1+\ln(2u)]^2}$$
$$= \frac{1+(\ln 2+\ln u)-\ln u}{u[1+\ln(2u)]^2} = \frac{1+\ln 2}{u[1+\ln(2u)]^2}$$

17. $y = \ln\left|2-x-5x^2\right| \quad \Rightarrow \quad y' = \dfrac{1}{2-x-5x^2}\cdot(-1-10x) = \dfrac{-10x-1}{2-x-5x^2}$ or $\dfrac{10x+1}{5x^2+x-2}$

19. $y = \ln\left(e^{-x}+xe^{-x}\right) = \ln\left(e^{-x}(1+x)\right) = \ln\left(e^{-x}\right) + \ln(1+x) = -x + \ln(1+x) \quad \Rightarrow$

$y' = -1 + \dfrac{1}{1+x} = \dfrac{-1-x+1}{1+x} = -\dfrac{x}{1+x}$

21. $y = x\ln x \quad \Rightarrow \quad y' = x(1/x) + (\ln x)\cdot 1 = 1 + \ln x \quad \Rightarrow \quad y'' = 1/x$

23. $y = \log_{10} x \quad \Rightarrow \quad y' = \dfrac{1}{x\ln 10} = \dfrac{1}{\ln 10}\left(\dfrac{1}{x}\right) \quad \Rightarrow \quad y'' = \dfrac{1}{\ln 10}\left(-\dfrac{1}{x^2}\right) = -\dfrac{1}{x^2\ln 10}$

25. $f(x) = \dfrac{x}{1-\ln(x-1)} \quad \Rightarrow$

$$f'(x) = \frac{[1-\ln(x-1)]\cdot 1 - x\cdot\dfrac{-1}{x-1}}{[1-\ln(x-1)]^2} = \frac{\dfrac{(x-1)[1-\ln(x-1)]+x}{x-1}}{[1-\ln(x-1)]^2} = \frac{x-1-(x-1)\ln(x-1)+x}{(x-1)[1-\ln(x-1)]^2}$$
$$= \frac{2x-1-(x-1)\ln(x-1)}{(x-1)[1-\ln(x-1)]^2}$$

$\text{Dom}(f) = \{x \mid x-1>0 \text{ and } 1-\ln(x-1)\neq 0\} = \{x \mid x>1 \text{ and } \ln(x-1)\neq 1\}$

$= \left\{x \mid x>1 \text{ and } x-1\neq e^1\right\} = \{x \mid x>1 \text{ and } x\neq 1+e\} = (1, 1+e)\cup(1+e,\infty)$

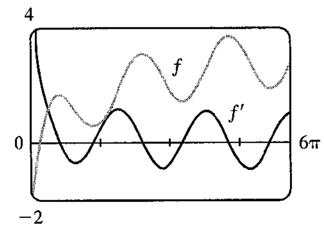
27. $f(x) = x^2\ln(1-x^2) \quad \Rightarrow \quad f'(x) = 2x\ln(1-x^2) + \dfrac{x^2(-2x)}{1-x^2} = 2x\ln(1-x^2) - \dfrac{2x^3}{1-x^2}$.

$\text{Dom}(f) = \left\{x \mid 1-x^2>0\right\} = \{x \mid |x|<1\} = (-1, 1)$.

29. $f(x) = \dfrac{x}{\ln x} \quad \Rightarrow \quad f'(x) = \dfrac{\ln x - x(1/x)}{(\ln x)^2} = \dfrac{\ln x - 1}{(\ln x)^2} \quad \Rightarrow \quad f'(e) = \dfrac{1-1}{1^2} = 0$

31. $y = f(x) = \ln \ln x \;\Rightarrow\; f'(x) = \dfrac{1}{\ln x}\left(\dfrac{1}{x}\right) \;\Rightarrow\; f'(e) = \dfrac{1}{e}$, so an equation of the tangent line at $(e, 0)$ is $y - 0 = \dfrac{1}{e}(x - e)$, or $y = \dfrac{1}{e}x - 1$, or $x - ey = e$.

33. $f(x) = \sin x + \ln x \;\Rightarrow\; f'(x) = \cos x + 1/x$. This is reasonable, because the graph shows that f increases when f' is positive, and $f'(x) = 0$ when f has a horizontal tangent.



35. $y = (2x+1)^5(x^4-3)^6 \;\Rightarrow\; \ln y = \ln\left((2x+1)^5(x^4-3)^6\right) \;\Rightarrow$

$\ln y = 5\ln(2x+1) + 6\ln(x^4-3) \;\Rightarrow\; \dfrac{1}{y}y' = 5\cdot\dfrac{1}{2x+1}\cdot 2 + 6\cdot\dfrac{1}{x^4-3}\cdot 4x^3 \;\Rightarrow$

$y' = y\left(\dfrac{10}{2x+1} + \dfrac{24x^3}{x^4-3}\right) = (2x+1)^5(x^4-3)^6\left(\dfrac{10}{2x+1} + \dfrac{24x^3}{x^4-3}\right)$.

[The answer could be simplified to $y' = 2(2x+1)^4(x^4-3)^5(29x^4+12x^3-15)$, but this is unnecessary.]

37. $y = \dfrac{\sin^2 x\,\tan^4 x}{(x^2+1)^2} \;\Rightarrow\; \ln y = \ln(\sin^2 x\,\tan^4 x) - \ln(x^2+1)^2 \;\Rightarrow$

$\ln y = \ln(\sin x)^2 + \ln(\tan x)^4 - \ln(x^2+1)^2 \;\Rightarrow\; \ln y = 2\ln|\sin x| + 4\ln|\tan x| - 2\ln(x^2+1) \;\Rightarrow$

$\dfrac{1}{y}y' = 2\cdot\dfrac{1}{\sin x}\cdot\cos x + 4\cdot\dfrac{1}{\tan x}\cdot\sec^2 x - 2\cdot\dfrac{1}{x^2+1}\cdot 2x \;\Rightarrow$

$y' = \dfrac{\sin^2 x\,\tan^4 x}{(x^2+1)^2}\left(2\cot x + \dfrac{4\sec^2 x}{\tan x} - \dfrac{4x}{x^2+1}\right)$

39. $y = x^x \;\Rightarrow\; \ln y = \ln x^x \;\Rightarrow\; \ln y = x\ln x \;\Rightarrow\; y'/y = x(1/x) + (\ln x)\cdot 1 \;\Rightarrow$

$y' = y(1+\ln x) \;\Rightarrow\; y' = x^x(1+\ln x)$

41. $y = x^{\sin x} \;\Rightarrow\; \ln y = \ln x^{\sin x} \;\Rightarrow\; \ln y = \sin x\ln x \;\Rightarrow\; \dfrac{y'}{y} = (\sin x)\cdot\dfrac{1}{x} + (\ln x)(\cos x) \;\Rightarrow$

$y' = y\left(\dfrac{\sin x}{x} + \ln x\cos x\right) \;\Rightarrow\; y' = x^{\sin x}\left(\dfrac{\sin x}{x} + \ln x\,\cos x\right)$

43. $y = (\ln x)^x \;\Rightarrow\; \ln y = \ln(\ln x)^x \;\Rightarrow\; \ln y = x\ln\ln x \;\Rightarrow\; \dfrac{y'}{y} = x\cdot\dfrac{1}{\ln x}\cdot\dfrac{1}{x} + (\ln\ln x)\cdot 1 \;\Rightarrow$

$y' = y\left(\dfrac{x}{x\ln x} + \ln\ln x\right) \;\Rightarrow\; y' = (\ln x)^x\left(\dfrac{1}{\ln x} + \ln\ln x\right)$

45. $y = x^{e^x} \;\Rightarrow\; \ln y = e^x\ln x \;\Rightarrow\; \dfrac{y'}{y} = e^x\cdot\dfrac{1}{x} + (\ln x)\cdot e^x \;\Rightarrow\; y' = x^{e^x}e^x\left(\ln x + \dfrac{1}{x}\right)$

47. $y = \ln(x^2+y^2) \;\Rightarrow\; y' = \dfrac{1}{x^2+y^2}\dfrac{d}{dx}(x^2+y^2) \;\Rightarrow\; y' = \dfrac{2x+2yy'}{x^2+y^2} \;\Rightarrow\; x^2y' + y^2y' = 2x + 2yy'$

$\Rightarrow\; x^2y' + y^2y' - 2yy' = 2x \;\Rightarrow\; (x^2+y^2-2y)y' = 2x \;\Rightarrow\; y' = \dfrac{2x}{x^2+y^2-2y}$

49. $f(x) = \ln(x-1) \Rightarrow f'(x) = 1/(x-1) = (x-1)^{-1} \Rightarrow f''(x) = -(x-1)^{-2} \Rightarrow$
$f'''(x) = 2(x-1)^{-3} \Rightarrow f^{(4)}(x) = -2 \cdot 3(x-1)^{-4} \Rightarrow \cdots \Rightarrow$
$f^{(n)}(x) = (-1)^{n-1} \cdot 2 \cdot 3 \cdot 4 \cdot \cdots \cdot (n-1)(x-1)^{-n} = (-1)^{n-1}\dfrac{(n-1)!}{(x-1)^n}$

51. If $f(x) = \ln(1+x)$, then $f'(x) = \dfrac{1}{1+x}$, so $f'(0) = 1$.

Thus, $\displaystyle\lim_{x\to 0}\frac{\ln(1+x)}{x} = \lim_{x\to 0}\frac{f(x)}{x} = \lim_{x\to 0}\frac{f(x)-f(0)}{x-0} = f'(0) = 1.$

3.9 Hyperbolic Functions

1. (a) $\sinh 0 = \frac{1}{2}(e^0 - e^0) = 0$ (b) $\cosh 0 = \frac{1}{2}(e^0 + e^0) = \frac{1}{2}(1+1) = 1$

3. (a) $\sinh(\ln 2) = \dfrac{e^{\ln 2} - e^{-\ln 2}}{2} = \dfrac{e^{\ln 2} - (e^{\ln 2})^{-1}}{2} = \dfrac{2 - 2^{-1}}{2} = \dfrac{2 - \frac{1}{2}}{2} = \dfrac{3}{4}$

(b) $\sinh 2 = \frac{1}{2}(e^2 - e^{-2}) \approx 3.62686$

5. (a) $\operatorname{sech} 0 = \dfrac{1}{\cosh 0} = \dfrac{1}{1} = 1$ (b) $\cosh^{-1} 1 = 0$ because $\cosh 0 = 1$.

7. $\sinh(-x) = \frac{1}{2}\left[e^{-x} - e^{-(-x)}\right] = \frac{1}{2}(e^{-x} - e^x) = -\frac{1}{2}(e^x - e^{-x}) = -\sinh x$

9. $\cosh x + \sinh x = \frac{1}{2}(e^x + e^{-x}) + \frac{1}{2}(e^x - e^{-x}) = \frac{1}{2}(2e^x) = e^x$

11. $\sinh x \cosh y + \cosh x \sinh y = \left[\frac{1}{2}(e^x - e^{-x})\right]\left[\frac{1}{2}(e^y + e^{-y})\right] + \left[\frac{1}{2}(e^x + e^{-x})\right]\left[\frac{1}{2}(e^y - e^{-y})\right]$

$= \frac{1}{4}\left[(e^{x+y} + e^{x-y} - e^{-x+y} - e^{-x-y}) + (e^{x+y} - e^{x-y} + e^{-x+y} - e^{-x-y})\right]$

$= \frac{1}{4}(2e^{x+y} - 2e^{-x-y}) = \frac{1}{2}\left[e^{x+y} - e^{-(x+y)}\right] = \sinh(x+y)$

13. Divide both sides of the identity $\cosh^2 x - \sinh^2 x = 1$ by $\sinh^2 x$:

$\dfrac{\cosh^2 x}{\sinh^2 x} - \dfrac{\sinh^2 x}{\sinh^2 x} = \dfrac{1}{\sinh^2 x} \Leftrightarrow \coth^2 x - 1 = \operatorname{csch}^2 x.$

15. Putting $y = x$ in the result from Exercise 11, we have
$\sinh 2x = \sinh(x+x) = \sinh x \cosh x + \cosh x \sinh x = 2 \sinh x \cosh x.$

17. $\tanh(\ln x) = \dfrac{\sinh(\ln x)}{\cosh(\ln x)} = \dfrac{(e^{\ln x} - e^{-\ln x})/2}{(e^{\ln x} + e^{-\ln x})/2} = \dfrac{x - (e^{\ln x})^{-1}}{x + (e^{\ln x})^{-1}} = \dfrac{x - x^{-1}}{x + x^{-1}}$

$= \dfrac{x - 1/x}{x + 1/x} = \dfrac{(x^2-1)/x}{(x^2+1)/x} = \dfrac{x^2-1}{x^2+1}$

19. By Exercise 9, $(\cosh x + \sinh x)^n = (e^x)^n = e^{nx} = \cosh nx + \sinh nx$.

21. $\tanh x = \frac{4}{5} > 0$, so $x > 0$. $\coth x = 1/\tanh x = \frac{5}{4}$, $\operatorname{sech}^2 x = 1 - \tanh^2 x = 1 - \left(\frac{4}{5}\right)^2 = \frac{9}{25} \Rightarrow$
$\operatorname{sech} x = \frac{3}{5}$ (since $\operatorname{sech} x > 0$), $\cosh x = 1/\operatorname{sech} x = \frac{5}{3}$, $\sinh x = \tanh x \cosh x = \frac{4}{5} \cdot \frac{5}{3} = \frac{4}{3}$, and
$\operatorname{csch} x = 1/\sinh x = \frac{3}{4}$.

23. (a) $\displaystyle\lim_{x\to\infty} \tanh x = \lim_{x\to\infty}\frac{e^x - e^{-x}}{e^x + e^{-x}} \cdot \frac{e^{-x}}{e^{-x}} = \lim_{x\to\infty}\frac{1 - e^{-2x}}{1 + e^{-2x}} = \frac{1-0}{1+0} = 1$

(b) $\displaystyle\lim_{x\to-\infty} \tanh x = \lim_{x\to-\infty}\frac{e^x - e^{-x}}{e^x + e^{-x}} \cdot \frac{e^x}{e^x} = \lim_{x\to-\infty}\frac{e^{2x} - 1}{e^{2x} + 1} = \frac{0-1}{0+1} = -1$

(c) $\displaystyle\lim_{x\to\infty} \sinh x = \lim_{x\to\infty}\frac{e^x - e^{-x}}{2} = \infty$

(d) $\lim_{x\to-\infty} \sinh x = \lim_{x\to-\infty} \dfrac{e^x - e^{-x}}{2} = -\infty$

(e) $\lim_{x\to\infty} \operatorname{sech} x = \lim_{x\to\infty} \dfrac{2}{e^x + e^{-x}} = 0$

(f) $\lim_{x\to\infty} \coth x = \lim_{x\to\infty} \dfrac{e^x + e^{-x}}{e^x - e^{-x}} \cdot \dfrac{e^{-x}}{e^{-x}} = \lim_{x\to\infty} \dfrac{1 + e^{-2x}}{1 - e^{-2x}} = \dfrac{1+0}{1-0} = 1$ [*Or:* Use part (a)]

(g) $\lim_{x\to0^+} \coth x = \lim_{x\to0^+} \dfrac{\cosh x}{\sinh x} = \infty$, since $\sinh x \to 0$ through positive values and $\cosh x \to 1$.

(h) $\lim_{x\to0^-} \coth x = \lim_{x\to0^-} \dfrac{\cosh x}{\sinh x} = -\infty$. since $\sinh x \to 0$ through negative values and $\cosh x \to 1$.

(i) $\lim_{x\to-\infty} \operatorname{csch} x = \lim_{x\to-\infty} \dfrac{2}{e^x - e^{-x}} = 0$

25. Let $y = \sinh^{-1} x$. Then $\sinh y = x$ and, by Example 1(a), $\cosh^2 y - \sinh^2 y = 1 \Rightarrow$ [with $\cosh y > 0$] $\cosh y = \sqrt{1 + \sinh^2 y} = \sqrt{1 + x^2}$. So by Exercise 9, $e^y = \sinh y + \cosh y = x + \sqrt{1+x^2} \Rightarrow$ $y = \ln\left(x + \sqrt{1+x^2}\right)$.

27. (a) Let $y = \tanh^{-1} x$. Then $x = \tanh y = \dfrac{\sinh y}{\cosh y} = \dfrac{(e^y - e^{-y})/2}{(e^y + e^{-y})/2} \cdot \dfrac{e^y}{e^y} = \dfrac{e^{2y} - 1}{e^{2y} + 1} \Rightarrow$

$xe^{2y} + x = e^{2y} - 1 \Rightarrow 1 + x = e^{2y} - xe^{2y} \Rightarrow 1 + x = e^{2y}(1-x) \Rightarrow$

$e^{2y} = \dfrac{1+x}{1-x} \Rightarrow 2y = \ln\left(\dfrac{1+x}{1-x}\right) \Rightarrow y = \frac{1}{2}\ln\left(\dfrac{1+x}{1-x}\right)$.

(b) Let $y = \tanh^{-1} x$. Then $x = \tanh y$, so from Exercise 18 we have

$e^{2y} = \dfrac{1 + \tanh y}{1 - \tanh y} = \dfrac{1+x}{1-x} \Rightarrow 2y = \ln\left(\dfrac{1+x}{1-x}\right) \Rightarrow y = \frac{1}{2}\ln\left(\dfrac{1+x}{1-x}\right)$.

29. (a) Let $y = \cosh^{-1} x$. Then $\cosh y = x$ and $y \geq 0 \Rightarrow \sinh y \dfrac{dy}{dx} = 1 \Rightarrow$

$\dfrac{dy}{dx} = \dfrac{1}{\sinh y} = \dfrac{1}{\sqrt{\cosh^2 y - 1}} = \dfrac{1}{\sqrt{x^2 - 1}}$ (since $\sinh y \geq 0$ for $y \geq 0$). *Or:* Use Formula 4.

(b) Let $y = \tanh^{-1} x$. Then $\tanh y = x \Rightarrow \operatorname{sech}^2 y \dfrac{dy}{dx} = 1 \Rightarrow \dfrac{dy}{dx} = \dfrac{1}{\operatorname{sech}^2 y} = \dfrac{1}{1 - \tanh^2 y} = \dfrac{1}{1 - x^2}$.

Or: Use Formula 5.

(c) Let $y = \operatorname{csch}^{-1} x$. Then $\operatorname{csch} y = x \Rightarrow -\operatorname{csch} y \coth y \dfrac{dy}{dx} = 1 \Rightarrow \dfrac{dy}{dx} = -\dfrac{1}{\operatorname{csch} y \coth y}$.

By Exercise 13, $\coth y = \pm\sqrt{\operatorname{csch}^2 y + 1} = \pm\sqrt{x^2 + 1}$. If $x > 0$, then $\coth y > 0$, so $\coth y = \sqrt{x^2+1}$. If $x < 0$, then $\coth y < 0$, so $\coth y = -\sqrt{x^2+1}$. In either case we have

$\dfrac{dy}{dx} = -\dfrac{1}{\operatorname{csch} y \coth y} = -\dfrac{1}{|x|\sqrt{x^2+1}}$.

(d) Let $y = \operatorname{sech}^{-1} x$. Then $\operatorname{sech} y = x \Rightarrow -\operatorname{sech} y \tanh y \dfrac{dy}{dx} = 1 \Rightarrow$

$\dfrac{dy}{dx} = -\dfrac{1}{\operatorname{sech} y \tanh y} = -\dfrac{1}{\operatorname{sech} y\sqrt{1 - \operatorname{sech}^2 y}} = -\dfrac{1}{x\sqrt{1-x^2}}$. (Note that $y > 0$ and so $\tanh y > 0$.)

(e) Let $y = \coth^{-1} x$. Then $\coth y = x \Rightarrow -\operatorname{csch}^2 y \dfrac{dy}{dx} = 1 \Rightarrow$

$\dfrac{dy}{dx} = -\dfrac{1}{\operatorname{csch}^2 y} = \dfrac{1}{1 - \coth^2 y} = \dfrac{1}{1 - x^2}$ by Exercise 13.

31. $f(x) = x\cosh x \quad \Rightarrow \quad f'(x) = x\,(\cosh x)' + (\cosh x)(x)' = x\sinh x + \cosh x$

33. $h(x) = \sinh(x^2) \quad \Rightarrow \quad h'(x) = \cosh(x^2)\cdot 2x = 2x\cosh(x^2)$

35. $G(x) = \dfrac{1-\cosh x}{1+\cosh x} \quad \Rightarrow$

$$G'(x) = \frac{(1+\cosh x)(-\sinh x) - (1-\cosh x)(\sinh x)}{(1+\cosh x)^2}$$
$$= \frac{-\sinh x - \sinh x\cosh x - \sinh x + \sinh x\cosh x}{(1+\cosh x)^2} = \frac{-2\sinh x}{(1+\cosh x)^2}$$

37. $h(t) = \coth\sqrt{1+t^2} \quad \Rightarrow \quad h'(t) = -\operatorname{csch}^2\sqrt{1+t^2}\cdot\frac{1}{2}(1+t^2)^{-1/2}(2t) = -\dfrac{t\operatorname{csch}^2\sqrt{1+t^2}}{\sqrt{1+t^2}}$

39. $H(t) = \tanh(e^t) \quad \Rightarrow \quad H'(t) = \operatorname{sech}^2(e^t)\cdot e^t = e^t\operatorname{sech}^2(e^t)$

41. $y = e^{\cosh 3x} \quad \Rightarrow \quad y' = e^{\cosh 3x}\cdot\sinh 3x\cdot 3 = 3e^{\cosh 3x}\sinh 3x$

43. $y = \tanh^{-1}\sqrt{x} \quad \Rightarrow \quad y' = \dfrac{1}{1-(\sqrt{x})^2}\cdot\frac{1}{2}x^{-1/2} = \dfrac{1}{2\sqrt{x}(1-x)}$

45. $y = x\sinh^{-1}(x/3) - \sqrt{9+x^2} \quad \Rightarrow$

$$y' = \sinh^{-1}\left(\frac{x}{3}\right) + x\frac{1/3}{\sqrt{1+(x/3)^2}} - \frac{2x}{2\sqrt{9+x^2}} = \sinh^{-1}\left(\frac{x}{3}\right) + \frac{x}{\sqrt{9+x^2}} - \frac{x}{\sqrt{9+x^2}} = \sinh^{-1}\left(\frac{x}{3}\right)$$

47. $y = \coth^{-1}\sqrt{x^2+1} \quad \Rightarrow \quad y' = \dfrac{1}{1-(x^2+1)}\dfrac{2x}{2\sqrt{x^2+1}} = -\dfrac{1}{x\sqrt{x^2+1}}$

49. (a) $y = 20\cosh(x/20) - 15 \quad \Rightarrow \quad y' = 20\sinh(x/20)\cdot\frac{1}{20} = \sinh(x/20)$. Since the right pole is positioned at $x = 7$, we have $y'(7) = \sinh\frac{7}{20} \approx 0.3572$.

(b) If α is the angle between the tangent line and the x-axis, then $\tan\alpha =$ slope of the line $= \sinh\frac{7}{20}$, so $\alpha = \tan^{-1}\left(\sinh\frac{7}{20}\right) \approx 0.343$ rad $\approx 19.66^\circ$. Thus, the angle between the line and the pole is $\theta = 90^\circ - \alpha \approx 70.34^\circ$.

51. (a) $y = A\sinh mx + B\cosh mx \quad \Rightarrow \quad y' = mA\cosh mx + mB\sinh mx \quad \Rightarrow$
$y'' = m^2A\sinh mx + m^2B\cosh mx = m^2(A\sinh mx + B\cosh mx) = m^2y$

(b) From part (a), a solution of $y'' = 9y$ is $y(x) = A\sinh 3x + B\cosh 3x$. So $-4 = y(0) = A\sinh 0 + B\cosh 0 = B$, so $B = -4$. Now $y'(x) = 3A\cosh 3x - 12\sinh 3x \quad \Rightarrow$ $6 = y'(0) = 3A \quad \Rightarrow \quad A = 2$, so $y = 2\sinh 3x - 4\cosh 3x$.

53. The tangent to $y = \cosh x$ has slope 1 when $y' = \sinh x = 1 \quad \Rightarrow \quad x = \sinh^{-1}1 = \ln(1+\sqrt{2})$, by Equation 3.

Since $\sinh x = 1$ and $y = \cosh x = \sqrt{1+\sinh^2 x}$, we have $\cosh x = \sqrt{2}$. The point is $\left(\ln(1+\sqrt{2}), \sqrt{2}\right)$.

55. If $ae^x + be^{-x} = \alpha\cosh(x+\beta)$ [or $\alpha\sinh(x+\beta)$], then

$ae^x + be^{-x} = \frac{\alpha}{2}\left(e^{x+\beta} \pm e^{-x-\beta}\right) = \frac{\alpha}{2}\left(e^xe^\beta \pm e^{-x}e^{-\beta}\right) = \left(\frac{\alpha}{2}e^\beta\right)e^x \pm \left(\frac{\alpha}{2}e^{-\beta}\right)e^{-x}$. Comparing coefficients of e^x and e^{-x}, we have $a = \frac{\alpha}{2}e^\beta$ **(1)** and $b = \pm\frac{\alpha}{2}e^{-\beta}$ **(2)**. We need to find α and β. Dividing equation **(1)** by equation (2) gives us $\frac{a}{b} = \pm e^{2\beta} \Rightarrow (*)\ \ 2\beta = \ln(\pm\frac{a}{b}) \Rightarrow \beta = \frac{1}{2}\ln(\pm\frac{a}{b})$. Solving equations (1) and (2) for e^β gives us $e^\beta = \frac{2a}{\alpha}$ and $e^\beta = \pm\frac{\alpha}{2b}$, so $\frac{2a}{\alpha} = \pm\frac{\alpha}{2b} \Rightarrow \alpha^2 = \pm 4ab \Rightarrow \alpha = 2\sqrt{\pm ab}$.

(*) If $\frac{a}{b} > 0$, we use the + sign and obtain a cosh function, whereas if $\frac{a}{b} < 0$, we use the − sign and obtain a sinh function.

In summary, if a and b have the same sign, we have $ae^x + be^{-x} = 2\sqrt{ab}\cosh(x + \frac{1}{2}\ln\frac{a}{b})$, whereas, if a and b have the opposite sign, then $ae^x + be^{-x} = 2\sqrt{-ab}\sinh(x + \frac{1}{2}\ln(-\frac{a}{b}))$.

3.10 Related Rates

1. $V = x^3 \Rightarrow \frac{dV}{dt} = \frac{dV}{dx}\frac{dx}{dt} = 3x^2\frac{dx}{dt}$

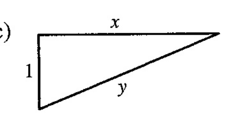
3. $y = x^3 + 2x \Rightarrow \frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt} = (3x^2 + 2)(5) = 5(3x^2 + 2)$. When $x = 2$, $\frac{dy}{dt} = 5(14) = 70$.

5. $z^2 = x^2 + y^2 \Rightarrow 2z\frac{dz}{dt} = 2x\frac{dx}{dt} + 2y\frac{dy}{dt} \Rightarrow \frac{dz}{dt} = \frac{1}{z}\left(x\frac{dx}{dt} + y\frac{dy}{dt}\right)$. When $x = 5$ and $y = 12$, $z^2 = 5^2 + 12^2 \Rightarrow z^2 = 169 \Rightarrow z = \pm 13$. For $\frac{dx}{dt} = 2$ and $\frac{dy}{dt} = 3$, $\frac{dz}{dt} = \frac{1}{\pm 13}(5 \cdot 2 + 12 \cdot 3) = \pm\frac{46}{13}$.

7. (a) Given: a plane flying horizontally at an altitude of 1 mi and a speed of 500 mi/h passes directly over a radar station. If we let t be time (in hours) and x be the horizontal distance traveled by the plane (in mi), then we are given that $dx/dt = 500$ mi/h.

(b) Unknown: the rate at which the distance from the plane to the station is increasing when it is 2 mi from the station. If we let y be the distance from the plane to the station, then we want to find dy/dt when $y = 2$ mi.

(c)



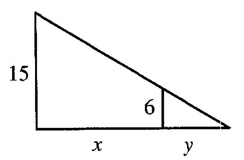
(d) By the Pythagorean Theorem, $y^2 = x^2 + 1 \Rightarrow 2y(dy/dt) = 2x(dx/dt)$.

(e) $\frac{dy}{dt} = \frac{x}{y}\frac{dx}{dt} = \frac{x}{y}(500)$. Since $y^2 = x^2 + 1$, when $y = 2$, $x = \sqrt{3}$, so $\frac{dy}{dt} = \frac{\sqrt{3}}{2}(500) = 250\sqrt{3} \approx 433$ mi/h.

9. (a) Given: a man 6 ft tall walks away from a street light mounted on a 15-ft-tall pole at a rate of 5 ft/s. If we let t be time (in s) and x be the distance from the pole to the man (in ft), then we are given that $dx/dt = 5$ ft/s.

(b) Unknown: the rate at which the tip of his shadow is moving when he is 40 ft from the pole. If we let y be the distance from the man to the tip of his shadow (in ft), then we want to find $\frac{d}{dt}(x + y)$ when $x = 40$ ft.

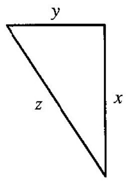
(c)



(d) By similar triangles, $\frac{15}{6} = \frac{x + y}{y} \Rightarrow 15y = 6x + 6y \Rightarrow 9y = 6x \Rightarrow y = \frac{2}{3}x$.

(e) The tip of the shadow moves at a rate of $\frac{d}{dt}(x + y) = \frac{d}{dt}(x + \frac{2}{3}x) = \frac{5}{3}\frac{dx}{dt} = \frac{5}{3}(5) = \frac{25}{3}$ ft/s.

11.



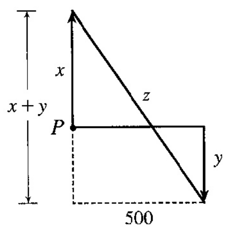
We are given that $\frac{dx}{dt} = 60$ mi/h and $\frac{dy}{dt} = 25$ mi/h. $z^2 = x^2 + y^2 \Rightarrow$

$2z\frac{dz}{dt} = 2x\frac{dx}{dt} + 2y\frac{dy}{dt} \Rightarrow z\frac{dz}{dt} = x\frac{dx}{dt} + y\frac{dy}{dt} \Rightarrow \frac{dz}{dt} = \frac{1}{z}\left(x\frac{dx}{dt} + y\frac{dy}{dt}\right).$

After 2 hours, $x = 2(60) = 120$ and $y = 2(25) = 50 \Rightarrow z = \sqrt{120^2 + 50^2} = 130$,

so $\frac{dz}{dt} = \frac{1}{z}\left(x\frac{dx}{dt} + y\frac{dy}{dt}\right) = \frac{120(60) + 50(25)}{130} = 65$ mi/h.

13.



We are given that $\frac{dx}{dt} = 4$ ft/s and $\frac{dy}{dt} = 5$ ft/s. $z^2 = (x+y)^2 + 500^2 \Rightarrow$

$2z\frac{dz}{dt} = 2(x+y)\left(\frac{dx}{dt} + \frac{dy}{dt}\right)$. 15 minutes after the woman starts, we have

$x = (4\text{ ft/s})(20\text{ min})(60\text{ s/min}) = 4800$ ft and $y = 5 \cdot 15 \cdot 60 = 4500 \Rightarrow$

$z = \sqrt{(4800 + 4500)^2 + 500^2} = \sqrt{86{,}740{,}000}$, so

$\frac{dz}{dt} = \frac{x+y}{z}\left(\frac{dx}{dt} + \frac{dy}{dt}\right) = \frac{4800 + 4500}{\sqrt{86{,}740{,}000}}(4+5) = \frac{837}{\sqrt{8674}} \approx 8.99$ ft/s.

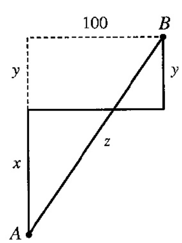
15. $A = \frac{1}{2}bh$, where b is the base and h is the altitude. We are given that $\frac{dh}{dt} = 1$ cm/min and $\frac{dA}{dt} = 2$ cm^2/min.

Using the Product Rule, we have $\frac{dA}{dt} = \frac{1}{2}\left(b\frac{dh}{dt} + h\frac{db}{dt}\right)$. When $h = 10$ and $A = 100$, we have

$100 = \frac{1}{2}b(10) \Rightarrow \frac{1}{2}b = 10 \Rightarrow b = 20$, so $2 = \frac{1}{2}\left(20 \cdot 1 + 10\frac{db}{dt}\right) \Rightarrow 4 = 20 + 10\frac{db}{dt} \Rightarrow$

$\frac{db}{dt} = \frac{4 - 20}{10} = -1.6$ cm/min.

17.



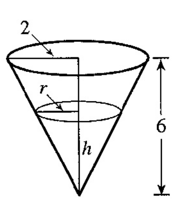
We are given that $\frac{dx}{dt} = 35$ km/h and $\frac{dy}{dt} = 25$ km/h. $z^2 = (x+y)^2 + 100^2$

$\Rightarrow 2z\frac{dz}{dt} = 2(x+y)\left(\frac{dx}{dt} + \frac{dy}{dt}\right)$. At 4:00 P.M., $x = 4(35) = 140$ and

$y = 4(25) = 100 \Rightarrow z = \sqrt{(140 + 100)^2 + 100^2} = \sqrt{67{,}600} = 260$, so

$\frac{dz}{dt} = \frac{x+y}{z}\left(\frac{dx}{dt} + \frac{dy}{dt}\right) = \frac{140 + 100}{260}(35 + 25) = \frac{720}{13} \approx 55.4$ km/h.

19.



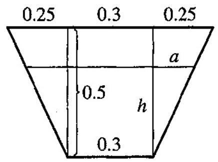
If $C =$ the rate at which water is pumped in, then $\frac{dV}{dt} = C - 10{,}000$, where

$V = \frac{1}{3}\pi r^2 h$ is the volume at time t. By similar triangles, $\frac{r}{2} = \frac{h}{6} \Rightarrow$

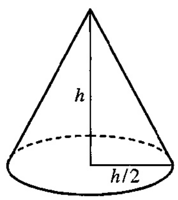
$r = \frac{1}{3}h \Rightarrow V = \frac{1}{3}\pi\left(\frac{1}{3}h\right)^2 h = \frac{\pi}{27}h^3 \Rightarrow \frac{dV}{dt} = \frac{\pi}{9}h^2\frac{dh}{dt}$.

When $h = 200$ cm, $\frac{dh}{dt} = 20$ cm/min, so $C - 10{,}000 = \frac{\pi}{9}(200)^2(20) \Rightarrow$

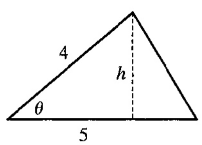
$C = 10{,}000 + \frac{800{,}000}{9}\pi \approx 289{,}253$ cm^3/min.

21. 

The figure is labeled in meters. The area A of a trapezoid is $\frac{1}{2}(\text{base}_1 + \text{base}_2)(\text{height})$, and the volume V of the 10-meter-long trough is $10A$. Thus, the volume of the trapezoid with height h is $V = (10)\frac{1}{2}\left[0.3 + (0.3 + 2a)\right] h$. By similar triangles, $\frac{a}{h} = \frac{0.25}{0.5} = \frac{1}{2}$, so $2a = h \Rightarrow V = 5(0.6 + h)h = 3h + 5h^2$. Now $\frac{dV}{dt} = \frac{dV}{dh}\frac{dh}{dt} \Rightarrow 0.2 = (3 + 10h)\frac{dh}{dt} \Rightarrow$ $\frac{dh}{dt} = \frac{0.2}{3 + 10h}$. When $h = 0.3$, $\frac{dh}{dt} = \frac{0.2}{3 + 10(0.3)} = \frac{0.2}{6}$ m/min $= \frac{1}{30}$ m/min or $\frac{10}{3}$ cm/min.

23. 

We are given that $\frac{dV}{dt} = 30\ \text{ft}^3/\text{min}$. $V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi\left(\frac{h}{2}\right)^2 h = \frac{\pi h^3}{12}$ $\Rightarrow \frac{dV}{dt} = \frac{dV}{dh}\frac{dh}{dt} \Rightarrow 30 = \frac{\pi h^2}{4}\frac{dh}{dt} \Rightarrow \frac{dh}{dt} = \frac{120}{\pi h^2}$. When $h = 10$ ft, $\frac{dh}{dt} = \frac{120}{10^2\pi} = \frac{6}{5\pi} \approx 0.38$ ft/min.

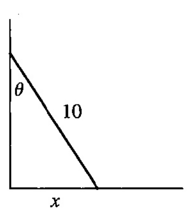
25. 

$A = \frac{1}{2}bh$, but $b = 5$ m and $\sin\theta = \frac{h}{4} \Rightarrow h = 4\sin\theta$, so $A = \frac{1}{2}(5)(4\sin\theta) = 10\sin\theta$. We are given $\frac{d\theta}{dt} = 0.06$ rad/s, so $\frac{dA}{dt} = \frac{dA}{d\theta}\frac{d\theta}{dt} = (10\cos\theta)(0.06) = 0.6\cos\theta$. When $\theta = \frac{\pi}{3}$, $\frac{dA}{dt} = 0.6\left(\cos\frac{\pi}{3}\right) = (0.6)\left(\frac{1}{2}\right) = 0.3\ \text{m}^2/\text{s}$.

27. Differentiating both sides of $PV = C$ with respect to t and using the Product Rule gives us $P\frac{dV}{dt} + V\frac{dP}{dt} = 0$ $\Rightarrow \frac{dV}{dt} = -\frac{V}{P}\frac{dP}{dt}$. When $V = 600$, $P = 150$ and $\frac{dP}{dt} = 20$, so we have $\frac{dV}{dt} = -\frac{600}{150}(20) = -80$. Thus, the volume is decreasing at a rate of 80 cm^3/min.

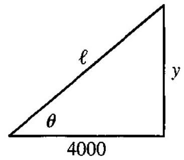
29. With $R_1 = 80$ and $R_2 = 100$, $\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2} = \frac{1}{80} + \frac{1}{100} = \frac{180}{8000} = \frac{9}{400}$, so $R = \frac{400}{9}$. Differentiating $\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2}$ with respect to t, we have $-\frac{1}{R^2}\frac{dR}{dt} = -\frac{1}{R_1^2}\frac{dR_1}{dt} - \frac{1}{R_2^2}\frac{dR_2}{dt} \Rightarrow$ $\frac{dR}{dt} = R^2\left(\frac{1}{R_1^2}\frac{dR_1}{dt} + \frac{1}{R_2^2}\frac{dR_2}{dt}\right)$. When $R_1 = 80$ and $R_2 = 100$,

$$\frac{dR}{dt} = \frac{400^2}{9^2}\left[\frac{1}{80^2}(0.3) + \frac{1}{100^2}(0.2)\right] = \frac{107}{810} \approx 0.132\ \Omega/\text{s}.$$

31. 

We are given that $\frac{dx}{dt} = 2$ ft/s. $\sin\theta = \frac{x}{10} \Rightarrow x = 10\sin\theta \Rightarrow$ $\frac{dx}{dt} = 10\cos\theta\frac{d\theta}{dt}$. When $\theta = \frac{\pi}{4}$, $2 = 10\cos\frac{\pi}{4}\frac{d\theta}{dt} \Rightarrow$ $\frac{d\theta}{dt} = \frac{2}{10(1/\sqrt{2})} = \frac{\sqrt{2}}{5}$ rad/s.

33. (a)



By the Pythagorean Theorem, $4000^2 + y^2 = \ell^2$. Differentiating with respect to t, we obtain $2y\,\frac{dy}{dt} = 2\ell\,\frac{d\ell}{dt}$. We know that $\frac{dy}{dt} = 600$ ft/s, so when $y = 3000$ ft, $\ell = \sqrt{4000^2 + 3000^2} = \sqrt{25{,}000{,}000} = 5000$ ft and

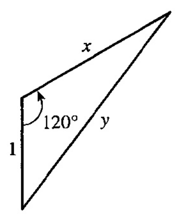
$$\frac{d\ell}{dt} = \frac{y}{\ell}\frac{dy}{dt} = \frac{3000}{5000}(600) = \frac{1800}{5} = 360 \text{ ft/s.}$$

(b) Here $\tan\theta = \frac{y}{4000}$ $\Rightarrow$ $\frac{d}{dt}(\tan\theta) = \frac{d}{dt}\left(\frac{y}{4000}\right)$ $\Rightarrow$ $\sec^2\theta\,\frac{d\theta}{dt} = \frac{1}{4000}\frac{dy}{dt}$ $\Rightarrow$ $\frac{d\theta}{dt} = \frac{\cos^2\theta}{4000}\frac{dy}{dt}$.

When $y = 3000$ ft, $\frac{dy}{dt} = 600$ ft/s, $\ell = 5000$ and $\cos\theta = \frac{4000}{\ell} = \frac{4000}{5000} = \frac{4}{5}$, so

$$\frac{d\theta}{dt} = \frac{(4/5)^2}{4000}(600) = 0.096 \text{ rad/s.}$$

35.



We are given that $\frac{dx}{dt} = 300$ km/h. By the Law of Cosines,

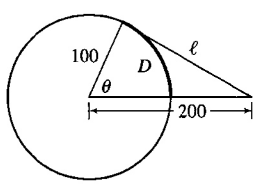
$y^2 = x^2 + 1^2 - 2(1)(x)\cos 120° = x^2 + 1 - 2x\left(-\frac{1}{2}\right) = x^2 + x + 1$, so

$2y\,\frac{dy}{dt} = 2x\,\frac{dx}{dt} + \frac{dx}{dt}$ $\Rightarrow$ $\frac{dy}{dt} = \frac{2x+1}{2y}\frac{dx}{dt}$. After 1 minute,

$x = \frac{300}{60} = 5$ km $\Rightarrow$ $y = \sqrt{5^2 + 5 + 1} = \sqrt{31}$ km $\Rightarrow$

$$\frac{dy}{dt} = \frac{2(5)+1}{2\sqrt{31}}(300) = \frac{1650}{\sqrt{31}} \approx 296 \text{ km/h.}$$

37.



Let the distance between the runner and the friend be ℓ. Then by the Law of Cosines,

$\ell^2 = 200^2 + 100^2 - 2\cdot 200\cdot 100\cdot\cos\theta = 50{,}000 - 40{,}000\cos\theta$ $(\star)$.

Differentiating implicitly with respect to t, we obtain

$2\ell\,\frac{d\ell}{dt} = -40{,}000(-\sin\theta)\,\frac{d\theta}{dt}$. Now if D is the distance run when the angle is θ radians, then by the formula for the length of an arc on a circle, $s = r\theta$, we have $D = 100\theta$, so

$\theta = \frac{1}{100}D$ $\Rightarrow$ $\frac{d\theta}{dt} = \frac{1}{100}\frac{dD}{dt} = \frac{7}{100}$. To substitute into the expression for $\frac{d\ell}{dt}$, we must know $\sin\theta$ at the time when $\ell = 200$, which we find from $(\star)$: $200^2 = 50{,}000 - 40{,}000\cos\theta$ $\Leftrightarrow$ $\cos\theta = \frac{1}{4}$ $\Rightarrow$

$\sin\theta = \sqrt{1 - \left(\frac{1}{4}\right)^2} = \frac{\sqrt{15}}{4}$. Substituting, we get $2(200)\,\frac{d\ell}{dt} = 40{,}000\frac{\sqrt{15}}{4}\left(\frac{7}{100}\right)$ $\Rightarrow$

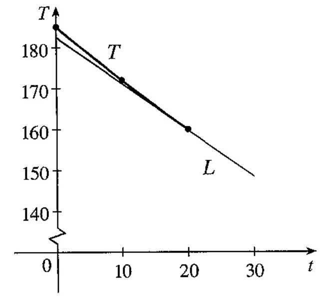
$d\ell/dt = \frac{7\sqrt{15}}{4} \approx 6.78$ m/s. Whether the distance between them is increasing or decreasing depends on the direction in which the runner is running.

3.11 Linear Approximations and Differentials

1. As in Example 1, $T(0) = 185$, $T(10) = 172$, $T(20) = 160$, and

$$T'(20) \approx \frac{T(10) - T(20)}{10 - 20} = \frac{172 - 160}{-10} = -1.2\ ^\circ\text{F/min}.$$

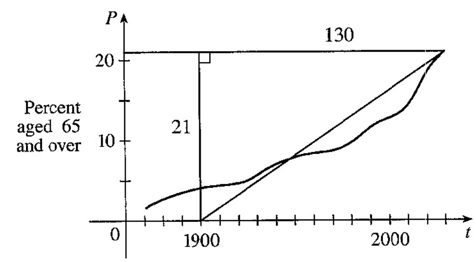
$T(30) \approx T(20) + T'(20)(30 - 20) \approx 160 - 1.2(10) = 148\ ^\circ\text{F}$.



We would expect the temperature of the turkey to get closer to 75 °F as time increases. Since the temperature decreased 13 °F in the first 10 minutes and 12 °F in the second 10 minutes, we can assume that the slopes of the tangent line are increasing through negative values: $-1.3, -1.2, \ldots$. Hence, the tangent lines are under the curve and 148 °F is an underestimate. From the figure, we estimate the slope of the tangent line at $t = 20$ to be $\frac{184-147}{0-30} = -\frac{37}{30}$.

Then the linear approximation becomes $T(30) \approx T(20) + T'(20) \cdot 10 \approx 160 - \frac{37}{30}(10) = 147\frac{2}{3} \approx 147.7$.

3. Extend the tangent line at the point (2030, 21) to the t-axis. Answers will vary based on this approximation—we'll use $t = 1900$ as our t-intercept. The linearization is then



$$P(t) \approx P(2030) + P'(2030)(t - 2030)$$
$$\approx 21 + \tfrac{21}{130}(t - 2030)$$

$P(2040) = 21 + \frac{21}{130}(2040 - 2030) \approx 22.6\%$

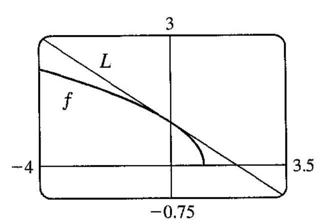
$P(2050) = 21 + \frac{21}{130}(2050 - 2030) \approx 24.2\%$

These predictions are probably too high since the tangent line lies above the graph at $t = 2030$.

5. $f(x) = x^3 \quad\Rightarrow\quad f'(x) = 3x^2$, so $f(1) = 1$ and $f'(1) = 3$. With $a = 1$, $L(x) = f(a) + f'(a)(x - a)$ becomes $L(x) = f(1) + f'(1)(x - 1) = 1 + 3(x - 1) = 3x - 2$.

7. $f(x) = \cos x \quad\Rightarrow\quad f'(x) = -\sin x$, so $f\left(\frac{\pi}{2}\right) = 0$ and $f'\left(\frac{\pi}{2}\right) = -1$. Thus,
$L(x) = f\left(\frac{\pi}{2}\right) + f'\left(\frac{\pi}{2}\right)\left(x - \frac{\pi}{2}\right) = 0 - 1\left(x - \frac{\pi}{2}\right) = -x + \frac{\pi}{2}$.

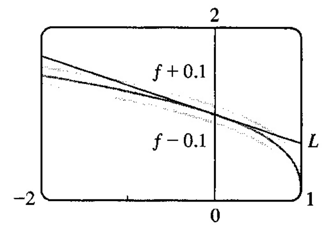
9. $f(x) = \sqrt{1-x} \quad\Rightarrow\quad f'(x) = \dfrac{-1}{2\sqrt{1-x}}$, so $f(0) = 1$ and $f'(0) = -\frac{1}{2}$. Therefore,



$$\sqrt{1-x} = f(x) \approx f(0) + f'(0)(x - 0)$$
$$= 1 + \left(-\tfrac{1}{2}\right)(x - 0) = 1 - \tfrac{1}{2}x$$

So $\sqrt{0.9} = \sqrt{1 - 0.1} \approx 1 - \frac{1}{2}(0.1) = 0.95$ and
$\sqrt{0.99} = \sqrt{1 - 0.01} \approx 1 - \frac{1}{2}(0.01) = 0.995$.

11. $f(x) = \sqrt[3]{1-x} = (1-x)^{1/3} \Rightarrow f'(x) = -\frac{1}{3}(1-x)^{-2/3}$, so $f(0) = 1$ and $f'(0) = -\frac{1}{3}$. Thus, $f(x) \approx f(0) + f'(0)(x-0) = 1 - \frac{1}{3}x$. We need $\sqrt[3]{1-x} - 0.1 < 1 - \frac{1}{3}x < \sqrt[3]{1-x} + 0.1$, which is true when $-1.204 < x < 0.706$.

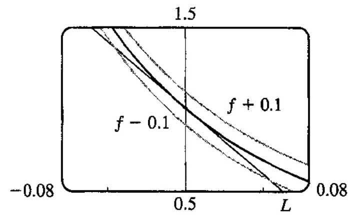


13. $f(x) = \dfrac{1}{(1+2x)^4} = (1+2x)^{-4} \Rightarrow$

$f'(x) = -4(1+2x)^{-5}(2) = \dfrac{-8}{(1+2x)^5}$, so $f(0) = 1$ and $f'(0) = -8$.

Thus, $f(x) \approx f(0) + f'(0)(x-0) = 1 + (-8)(x-0) = 1 - 8x$.

We need $1/(1+2x)^4 - 0.1 < 1 - 8x < 1/(1+2x)^4 + 0.1$, which is true when $-0.045 < x < 0.055$.



15. If $y = f(x)$, then the differential dy is equal to $f'(x)\,dx$. $y = x^4 + 5x \Rightarrow dy = (4x^3 + 5)\,dx$.

17. $y = x \ln x \Rightarrow dy = \left(x \cdot \dfrac{1}{x} + \ln x \cdot 1\right) dx = (1 + \ln x)\,dx$

19. $y = \dfrac{u+1}{u-1} \Rightarrow dy = \dfrac{(u-1)(1) - (u+1)(1)}{(u-1)^2}\,du = \dfrac{-2}{(u-1)^2}\,du$

21. (a) $y = x^2 + 2x \Rightarrow dy = (2x+2)\,dx$

(b) When $x = 3$ and $dx = \frac{1}{2}$, $dy = [2(3) + 2]\left(\frac{1}{2}\right) = 4$.

23. (a) $y = \sqrt{4+5x} \Rightarrow dy = \frac{1}{2}(4+5x)^{-1/2} \cdot 5\,dx = \dfrac{5}{2\sqrt{4+5x}}\,dx$

(b) When $x = 0$ and $dx = 0.04$, $dy = \dfrac{5}{2\sqrt{4}}(0.04) = \frac{5}{4} \cdot \frac{1}{25} = \frac{1}{20} = 0.05$.

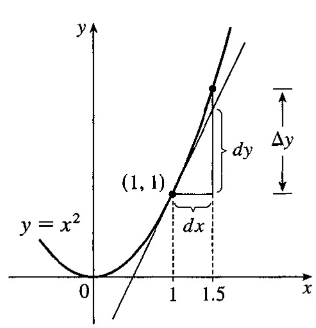
25. (a) $y = \tan x \Rightarrow dy = \sec^2 x\,dx$

(b) When $x = \pi/4$ and $dx = -0.1$, $dy = [\sec(\pi/4)]^2(-0.1) = \left(\sqrt{2}\right)^2(-0.1) = -0.2$.

27. $y = x^2$, $x = 1$, $\Delta x = 0.5 \Rightarrow$

$\Delta y = (1.5)^2 - 1^2 = 1.25$.

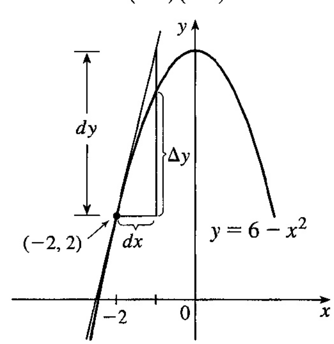
$dy = 2x\,dx = 2(1)(0.5) = 1$



29. $y = 6 - x^2$, $x = -2$, $\Delta x = 0.4 \Rightarrow$

$\Delta y = \left(6 - (-1.6)^2\right) - \left(6 - (-2)^2\right) = 1.44$

$dy = -2x\,dx = -2(-2)(0.4) = 1.6$



31. $y = f(x) = x^5 \Rightarrow dy = 5x^4\,dx$. When $x = 2$ and $dx = 0.001$, $dy = 5(2)^4(0.001) = 0.08$, so $(2.001)^5 = f(2.001) \approx f(2) + dy = 32 + 0.08 = 32.08$.

33. $y = f(x) = x^{2/3} \Rightarrow dy = \dfrac{2}{3\sqrt[3]{x}}\,dx$. When $x = 8$ and $dx = 0.06$, $dy = \dfrac{2}{3\sqrt[3]{8}}(0.06) = 0.02$, so $(8.06)^{2/3} = f(8.06) \approx f(8) + dy = 4 + 0.02 = 4.02$.

35. $y = f(x) = \tan x \Rightarrow dy = \sec^2 x\,dx$. When $x = 45°$ and $dx = -1°$,
$dy = \sec^2 45°(-\pi/180) = \left(\sqrt{2}\right)^2(-\pi/180) = -\pi/90$, so
$\tan 44° = f(44°) \approx f(45°) + dy = 1 - \pi/90 \approx 0.965$.

37. $y = f(x) = \sec x \Rightarrow f'(x) = \sec x\,\tan x$, so $f(0) = 1$ and $f'(0) = 1 \cdot 0 = 0$. The linear approximation of f at 0 is $f(0) + f'(0)(x - 0) = 1 + 0(x) = 1$. Since 0.08 is close to 0, approximating sec 0.08 with 1 is reasonable.

39. $y = f(x) = \ln x \Rightarrow f'(x) = 1/x$, so $f(1) = 0$ and $f'(1) = 1$. The linear approximation of f at 1 is $f(1) + f'(1)(x - 1) = 0 + 1(x - 1) = x - 1$. Now $f(1.05) = \ln 1.05 \approx 1.05 - 1 = 0.05$, so the approximation is reasonable.

41. (a) If x is the edge length, then $V = x^3 \Rightarrow dV = 3x^2\,dx$. When $x = 30$ and $dx = 0.1$, $dV = 3(30)^2(0.1) = 270$, so the maximum possible error in computing the volume of the cube is about 270 cm^3. The relative error is calculated by dividing the change in V, ΔV, by V. We approximate ΔV with dV.

Relative error $= \dfrac{\Delta V}{V} \approx \dfrac{dV}{V} = \dfrac{3x^2\,dx}{x^3} = 3\,\dfrac{dx}{x} = 3\left(\dfrac{0.1}{30}\right) = 0.01$.

Percentage error = relative error $\times$ 100% = 0.01 $\times$ 100% = 1%.

(b) $S = 6x^2 \Rightarrow dS = 12x\,dx$. When $x = 30$ and $dx = 0.1$, $dS = 12(30)(0.1) = 36$, so the maximum possible error in computing the surface area of the cube is about 36 cm^2.

Relative error $= \dfrac{\Delta S}{S} \approx \dfrac{dS}{S} = \dfrac{12x\,dx}{6x^2} = 2\,\dfrac{dx}{x} = 2\left(\dfrac{0.1}{30}\right) = 0.00\overline{6}$.

Percentage error = relative error $\times$ 100% = $0.00\overline{6} \times 100\% = 0.\overline{6}\%$.

43. (a) For a sphere of radius r, the circumference is $C = 2\pi r$ and the surface area is $S = 4\pi r^2$, so $r = C/(2\pi) \Rightarrow$ $S = 4\pi(C/2\pi)^2 = C^2/\pi \Rightarrow dS = (2/\pi)C\,dC$. When $C = 84$ and $dC = 0.5$, $dS = \dfrac{2}{\pi}(84)(0.5) = \dfrac{84}{\pi}$, so the maximum error is about $\dfrac{84}{\pi} \approx 27$ cm^2. Relative error $\approx \dfrac{dS}{S} = \dfrac{84/\pi}{84^2/\pi} = \dfrac{1}{84} \approx 0.012$

(b) $V = \dfrac{4}{3}\pi r^3 = \dfrac{4}{3}\pi\left(\dfrac{C}{2\pi}\right)^3 = \dfrac{C^3}{6\pi^2} \Rightarrow dV = \dfrac{1}{2\pi^2}C^2\,dC$. When $C = 84$ and $dC = 0.5$, $dV = \dfrac{1}{2\pi^2}(84)^2(0.5) = \dfrac{1764}{\pi^2}$, so the maximum error is about $\dfrac{1764}{\pi^2} \approx 179$ cm^3. The relative error is approximately $\dfrac{dV}{V} = \dfrac{1764/\pi^2}{(84)^3/(6\pi^2)} = \dfrac{1}{56} \approx 0.018$.

45. (a) $V = \pi r^2 h \Rightarrow \Delta V \approx dV = 2\pi rh\,dr = 2\pi rh\,\Delta r$

(b) The error is

$$\begin{aligned}\Delta V - dV &= [\pi(r + \Delta r)^2 h - \pi r^2 h] - 2\pi rh\,\Delta r = \pi r^2 h + 2\pi rh\,\Delta r + \pi(\Delta r)^2 h - \pi r^2 h - 2\pi rh\,\Delta r \\ &= \pi(\Delta r)^2 h\end{aligned}$$

47. (a) $dc = \frac{dc}{dx}\,dx = 0\,dx = 0$

(b) $d(cu) = \frac{d}{dx}(cu)\,dx = c\,\frac{du}{dx}\,dx = c\,du$

(c) $d(u+v) = \frac{d}{dx}(u+v)\,dx = \left(\frac{du}{dx} + \frac{dv}{dx}\right)dx = \frac{du}{dx}\,dx + \frac{dv}{dx}\,dx = du + dv$

(d) $d(uv) = \frac{d}{dx}(uv)\,dx = \left(u\,\frac{dv}{dx} + v\,\frac{du}{dx}\right)dx = u\,\frac{dv}{dx}\,dx + v\,\frac{du}{dx}\,dx = u\,dv + v\,du$

(e) $d\left(\frac{u}{v}\right) = \frac{d}{dx}\left(\frac{u}{v}\right)dx = \frac{v\,\frac{du}{dx} - u\,\frac{dv}{dx}}{v^2}\,dx = \frac{v\,\frac{du}{dx}\,dx - u\,\frac{dv}{dx}\,dx}{v^2} = \frac{v\,du - u\,dv}{v^2}$

(f) $d\,(x^n) = \frac{d}{dx}\,(x^n)\,dx = nx^{n-1}\,dx$

49. (a) The graph shows that $f'(1) = 2$, so $L(x) = f(1) + f'(1)(x-1) = 5 + 2(x-1) = 2x + 3$. $f(0.9) \approx L(0.9) = 4.8$ and $f(1.1) \approx L(1.1) = 5.2$.

(b) From the graph, we see that $f'(x)$ is positive and decreasing. This means that the slopes of the tangent lines are positive, but the tangents are becoming less steep. So the tangent lines lie *above* the curve. Thus, the estimates in part (a) are too large.

3 Review

CONCEPT CHECK

1. (a) The Power Rule: If n is any real number, then $\frac{d}{dx}(x^n) = nx^{n-1}$. The derivative of a variable base raised to a constant power is the power times the base raised to the power minus one.

(b) The Constant Multiple Rule: If c is a constant and f is a differentiable function, then $\frac{d}{dx}\,[cf(x)] = c\,\frac{d}{dx}\,f(x)$. The derivative of a constant times a function is the constant times the derivative of the function.

(c) The Sum Rule: If f and g are both differentiable, then $\frac{d}{dx}\,[f(x) + g(x)] = \frac{d}{dx}\,f(x) + \frac{d}{dx}\,g(x)$. The derivative of a sum of functions is the sum of the derivatives.

(d) The Difference Rule: If f and g are both differentiable, then $\frac{d}{dx}\,[f(x) - g(x)] = \frac{d}{dx}\,f(x) - \frac{d}{dx}\,g(x)$. The derivative of a difference of functions is the difference of the derivatives.

(e) The Product Rule: If f and g are both differentiable, then $\frac{d}{dx}\,[f(x)g(x)] = f(x)\,\frac{d}{dx}\,g(x) + g(x)\,\frac{d}{dx}\,f(x)$. The derivative of a product of two functions is the first function times the derivative of the second function plus the second function times the derivative of the first function.

(f) The Quotient Rule: If f and g are both differentiable, then $\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)\,\frac{d}{dx}\,f(x) - f(x)\,\frac{d}{dx}\,g(x)}{[g(x)]^2}$. The derivative of a quotient of functions is the denominator times the derivative of the numerator minus the numerator times the derivative of the denominator, all divided by the square of the denominator.

(g) The Chain Rule: If f and g are both differentiable and $F = f \circ g$ is the composite function defined by $F(x) = f(g(x))$, then F is differentiable and F' is given by the product $F'(x) = f'(g(x))g'(x)$. The derivative of a composite function is the derivative of the outer function evaluated at the inner function times the derivative of the inner function.

2. (a) $y = x^n \Rightarrow y' = nx^{n-1}$ (b) $y = e^x \Rightarrow y' = e^x$

(c) $y = a^x \Rightarrow y' = a^x \ln a$ (d) $y = \ln x \Rightarrow y' = 1/x$

(e) $y = \log_a x \Rightarrow y' = 1/(x \ln a)$ (f) $y = \sin x \Rightarrow y' = \cos x$

(g) $y = \cos x \Rightarrow y' = -\sin x$ (h) $y = \tan x \Rightarrow y' = \sec^2 x$

(i) $y = \csc x \Rightarrow y' = -\csc x \cot x$ (j) $y = \sec x \Rightarrow y' = \sec x \tan x$

(k) $y = \cot x \Rightarrow y' = -\csc^2 x$ (l) $y = \sin^{-1} x \Rightarrow y' = 1/\sqrt{1-x^2}$

(m) $y = \cos^{-1} x \Rightarrow y' = -1/\sqrt{1-x^2}$ (n) $y = \tan^{-1} x \Rightarrow y' = 1/(1+x^2)$

(o) $y = \sinh x \Rightarrow y' = \cosh x$ (p) $y = \cosh x \Rightarrow y' = \sinh x$

(q) $y = \tanh x \Rightarrow y' = \operatorname{sech}^2 x$ (r) $y = \sinh^{-1} x \Rightarrow y' = 1/\sqrt{1+x^2}$

(s) $y = \cosh^{-1} x \Rightarrow y' = 1/\sqrt{x^2-1}$ (t) $y = \tanh^{-1} x \Rightarrow y' = 1/(1-x^2)$

3. (a) e is the number such that $\lim\limits_{h\to 0} \dfrac{e^h - 1}{h} = 1$.

(b) $e = \lim\limits_{x\to 0} (1+x)^{1/x}$

(c) The differentiation formula for $y = a^x$ $[y' = a^x \ln a]$ is simplest when $a = e$ because $\ln e = 1$.

(d) The differentiation formula for $y = \log_a x$ $[y' = 1/(x \ln a)]$ is simplest when $a = e$ because $\ln e = 1$.

4. (a) Implicit differentiation consists of differentiating both sides of an equation involving x and y with respect to x, and then solving the resulting equation for y'.

(b) Logarithmic differentiation consists of taking natural logarithms of both sides of an equation $y = f(x)$, simplifying, differentiating implicitly with respect to x, and then solving the resulting equation for y'.

5. The second derivative of a function f is the rate of change of the first derivative f'. The third derivative is the derivative (rate of change) of the second derivative. If f is the position function of an object, f' is its velocity function, f'' is its acceleration function, and f''' is its jerk function.

6. (a) The linearization L of f at $x = a$ is $L(x) = f(a) + f'(a)(x-a)$.

(b) If $y = f(x)$, then the differential dy is given by $dy = f'(x)\,dx$.

(c) See Figure 6 in Section 3.11.

TRUE-FALSE QUIZ

1. True. This is the Sum Rule.

3. True. This is the Chain Rule.

5. False. $\dfrac{d}{dx} f(\sqrt{x}) = \dfrac{f'(\sqrt{x})}{2\sqrt{x}}$ by the Chain Rule.

7. False. $\dfrac{d}{dx} 10^x = 10^x \ln 10$

9. True. $\dfrac{d}{dx}(\tan^2 x) = 2 \tan x \sec^2 x$, and $\dfrac{d}{dx}(\sec^2 x) = 2 \sec x (\sec x \tan x) = 2 \tan x \sec^2 x$.

11. True. $g(x) = x^5 \Rightarrow g'(x) = 5x^4 \Rightarrow g'(2) = 5(2)^4 = 80$, and by the definition of the derivative,

$$\lim_{x\to 2} \frac{g(x) - g(2)}{x - 2} = g'(2) = 80.$$

13. False. A tangent line to the parabola $y = x^2$ has slope $dy/dx = 2x$, so at $(-2, 4)$ the slope of the tangent is $2(-2) = -4$ and an equation of the tangent line is $y - 4 = -4(x + 2)$. [The given equation, $y - 4 = 2x(x + 2)$, is not even linear!]

EXERCISES

1. $y = \left(x^4 - 3x^2 + 5\right)^3 \Rightarrow$

$$y' = 3\left(x^4 - 3x^2 + 5\right)^2 \frac{d}{dx}\left(x^4 - 3x^2 + 5\right) = 3\left(x^4 - 3x^2 + 5\right)^2\left(4x^3 - 6x\right) = 6x\left(x^4 - 3x^2 + 5\right)^2\left(2x^2 - 3\right)$$

3. $y = \sqrt{x} + \dfrac{1}{\sqrt[3]{x^4}} = x^{1/2} + x^{-4/3} \Rightarrow y' = \frac{1}{2}x^{-1/2} - \frac{4}{3}x^{-7/3} = \dfrac{1}{2\sqrt{x}} - \dfrac{4}{3\sqrt[3]{x^7}}$

5. $y = 2x\sqrt{x^2 + 1} \Rightarrow$

$$y' = 2x \cdot \tfrac{1}{2}\left(x^2 + 1\right)^{-1/2}(2x) + \sqrt{x^2 + 1}\,(2) = \frac{2x^2}{\sqrt{x^2 + 1}} + 2\sqrt{x^2 + 1} = \frac{2x^2 + 2\left(x^2 + 1\right)}{\sqrt{x^2 + 1}} = \frac{2\left(2x^2 + 1\right)}{\sqrt{x^2 + 1}}$$

7. $y = e^{\sin 2\theta} \Rightarrow y' = e^{\sin 2\theta}\dfrac{d}{d\theta}(\sin 2\theta) = e^{\sin 2\theta}(\cos 2\theta)(2) = 2\cos 2\theta\, e^{\sin 2\theta}$

9. $y = \dfrac{t}{1 - t^2} \Rightarrow y' = \dfrac{(1 - t^2)(1) - t(-2t)}{(1 - t^2)^2} = \dfrac{1 - t^2 + 2t^2}{(1 - t^2)^2} = \dfrac{t^2 + 1}{(1 - t^2)^2}$

11. $y = xe^{-1/x} \Rightarrow y' = xe^{-1/x}(1/x^2) + e^{-1/x} \cdot 1 = e^{-1/x}(1/x + 1)$

13. $y = \tan\sqrt{1 - x} \Rightarrow y' = \left(\sec^2\sqrt{1 - x}\right)\left(\dfrac{1}{2\sqrt{1 - x}}\right)(-1) = -\dfrac{\sec^2\sqrt{1 - x}}{2\sqrt{1 - x}}$

15. $\dfrac{d}{dx}\left(xy^4 + x^2y\right) = \dfrac{d}{dx}(x + 3y) \Rightarrow x \cdot 4y^3y' + y^4 \cdot 1 + x^2 \cdot y' + y \cdot 2x = 1 + 3y' \Rightarrow$

$y'\left(4xy^3 + x^2 - 3\right) = 1 - y^4 - 2xy \Rightarrow y' = \dfrac{1 - y^4 - 2xy}{4xy^3 + x^2 - 3}$

17. $y = \dfrac{\sec 2\theta}{1 + \tan 2\theta} \Rightarrow$

$$y' = \frac{(1 + \tan 2\theta)(\sec 2\theta\, \tan 2\theta \cdot 2) - (\sec 2\theta)(\sec^2 2\theta \cdot 2)}{(1 + \tan 2\theta)^2} = \frac{2\sec 2\theta\left[(1 + \tan 2\theta)\tan 2\theta - \sec^2 2\theta\right]}{(1 + \tan 2\theta)^2}$$

$$= \frac{2\sec 2\theta\left(\tan 2\theta + \tan^2 2\theta - \sec^2 2\theta\right)}{(1 + \tan 2\theta)^2} = \frac{2\sec 2\theta\,(\tan 2\theta - 1)}{(1 + \tan 2\theta)^2} \qquad \left[1 + \tan^2 x = \sec^2 x\right]$$

19. $y = e^{cx}(c\sin x - \cos x) \Rightarrow$

$y' = e^{cx}(c\cos x + \sin x) + ce^{cx}(c\sin x - \cos x)$

$= e^{cx}\left(c^2\sin x - c\cos x + c\cos x + \sin x\right) = e^{cx}\left(c^2\sin x + \sin x\right) = e^{cx}\sin x\left(c^2 + 1\right)$

21. $y = e^{e^x} \Rightarrow y' = e^{e^x}\dfrac{d}{dx}(e^x) = e^{e^x}e^x = e^{x + e^x}$

23. $y = \left(1 - x^{-1}\right)^{-1} \Rightarrow$

$y' = -1(1 - x^{-1})^{-2}[-(-1x^{-2})] = -(1 - 1/x)^{-2}x^{-2} = -((x - 1)/x)^{-2}x^{-2} = -(x - 1)^{-2}$

25. $\sin(xy) = x^2 - y \quad\Rightarrow\quad \cos(xy)(xy' + y \cdot 1) = 2x - y' \quad\Rightarrow\quad x\cos(xy)y' + y' = 2x - y\cos(xy) \quad\Rightarrow$
$y'[x\cos(xy) + 1] = 2x - y\cos(xy) \quad\Rightarrow\quad y' = \dfrac{2x - y\cos(xy)}{x\cos(xy) + 1}$

27. $y = \log_5(1 + 2x) \quad\Rightarrow\quad y' = \dfrac{1}{(1+2x)\ln 5}\dfrac{d}{dx}(1 + 2x) = \dfrac{2}{(1+2x)\ln 5}$

29. $y = \ln\sin x - \frac{1}{2}\sin^2 x \quad\Rightarrow\quad y' = \dfrac{1}{\sin x} \cdot \cos x - \frac{1}{2} \cdot 2\sin x \cdot \cos x = \cot x - \sin x\cos x$

31. $y = x\tan^{-1}(4x) \quad\Rightarrow\quad y' = x \cdot \dfrac{1}{1 + (4x)^2} \cdot 4 + \tan^{-1}(4x) \cdot 1 = \dfrac{4x}{1 + 16x^2} + \tan^{-1}(4x)$

33. $y = \ln|\sec 5x + \tan 5x| \quad\Rightarrow$
$y' = \dfrac{1}{\sec 5x + \tan 5x}(\sec 5x\,\tan 5x \cdot 5 + \sec^2 5x \cdot 5) = \dfrac{5\sec 5x\,(\tan 5x + \sec 5x)}{\sec 5x + \tan 5x} = 5\sec 5x$

35. $y = \cot(3x^2 + 5) \quad\Rightarrow\quad y' = -\csc^2(3x^2 + 5)(6x) = -6x\csc^2(3x^2 + 5)$

37. $y = \sin\left(\tan\sqrt{1 + x^3}\right) \quad\Rightarrow\quad y' = \cos\left(\tan\sqrt{1 + x^3}\right)\left(\sec^2\sqrt{1 + x^3}\right)\left[3x^2/\left(2\sqrt{1 + x^3}\right)\right]$

39. $y = \tan^2(\sin\theta) = [\tan(\sin\theta)]^2 \quad\Rightarrow\quad y' = 2[\tan(\sin\theta)] \cdot \sec^2(\sin\theta) \cdot \cos\theta$

41. $y = \dfrac{\sqrt{x+1}(2-x)^5}{(x+3)^7} \quad\Rightarrow\quad \ln y = \frac{1}{2}\ln(x+1) + 5\ln(2-x) - 7\ln(x+3) \quad\Rightarrow$
$\dfrac{y'}{y} = \dfrac{1}{2(x+1)} + \dfrac{-5}{2-x} - \dfrac{7}{x+3} \quad\Rightarrow\quad y' = \dfrac{\sqrt{x+1}(2-x)^5}{(x+3)^7}\left[\dfrac{1}{2(x+1)} - \dfrac{5}{2-x} - \dfrac{7}{x+3}\right]$ or
$y' = \dfrac{(2-x)^4(3x^2 - 55x - 52)}{2\sqrt{x+1}(x+3)^8}$.

43. $y = x\sinh(x^2) \quad\Rightarrow\quad y' = x\cosh(x^2) \cdot 2x + \sinh(x^2) \cdot 1 = 2x^2\cosh(x^2) + \sinh(x^2)$

45. $y = \ln(\cosh 3x) \quad\Rightarrow\quad y' = (1/\cosh 3x)(\sinh 3x)(3) = 3\tanh 3x$

47. $y = \cosh^{-1}(\sinh x) \quad\Rightarrow\quad y' = \dfrac{1}{\sqrt{(\sinh x)^2 - 1}} \cdot \cosh x = \dfrac{\cosh x}{\sqrt{\sinh^2 x - 1}}$

49. $f(t) = \sqrt{4t + 1} \quad\Rightarrow\quad f'(t) = \frac{1}{2}(4t + 1)^{-1/2} \cdot 4 = 2(4t + 1)^{-1/2} \quad\Rightarrow$
$f''(t) = 2(-\frac{1}{2})(4t + 1)^{-3/2} \cdot 4 = -4/(4t + 1)^{3/2}$, so $f''(2) = -4/9^{3/2} = -\frac{4}{27}$.

51. $x^6 + y^6 = 1 \quad\Rightarrow\quad 6x^5 + 6y^5y' = 0 \quad\Rightarrow\quad y' = -x^5/y^5 \quad\Rightarrow$
$y'' = -\dfrac{y^5(5x^4) - x^5(5y^4y')}{(y^5)^2} = -\dfrac{5x^4y^4\left[y - x(-x^5/y^5)\right]}{y^{10}} = -\dfrac{5x^4\left[(y^6 + x^6)/y^5\right]}{y^6} = -\dfrac{5x^4}{y^{11}}$

53. We first show it is true for $n = 1$: $f(x) = xe^x \quad\Rightarrow\quad f'(x) = xe^x + e^x = (x + 1)e^x$. We now assume it is true for $n = k$: $f^{(k)}(x) = (x + k)e^x$. With this assumption, we must show it is true for $n = k + 1$:
$f^{(k+1)}(x) = \dfrac{d}{dx}\left[f^{(k)}(x)\right] = \dfrac{d}{dx}[(x + k)e^x] = (x + k)e^x + e^x = [(x + k) + 1]e^x = [x + (k + 1)]e^x$.
Therefore, $f^{(n)}(x) = (x + n)e^x$ by mathematical induction.

55. $y = 4\sin^2 x \quad\Rightarrow\quad y' = 4 \cdot 2\sin x \cos x$. At $\left(\frac{\pi}{6}, 1\right)$, $y' = 8 \cdot \frac{1}{2} \cdot \frac{\sqrt{3}}{2} = 2\sqrt{3}$, so an equation of the tangent line is $y - 1 = 2\sqrt{3}\left(x - \frac{\pi}{6}\right)$, or $y = 2\sqrt{3}x + 1 - \pi\sqrt{3}/3$.

57. $y = \sqrt{1 + 4\sin x} \quad\Rightarrow\quad y' = \frac{1}{2}(1 + 4\sin x)^{-1/2} \cdot 4\cos x = \dfrac{2\cos x}{\sqrt{1 + 4\sin x}}$. At $(0, 1)$, $y' = \dfrac{2}{\sqrt{1}} = 2$, so an equation of the tangent line is $y - 1 = 2(x - 0)$, or $y = 2x + 1$.

59. $y = (2 + x)e^{-x} \quad\Rightarrow\quad y' = (2 + x)(-e^{-x}) + e^{-x} \cdot 1 = e^{-x}[-(2 + x) + 1] = e^{-x}(-x - 1)$. At $(0, 2)$, $y' = 1(-1) = -1$, so an equation of the tangent line is $y - 2 = -1(x - 0)$, or $y = -x + 2$.

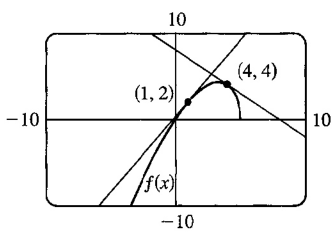
61. (a) $f(x) = x\sqrt{5 - x} \quad\Rightarrow$

$$f'(x) = x\left[\tfrac{1}{2}(5 - x)^{-1/2}(-1)\right] + \sqrt{5 - x} = \frac{-x}{2\sqrt{5 - x}} + \sqrt{5 - x} \cdot \frac{2\sqrt{5 - x}}{2\sqrt{5 - x}}$$

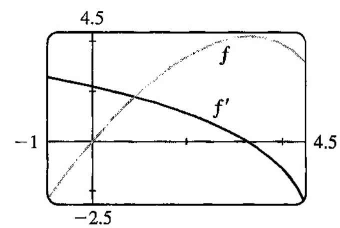
$$= \frac{-x}{2\sqrt{5 - x}} + \frac{2(5 - x)}{2\sqrt{5 - x}} = \frac{-x + 10 - 2x}{2\sqrt{5 - x}} = \frac{10 - 3x}{2\sqrt{5 - x}}$$

(b) At $(1, 2)$: $f'(1) = \frac{7}{4}$. So an equation of the tangent line is $y - 2 = \frac{7}{4}(x - 1)$ or $y = \frac{7}{4}x + \frac{1}{4}$.
At $(4, 4)$: $f'(4) = -\frac{2}{2} = -1$. So an equation of the tangent line is $y - 4 = -1(x - 4)$ or $y = -x + 8$.

(c)



(d)



The graphs look reasonable, since f' is positive where f has tangents with positive slope, and f' is negative where f has tangents with negative slope.

63. $y = \sin x + \cos x \quad\Rightarrow\quad y' = \cos x - \sin x = 0 \quad\Leftrightarrow\quad \cos x = \sin x$ and $0 \le x \le 2\pi \quad\Leftrightarrow\quad x = \frac{\pi}{4}$ or $\frac{5\pi}{4}$, so the points are $\left(\frac{\pi}{4}, \sqrt{2}\right)$ and $\left(\frac{5\pi}{4}, -\sqrt{2}\right)$.

65. $f(x) = (x - a)(x - b)(x - c) \quad\Rightarrow\quad f'(x) = (x - b)(x - c) + (x - a)(x - c) + (x - a)(x - b)$. So

$$\frac{f'(x)}{f(x)} = \frac{(x - b)(x - c) + (x - a)(x - c) + (x - a)(x - b)}{(x - a)(x - b)(x - c)} = \frac{1}{x - a} + \frac{1}{x - b} + \frac{1}{x - c}.$$

Or: $f(x) = (x - a)(x - b)(x - c) \quad\Rightarrow\quad \ln|f(x)| = \ln|x - a| + \ln|x - b| + \ln|x - c| \quad\Rightarrow$

$$\frac{f'(x)}{f(x)} = \frac{1}{x - a} + \frac{1}{x - b} + \frac{1}{x - c}$$

67. (a) $h(x) = f(x)g(x) \quad\Rightarrow\quad h'(x) = f(x)g'(x) + g(x)f'(x) \quad\Rightarrow$
$h'(2) = f(2)g'(2) + g(2)f'(2) = (3)(4) + (5)(-2) = 12 - 10 = 2$

(b) $F(x) = f(g(x)) \quad\Rightarrow\quad F'(x) = f'(g(x))g'(x) \quad\Rightarrow\quad F'(2) = f'(g(2))g'(2) = f'(5)(4) = 11 \cdot 4 = 44$

69. $f(x) = x^2 g(x) \quad\Rightarrow\quad f'(x) = x^2 g'(x) + g(x)(2x) = x\,[xg'(x) + 2g(x)]$

71. $f(x) = [g(x)]^2 \quad\Rightarrow\quad f'(x) = 2[g(x)]^1 \cdot g'(x) = 2g(x)g'(x)$

73. $f(x) = g(e^x) \Rightarrow f'(x) = g'(e^x)e^x$

75. $f(x) = \ln|g(x)| \Rightarrow f'(x) = \dfrac{1}{g(x)}g'(x) = \dfrac{g'(x)}{g(x)}$

77. $h(x) = \dfrac{f(x)g(x)}{f(x)+g(x)} \Rightarrow$

$$h'(x) = \frac{[f(x)+g(x)]\,[f(x)g'(x)+g(x)f'(x)] - f(x)g(x)\,[f'(x)+g'(x)]}{[f(x)+g(x)]^2}$$

$$= \frac{[f(x)]^2 g'(x) + f(x)g(x)f'(x) + f(x)g(x)g'(x) + [g(x)]^2 f'(x) - f(x)g(x)f'(x) - f(x)g(x)g'(x)}{[f(x)+g(x)]^2}$$

$$= \frac{f'(x)[g(x)]^2 + g'(x)[f(x)]^2}{[f(x)+g(x)]^2}$$

79. Using the Chain Rule repeatedly, $h(x) = f(g(\sin 4x)) \Rightarrow$

$$h'(x) = f'(g(\sin 4x)) \cdot \frac{d}{dx}(g(\sin 4x)) = f'(g(\sin 4x)) \cdot g'(\sin 4x) \cdot \frac{d}{dx}(\sin 4x)$$
$$= f'(g(\sin 4x))g'(\sin 4x)(\cos 4x)(4)$$

81. $y = [\ln(x+4)]^2 \Rightarrow y' = 2[\ln(x+4)]^1 \cdot \dfrac{1}{x+4} \cdot 1 = 2\,\dfrac{\ln(x+4)}{x+4}$ and $y' = 0 \Leftrightarrow \ln(x+4) = 0 \Leftrightarrow$ $x + 4 = e^0 \Rightarrow x + 4 = 1 \Leftrightarrow x = -3$, so the tangent is horizontal at the point $(-3, 0)$.

83. $y = f(x) = ax^2 + bx + c \Rightarrow f'(x) = 2ax + b$. We know that $f'(-1) = 6$ and $f'(5) = -2$, so $-2a + b = 6$ and $10a + b = -2$. Subtracting the first equation from the second gives $12a = -8 \Rightarrow a = -\frac{2}{3}$. Substituting $-\frac{2}{3}$ for a in the first equation gives $b = \frac{14}{3}$. Now $f(1) = 4 \Rightarrow 4 = a + b + c$, so $c = 4 + \frac{2}{3} - \frac{14}{3} = 0$ and hence, $f(x) = -\frac{2}{3}x^2 + \frac{14}{3}x$.

85. $s(t) = Ae^{-ct}\cos(\omega t + \delta) \Rightarrow$

$$v(t) = s'(t) = A\{e^{-ct}[-\omega\sin(\omega t+\delta)] + \cos(\omega t+\delta)(-ce^{-ct})\}$$
$$= -Ae^{-ct}[\omega\sin(\omega t+\delta) + c\cos(\omega t+\delta)] \Rightarrow$$

$$a(t) = v'(t) = -A\{e^{-ct}[\omega^2\cos(\omega t+\delta) - c\omega\sin(\omega t+\delta)] + [\omega\sin(\omega t+\delta) + c\cos(\omega t+\delta)](-ce^{-ct})\}$$
$$= -Ae^{-ct}[\omega^2\cos(\omega t+\delta) - c\omega\sin(\omega t+\delta) - c\omega\sin(\omega t+\delta) - c^2\cos(\omega t+\delta)]$$
$$= -Ae^{-ct}[(\omega^2 - c^2)\cos(\omega t+\delta) - 2c\omega\sin(\omega t+\delta)]$$
$$= Ae^{-ct}[(c^2 - \omega^2)\cos(\omega t+\delta) + 2c\omega\sin(\omega t+\delta)]$$

87. (a) $y = t^3 - 12t + 3 \Rightarrow v(t) = y' = 3t^2 - 12 \Rightarrow a(t) = v'(t) = 6t$

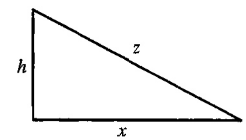
(b) $v(t) = 3(t^2 - 4) > 0$ when $t > 2$, so it moves upward when $t > 2$ and downward when $0 \le t < 2$.

(c) Distance upward $= y(3) - y(2) = -6 - (-13) = 7$,
Distance downward $= y(0) - y(2) = 3 - (-13) = 16$. Total distance $= 7 + 16 = 23$.

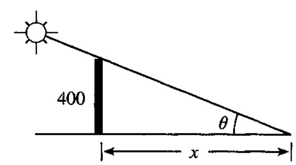
89. The linear density ρ is the rate of change of mass m with respect to length x. $m = x(1+\sqrt{x}) = x + x^{3/2} \Rightarrow$ $\rho = dm/dx = 1 + \frac{3}{2}\sqrt{x}$, so the linear density when $x = 4$ is $1 + \frac{3}{2}\sqrt{4} = 4$ kg/m.

91. If $x =$ edge length, then $V = x^3 \Rightarrow dV/dt = 3x^2\,dx/dt = 10 \Rightarrow dx/dt = 10/(3x^2)$ and $S = 6x^2 \Rightarrow$ $dS/dt = (12x)\,dx/dt = 12x[10/(3x^2)] = 40/x$. When $x = 30$, $dS/dt = \frac{40}{30} = \frac{4}{3}$ cm^2/min.

93. Given $dh/dt = 5$ and $dx/dt = 15$, find dz/dt. $z^2 = x^2 + h^2 \Rightarrow$ $2z\,\dfrac{dz}{dt} = 2x\,\dfrac{dx}{dt} + 2h\,\dfrac{dh}{dt} \Rightarrow \dfrac{dz}{dt} = \dfrac{1}{z}(15x + 5h)$. When $t = 3$, $h = 45 + 3(5) = 60$ and $x = 15(3) = 45 \Rightarrow z = \sqrt{45^2 + 60^2} = 75$, so $\dfrac{dz}{dt} = \frac{1}{75}[15(45) + 5(60)] = 13$ ft/s.

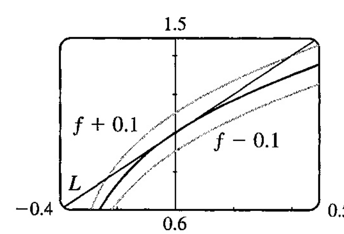


95. We are given $d\theta/dt = -0.25$ rad/h. $\tan\theta = 400/x \Rightarrow$ $x = 400\cot\theta \Rightarrow \dfrac{dx}{dt} = -400\csc^2\theta\,\dfrac{d\theta}{dt}$. When $\theta = \frac{\pi}{6}$, $\dfrac{dx}{dt} = -400(2)^2(-0.25) = 400$ ft/h.

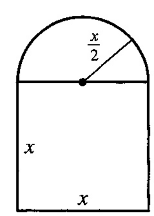


97. (a) $f(x) = \sqrt[3]{1+3x} = (1+3x)^{1/3} \Rightarrow f'(x) = (1+3x)^{-2/3}$, so the linearization of f at $a = 0$ is $L(x) = f(0) + f'(0)(x-0) = 1^{1/3} + 1^{-2/3}x = 1 + x$. Thus, $\sqrt[3]{1+3x} \approx 1 + x \Rightarrow$ $\sqrt[3]{1.03} = \sqrt[3]{1+3(0.01)} \approx 1 + (0.01) = 1.01$.

(b) The linear approximation is $\sqrt[3]{1+3x} \approx 1 + x$, so for the required accuracy we want $\sqrt[3]{1+3x} - 0.1 < 1 + x < \sqrt[3]{1+3x} + 0.1$. From the graph, it appears that this is true when $-0.23 < x < 0.40$.



99. $A = x^2 + \frac{1}{2}\pi\left(\frac{1}{2}x\right)^2 = \left(1 + \frac{\pi}{8}\right)x^2 \Rightarrow dA = \left(2 + \frac{\pi}{4}\right)x\,dx$. When $x = 60$ and $dx = 0.1$, $dA = \left(2 + \frac{\pi}{4}\right)60(0.1) = 12 + \frac{3\pi}{2}$, so the maximum error is approximately $12 + \frac{3\pi}{2} \approx 16.7$ cm^2.



101. $\displaystyle\lim_{h\to 0}\frac{\sqrt[4]{16+h} - 2}{h} = \left[\frac{d}{dx}\sqrt[4]{x}\right]_{x=16} = \frac{1}{4}x^{-3/4}\Big|_{x=16} = \frac{1}{4\left(\sqrt[4]{16}\right)^3} = \frac{1}{32}$

103. $$\lim_{x\to 0}\frac{\sqrt{1+\tan x}-\sqrt{1+\sin x}}{x^3}=\lim_{x\to 0}\frac{\left(\sqrt{1+\tan x}-\sqrt{1+\sin x}\right)\left(\sqrt{1+\tan x}+\sqrt{1+\sin x}\right)}{x^3\left(\sqrt{1+\tan x}+\sqrt{1+\sin x}\right)}$$

$$=\lim_{x\to 0}\frac{(1+\tan x)-(1+\sin x)}{x^3\left(\sqrt{1+\tan x}+\sqrt{1+\sin x}\right)}=\lim_{x\to 0}\frac{\sin x\,(1/\cos x-1)}{x^3\left(\sqrt{1+\tan x}+\sqrt{1+\sin x}\right)}\cdot\frac{\cos x}{\cos x}$$

$$=\lim_{x\to 0}\frac{\sin x\,(1-\cos x)}{x^3\left(\sqrt{1+\tan x}+\sqrt{1+\sin x}\right)\cos x}\cdot\frac{1+\cos x}{1+\cos x}$$

$$=\lim_{x\to 0}\frac{\sin x\cdot\sin^2 x}{x^3\left(\sqrt{1+\tan x}+\sqrt{1+\sin x}\right)\cos x\,(1+\cos x)}$$

$$=\left(\lim_{x\to 0}\frac{\sin x}{x}\right)^3\lim_{x\to 0}\frac{1}{\left(\sqrt{1+\tan x}+\sqrt{1+\sin x}\right)\cos x\,(1+\cos x)}$$

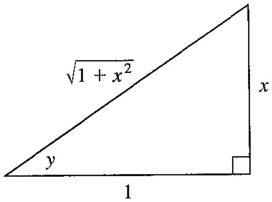
$$=1^3\cdot\frac{1}{\left(\sqrt{1}+\sqrt{1}\right)\cdot 1\cdot(1+1)}=\frac{1}{4}$$

105. $\dfrac{d}{dx}\left[f(2x)\right]=x^2 \;\Rightarrow\; f'(2x)\cdot 2=x^2 \;\Rightarrow\; f'(2x)=\frac{1}{2}x^2$. Let $t=2x$. Then $f'(t)=\frac{1}{2}\left(\frac{1}{2}t\right)^2=\frac{1}{8}t^2$, so $f'(x)=\frac{1}{8}x^2$.

PROBLEMS PLUS

1. Let a be the x-coordinate of Q. Since the derivative of $y = 1 - x^2$ is $y' = -2x$, the slope at Q is $-2a$. But since the triangle is equilateral, $\overline{AO}/\overline{OC} = \sqrt{3}/1$, so the slope at Q is $-\sqrt{3}$. Therefore, we must have that $-2a = -\sqrt{3}$ $\Rightarrow$ $a = \frac{\sqrt{3}}{2}$. Thus, the point Q has coordinates $\left(\frac{\sqrt{3}}{2}, 1 - \left(\frac{\sqrt{3}}{2}\right)^2\right) = \left(\frac{\sqrt{3}}{2}, \frac{1}{4}\right)$ and by symmetry, P has coordinates $\left(-\frac{\sqrt{3}}{2}, \frac{1}{4}\right)$.

3. Let $y = \tan^{-1} x$. Then $\tan y = x$, so from the triangle we see that $\sin(\tan^{-1} x) = \sin y = \dfrac{x}{\sqrt{1+x^2}}$. Using this fact we have that $\sin(\tan^{-1}(\sinh x)) = \dfrac{\sinh x}{\sqrt{1+\sinh^2 x}} = \dfrac{\sinh x}{\cosh x} = \tanh x$. Hence, $\sin^{-1}(\tanh x) = \sin^{-1}\left(\sin(\tan^{-1}(\sinh x))\right) = \tan^{-1}(\sinh x)$.



5. We use mathematical induction. Let S_n be the statement that $\dfrac{d^n}{dx^n}(\sin^4 x + \cos^4 x) = 4^{n-1}\cos(4x + n\pi/2)$.

S_1 is true because

$$\begin{aligned}\frac{d}{dx}(\sin^4 x + \cos^4 x) &= 4\sin^3 x\cos x - 4\cos^3 x\sin x = 4\sin x\cos x\,(\sin^2 x - \cos^2 x)\\ &= -4\sin x\cos x\cos 2x = -2\sin 2x\cos 2x = -\sin 4x = \sin(-4x)\\ &= \cos\left(\tfrac{\pi}{2} - (-4x)\right) = \cos\left(\tfrac{\pi}{2} + 4x\right) = 4^{n-1}\cos\left(4x + n\tfrac{\pi}{2}\right) \text{ when } n = 1\end{aligned}$$

Now assume S_k is true, that is, $\dfrac{d^k}{dx^k}(\sin^4 x + \cos^4 x) = 4^{k-1}\cos\left(4x + k\frac{\pi}{2}\right)$. Then

$$\begin{aligned}\frac{d^{k+1}}{dx^{k+1}}(\sin^4 x + \cos^4 x) &= \frac{d}{dx}\left[\frac{d^k}{dx^k}(\sin^4 x + \cos^4 x)\right] = \frac{d}{dx}\left[4^{k-1}\cos\left(4x + k\tfrac{\pi}{2}\right)\right]\\ &= -4^{k-1}\sin\left(4x + k\tfrac{\pi}{2}\right)\cdot\frac{d}{dx}\left(4x + k\tfrac{\pi}{2}\right) = -4^k\sin\left(4x + k\tfrac{\pi}{2}\right)\\ &= 4^k\sin\left(-4x - k\tfrac{\pi}{2}\right) = 4^k\cos\left(\tfrac{\pi}{2} - \left(-4x - k\tfrac{\pi}{2}\right)\right)\\ &= 4^k\cos\left(4x + (k+1)\tfrac{\pi}{2}\right)\end{aligned}$$

which shows that S_{k+1} is true.

Therefore, $\dfrac{d^n}{dx^n}(\sin^4 x + \cos^4 x) = 4^{n-1}\cos\left(4x + n\frac{\pi}{2}\right)$ for every positive integer n, by mathematical induction.

Another proof: First write

$\sin^4 x + \cos^4 x = (\sin^2 x + \cos^2 x)^2 - 2\sin^2 x\cos^2 x = 1 - \frac{1}{2}\sin^2 2x = 1 - \frac{1}{4}(1 - \cos 4x) = \frac{3}{4} + \frac{1}{4}\cos 4x$.

Then we have $\dfrac{d^n}{dx^n}(\sin^4 x + \cos^4 x) = \dfrac{d^n}{dx^n}\left(\frac{3}{4} + \frac{1}{4}\cos 4x\right) = \frac{1}{4}\cdot 4^n\cos\left(4x + n\frac{\pi}{2}\right) = 4^{n-1}\cos\left(4x + n\frac{\pi}{2}\right)$.

7. We must find a value x_0 such that the normal lines to the parabola $y = x^2$ at $x = \pm x_0$ intersect at a point one unit from the points $(\pm x_0, x_0^2)$. The normals to $y = x^2$ at $x = \pm x_0$ have slopes $-\dfrac{1}{\pm 2x_0}$ and pass through $(\pm x_0, x_0^2)$ respectively, so the normals have the equations $y - x_0^2 = -\dfrac{1}{2x_0}(x - x_0)$ and $y - x_0^2 = \dfrac{1}{2x_0}(x + x_0)$. The common y-intercept is $x_0^2 + \frac{1}{2}$. We want to find the value of x_0 for which the distance from $\left(0, x_0^2 + \frac{1}{2}\right)$ to (x_0, x_0^2) equals 1. The square of the distance is $(x_0 - 0)^2 + \left[x_0^2 - \left(x_0^2 + \frac{1}{2}\right)\right]^2 = x_0^2 + \frac{1}{4} = 1 \quad \Leftrightarrow \quad x_0 = \pm\frac{\sqrt{3}}{2}$. For these values of x_0, the y-intercept is $x_0^2 + \frac{1}{2} = \frac{5}{4}$, so the center of the circle is at $\left(0, \frac{5}{4}\right)$.

Another solution: Let the center of the circle be $(0, a)$. Then the equation of the circle is $x^2 + (y - a)^2 = 1$. Solving with the equation of the parabola, $y = x^2$, we get $x^2 + \left(x^2 - a\right)^2 = 1 \quad \Leftrightarrow \quad x^2 + x^4 - 2ax^2 + a^2 = 1$ $\Leftrightarrow \quad x^4 + (1 - 2a)x^2 + a^2 - 1 = 0$. The parabola and the circle will be tangent to each other when this quadratic equation in x^2 has equal roots; that is, when the discriminant is 0. Thus, $(1 - 2a)^2 - 4\left(a^2 - 1\right) = 0 \quad \Leftrightarrow$ $1 - 4a + 4a^2 - 4a^2 + 4 = 0 \quad \Leftrightarrow \quad 4a = 5$, so $a = \frac{5}{4}$. The center of the circle is $\left(0, \frac{5}{4}\right)$.

9. We can assume without loss of generality that $\theta = 0$ at time $t = 0$, so that $\theta = 12\pi t$ rad. [The angular velocity of the wheel is 360 rpm $= 360 \cdot (2\pi \text{ rad})/(60 \text{ s}) = 12\pi$ rad/s.] Then the position of A as a function of time is

$$A = (40\cos\theta, 40\sin\theta) = (40\cos 12\pi t, 40\sin 12\pi t), \text{ so } \sin\alpha = \frac{y}{1.2 \text{ m}} = \frac{40\sin\theta}{120} = \frac{\sin\theta}{3} = \tfrac{1}{3}\sin 12\pi t.$$

(a) Differentiating the expression for $\sin\alpha$, we get $\cos\alpha \cdot \dfrac{d\alpha}{dt} = \frac{1}{3} \cdot 12\pi \cdot \cos 12\pi t = 4\pi\cos\theta$.

When $\theta = \frac{\pi}{3}$, we have $\sin\alpha = \frac{1}{3}\sin\theta = \frac{\sqrt{3}}{6}$, so $\cos\alpha = \sqrt{1 - \left(\frac{\sqrt{3}}{6}\right)^2} = \sqrt{\frac{11}{12}}$ and

$$\frac{d\alpha}{dt} = \frac{4\pi\cos\frac{\pi}{3}}{\cos\alpha} = \frac{2\pi}{\sqrt{11/12}} = \frac{4\pi\sqrt{3}}{\sqrt{11}} \approx 6.56 \text{ rad/s}.$$

(b) By the Law of Cosines, $|AP|^2 = |OA|^2 + |OP|^2 - 2\,|OA|\,|OP|\cos\theta \quad \Rightarrow$

$120^2 = 40^2 + |OP|^2 - 2 \cdot 40\,|OP|\cos\theta \quad \Rightarrow \quad |OP|^2 - (80\cos\theta)\,|OP| - 12{,}800 = 0 \quad \Rightarrow$

$$|OP| = \tfrac{1}{2}\left(80\cos\theta \pm \sqrt{6400\cos^2\theta + 51{,}200}\right) = 40\cos\theta \pm 40\sqrt{\cos^2\theta + 8}$$

$$= 40\left(\cos\theta + \sqrt{8 + \cos^2\theta}\right) \text{ cm} \quad (\text{since } |OP| > 0)$$

As a check, note that $|OP| = 160$ cm when $\theta = 0$ and $|OP| = 80\sqrt{2}$ cm when $\theta = \frac{\pi}{2}$.

(c) By part (b), the x-coordinate of P is given by $x = 40\left(\cos\theta + \sqrt{8 + \cos^2\theta}\right)$, so

$$\frac{dx}{dt} = \frac{dx}{d\theta}\frac{d\theta}{dt} = 40\left(-\sin\theta - \frac{2\cos\theta\sin\theta}{2\sqrt{8 + \cos^2\theta}}\right) \cdot 12\pi = -480\pi\sin\theta\left(1 + \frac{\cos\theta}{\sqrt{8 + \cos^2\theta}}\right) \text{ cm/s}.$$

In particular, $dx/dt = 0$ cm/s when $\theta = 0$ and $dx/dt = -480\pi$ cm/s when $\theta = \frac{\pi}{2}$.

11. Consider the statement that $\frac{d^n}{dx^n}(e^{ax}\sin bx) = r^n e^{ax}\sin(bx + n\theta)$. For $n = 1$,

$\frac{d}{dx}(e^{ax}\sin bx) = ae^{ax}\sin bx + be^{ax}\cos bx$, and

$$re^{ax}\sin(bx+\theta) = re^{ax}[\sin bx\cos\theta + \cos bx\sin\theta] = re^{ax}\left(\frac{a}{r}\sin bx + \frac{b}{r}\cos bx\right)$$

$$= ae^{ax}\sin bx + be^{ax}\cos bx$$

since $\tan\theta = \frac{b}{a} \Rightarrow \sin\theta = \frac{b}{r}$ and $\cos\theta = \frac{a}{r}$.

So the statement is true for $n = 1$. Assume it is true for $n = k$. Then

$$\frac{d^{k+1}}{dx^{k+1}}(e^{ax}\sin bx) = \frac{d}{dx}\left[r^k e^{ax}\sin(bx+k\theta)\right] = r^k ae^{ax}\sin(bx+k\theta) + r^k e^{ax} b\cos(bx+k\theta)$$

$$= r^k e^{ax}[a\sin(bx+k\theta) + b\cos(bx+k\theta)]$$

But

$$\sin[bx+(k+1)\theta] = \sin[(bx+k\theta)+\theta] = \sin(bx+k\theta)\cos\theta + \sin\theta\cos(bx+k\theta)$$

$$= \frac{a}{r}\sin(bx+k\theta) + \frac{b}{r}\cos(bx+k\theta)$$

Hence, $a\sin(bx+k\theta) + b\cos(bx+k\theta) = r\sin[bx+(k+1)\theta]$. So

$$\frac{d^{k+1}}{dx^{k+1}}(e^{ax}\sin bx) = r^k e^{ax}[a\sin(bx+k\theta) + b\cos(bx+k\theta)] = r^k e^{ax}[r\sin(bx+(k+1)\theta)]$$

$$= r^{k+1}e^{ax}[\sin(bx+(k+1)\theta)]$$

Therefore, the statement is true for all n by mathematical induction.

13. It seems from the figure that as P approaches the point $(0,2)$ from the right, $x_T \to \infty$ and $y_T \to 2^+$. As P approaches the point $(3,0)$ from the left, it appears that $x_T \to 3^+$ and $y_T \to \infty$. So we guess that $x_T \in (3,\infty)$ and $y_T \in (2,\infty)$. It is more difficult to estimate the range of values for x_N and y_N. We might perhaps guess that $x_N \in (0,3)$, and $y_N \in (-\infty,0)$ or $(-2,0)$.

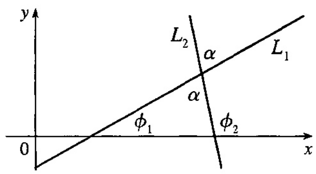
In order to actually solve the problem, we implicitly differentiate the equation of the ellipse to find the equation of the tangent line: $\frac{x^2}{9} + \frac{y^2}{4} = 1 \Rightarrow \frac{2x}{9} + \frac{2y}{4}y' = 0$, so $y' = -\frac{4}{9}\frac{x}{y}$. So at the point (x_0, y_0) on the ellipse, an equation of the tangent line is $y - y_0 = -\frac{4}{9}\frac{x_0}{y_0}(x - x_0)$ or $4x_0x + 9y_0y = 4x_0^2 + 9y_0^2$. This can be written as $\frac{x_0x}{9} + \frac{y_0y}{4} = \frac{x_0^2}{9} + \frac{y_0^2}{4} = 1$, because (x_0, y_0) lies on the ellipse. So an equation of the tangent line is $\frac{x_0x}{9} + \frac{y_0y}{4} = 1$.

Therefore, the x-intercept x_T for the tangent line is given by $\frac{x_0 x_T}{9} = 1 \quad \Leftrightarrow \quad x_T = \frac{9}{x_0}$, and the y-intercept y_T is given by $\frac{y_0 y_T}{4} = 1 \quad \Leftrightarrow \quad y_T = \frac{4}{y_0}$.

So as x_0 takes on all values in $(0, 3)$, x_T takes on all values in $(3, \infty)$, and as y_0 takes on all values in $(0, 2)$, y_T takes on all values in $(2, \infty)$. At the point (x_0, y_0) on the ellipse, the slope of the normal line is $-\frac{1}{y'(x_0, y_0)} = \frac{9}{4}\frac{y_0}{x_0}$, and its equation is $y - y_0 = \frac{9}{4}\frac{y_0}{x_0}(x - x_0)$. So the x-intercept x_N for the normal line is given by $0 - y_0 = \frac{9}{4}\frac{y_0}{x_0}(x_N - x_0) \quad \Rightarrow \quad x_N = -\frac{4x_0}{9} + x_0 = \frac{5x_0}{9}$, and the y-intercept y_N is given by $y_N - y_0 = \frac{9}{4}\frac{y_0}{x_0}(0 - x_0) \quad \Rightarrow \quad y_N = -\frac{9y_0}{4} + y_0 = -\frac{5y_0}{4}$.

So as x_0 takes on all values in $(0, 3)$, x_N takes on all values in $\left(0, \frac{5}{3}\right)$, and as y_0 takes on all values in $(0, 2)$, y_N takes on all values in $\left(-\frac{5}{2}, 0\right)$.

15. (a)



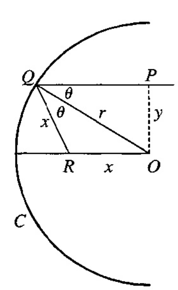
If the two lines L_1 and L_2 have slopes m_1 and m_2 and angles of inclination ϕ_1 and ϕ_2, then $m_1 = \tan\,\phi_1$ and $m_2 = \tan\,\phi_2$. The triangle in the figure shows that $\phi_1 + \alpha + (180° - \phi_2) = 180°$ and so $\alpha = \phi_2 - \phi_1$. Therefore, using the identity for $\tan(x - y)$, we have $\tan\alpha = \tan(\phi_2 - \phi_1) = \frac{\tan\phi_2 - \tan\phi_1}{1 + \tan\phi_2 \tan\phi_1}$ and so $\tan\,\alpha = \frac{m_2 - m_1}{1 + m_1 m_2}$.

(b) (i) The parabolas intersect when $x^2 = (x - 2)^2 \quad \Rightarrow \quad x = 1$. If $y = x^2$, then $y' = 2x$, so the slope of the tangent to $y = x^2$ at $(1, 1)$ is $m_1 = 2(1) = 2$. If $y = (x - 2)^2$, then $y' = 2(x - 2)$, so the slope of the tangent to $y = (x - 2)^2$ at $(1, 1)$ is $m_2 = 2(1 - 2) = -2$. Therefore,

$\tan\alpha = \frac{m_2 - m_1}{1 + m_1 m_2} = \frac{-2 - 2}{1 + 2(-2)} = \frac{4}{3}$ and so $\alpha = \tan^{-1}\left(\frac{4}{3}\right) \approx 53°$ (or $127°$).

(ii) $x^2 - y^2 = 3$ and $x^2 - 4x + y^2 + 3 = 0$ intersect when $x^2 - 4x + \left(x^2 - 3\right) + 3 = 0 \quad \Leftrightarrow \quad 2x(x - 2) = 0 \quad \Rightarrow \quad x = 0$ or 2, but 0 is extraneous. If $x = 2$, then $y = \pm 1$. If $x^2 - y^2 = 3$ then $2x - 2yy' = 0 \quad \Rightarrow \quad y' = x/y$ and $x^2 - 4x + y^2 + 3 = 0 \quad \Rightarrow \quad 2x - 4 + 2yy' = 0 \quad \Rightarrow \quad y' = \frac{2 - x}{y}$. At $(2, 1)$ the slopes are $m_1 = 2$ and $m_2 = 0$, so $\tan\alpha = \frac{0 - 2}{1 + 2 \cdot 0} = -2 \quad \Rightarrow \quad \alpha \approx 117°$. At $(2, -1)$ the slopes are $m_1 = -2$ and $m_2 = 0$, so $\tan\alpha = \frac{0 - (-2)}{1 + (-2)(0)} = 2 \quad \Rightarrow \quad \alpha \approx 63°$ (or $117°$).

17. Since $\angle ROQ = \angle OQP = \theta$, the triangle QOR is isosceles, so $|QR| = |RO| = x$. By the Law of Cosines, $x^2 = x^2 + r^2 - 2rx\cos\theta$. Hence, $2rx\cos\theta = r^2$, so $x = \frac{r^2}{2r\cos\theta} = \frac{r}{2\cos\theta}$. Note that as $y \to 0^+$, $\theta \to 0^+$ (since $\sin\theta = y/r$), and hence $x \to \frac{r}{2\cos 0} = \frac{r}{2}$. Thus, as P is taken closer and closer to the x-axis, the point R approaches the midpoint of the radius AO.



19. $\lim_{x\to 0} \dfrac{\sin(a+2x) - 2\sin(a+x) + \sin a}{x^2}$

$$= \lim_{x\to 0} \frac{\sin a \cos 2x + \cos a \sin 2x - 2\sin a \cos x - 2\cos a \sin x + \sin a}{x^2}$$

$$= \lim_{x\to 0} \frac{\sin a\,(\cos 2x - 2\cos x + 1) + \cos a\,(\sin 2x - 2\sin x)}{x^2}$$

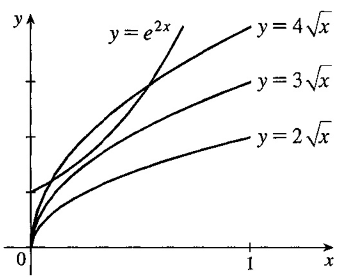
$$= \lim_{x\to 0} \frac{\sin a\,(2\cos^2 x - 1 - 2\cos x + 1) + \cos a\,(2\sin x\cos x - 2\sin x)}{x^2}$$

$$= \lim_{x\to 0} \frac{\sin a\,(2\cos x)(\cos x - 1) + \cos a\,(2\sin x)(\cos x - 1)}{x^2}$$

$$= \lim_{x\to 0} \frac{2(\cos x - 1)[\sin a\cos x + \cos a \sin x](\cos x + 1)}{x^2(\cos x + 1)}$$

$$= \lim_{x\to 0} \frac{-2\sin^2 x\,[\sin(a+x)]}{x^2(\cos x + 1)} = -2\lim_{x\to 0}\left(\frac{\sin x}{x}\right)^2 \cdot \frac{\sin(a+x)}{\cos x + 1} = -2(1)^2 \frac{\sin(a+0)}{\cos 0 + 1} = -\sin a$$

21.



Let $f(x) = e^{2x}$ and $g(x) = k\sqrt{x}$ $(k > 0)$. From the graphs of f and g, we see that f will intersect g exactly once when f and g share a tangent line. Thus, we must have $f = g$ and $f' = g'$ at $x = a$. $f(a) = g(a) \Rightarrow e^{2a} = k\sqrt{a}$ **(1)** and $f'(a) = g'(a) \Rightarrow 2e^{2a} = \dfrac{k}{2\sqrt{a}} \Rightarrow e^{2a} = \dfrac{k}{4\sqrt{a}}$. So we must have $k\sqrt{a} = \dfrac{k}{4\sqrt{a}} \Rightarrow (\sqrt{a})^2 = \dfrac{k}{4k} \Rightarrow a = \frac{1}{4}$. From **(1)**, $e^{2(1/4)} = k\sqrt{1/4} \Rightarrow k = 2e^{1/2} = 2\sqrt{e} \approx 3.297$.

23. $y = \dfrac{x}{\sqrt{a^2-1}} - \dfrac{2}{\sqrt{a^2-1}} \arctan \dfrac{\sin x}{a + \sqrt{a^2-1} + \cos x}$. Let $k = a + \sqrt{a^2-1}$. Then

$$y' = \frac{1}{\sqrt{a^2-1}} - \frac{2}{\sqrt{a^2-1}} \cdot \frac{1}{1 + \sin^2 x/(k+\cos x)^2} \cdot \frac{\cos x(k+\cos x) + \sin^2 x}{(k+\cos x)^2}$$

$$= \frac{1}{\sqrt{a^2-1}} - \frac{2}{\sqrt{a^2-1}} \cdot \frac{k\cos x + \cos^2 x + \sin^2 x}{(k+\cos x)^2 + \sin^2 x} = \frac{1}{\sqrt{a^2-1}} - \frac{2}{\sqrt{a^2-1}} \cdot \frac{k\cos x + 1}{k^2 + 2k\cos x + 1}$$

$$= \frac{k^2 + 2k\cos x + 1 - 2k\cos x - 2}{\sqrt{a^2-1}\,(k^2 + 2k\cos x + 1)} = \frac{k^2 - 1}{\sqrt{a^2-1}\,(k^2 + 2k\cos x + 1)}$$

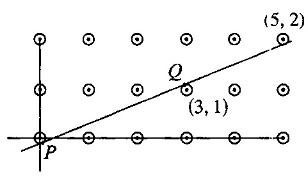
But $k^2 = 2a^2 + 2a\sqrt{a^2-1} - 1 = 2a\left(a + \sqrt{a^2-1}\right) - 1 = 2ak - 1$, so $k^2 + 1 = 2ak$, and $k^2 - 1 = 2(ak - 1)$. So $y' = \dfrac{2(ak-1)}{\sqrt{a^2-1}\,(2ak + 2k\cos x)} = \dfrac{ak-1}{\sqrt{a^2-1}k\,(a + \cos x)}$. But $ak - 1 = a^2 + a\sqrt{a^2-1} - 1 = k\sqrt{a^2-1}$, so $y' = 1/(a + \cos x)$.

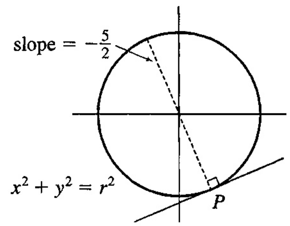
25. $y = x^4 - 2x^2 - x \Rightarrow y' = 4x^3 - 4x - 1$. The equation of the tangent line at $x = a$ is $y - \left(a^4 - 2a^2 - a\right) = (4a^3 - 4a - 1)(x - a)$ or $y = (4a^3 - 4a - 1)x + \left(-3a^4 + 2a^2\right)$ and similarly for $x = b$. So if at $x = a$ and $x = b$ we have the same tangent line, then $4a^3 - 4a - 1 = 4b^3 - 4b - 1$ and $-3a^4 + 2a^2 = -3b^4 + 2b^2$. The first equation gives $a^3 - b^3 = a - b \Rightarrow (a-b)(a^2 + ab + b^2) = (a - b)$. Assuming $a \neq b$, we have $1 = a^2 + ab + b^2$. The second equation gives $3(a^4 - b^4) = 2(a^2 - b^2) \Rightarrow 3(a^2 - b^2)(a^2 + b^2) = 2(a^2 - b^2)$ which is true if $a = -b$. Substituting into $1 = a^2 + ab + b^2$ gives

$1 = a^2 - a^2 + a^2 \Rightarrow a = \pm 1$ so that $a = 1$ and $b = -1$ or vice versa. Thus, the points $(1, -2)$ and $(-1, 0)$ have a common tangent line.

As long as there are only two such points, we are done. So we show that these are in fact the only two such points. Suppose that $a^2 - b^2 \neq 0$. Then $3(a^2 - b^2)(a^2 + b^2) = 2(a^2 - b^2)$ gives $3(a^2 + b^2) = 2$ or $a^2 + b^2 = \frac{2}{3}$. Thus, $ab = (a^2 + ab + b^2) - (a^2 + b^2) = 1 - \frac{2}{3} = \frac{1}{3}$, so $b = \dfrac{1}{3a}$. Hence, $a^2 + \dfrac{1}{9a^2} = \dfrac{2}{3}$, so $9a^4 + 1 = 6a^2 \Rightarrow$ $0 = 9a^4 - 6a^2 + 1 = (3a^2 - 1)^2$. So $3a^2 - 1 = 0 \Rightarrow a^2 = \frac{1}{3} \Rightarrow b^2 = \dfrac{1}{9a^2} = \frac{1}{3} = a^2$, contradicting our assumption that $a^2 \neq b^2$.

27.



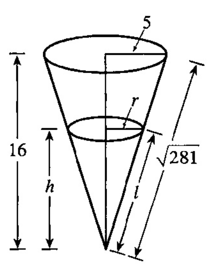


Because of the periodic nature of the lattice points, it suffices to consider the points in the 5×2 grid shown. We can see that the minimum value of r occurs when there is a line with slope $\frac{2}{5}$ which touches the circle centered at $(3, 1)$ and the circles centered at $(0, 0)$ and $(5, 2)$. To find P, the point at which the line is tangent to the circle at $(0, 0)$, we simultaneously solve $x^2 + y^2 = r^2$ and $y = -\frac{5}{2}x \Rightarrow x^2 + \frac{25}{4}x^2 = r^2 \Rightarrow x^2 = \frac{4}{29}r^2 \Rightarrow$ $x = \frac{2}{\sqrt{29}}r$, $y = -\frac{5}{\sqrt{29}}r$. To find Q, we either use symmetry or solve $(x - 3)^2 + (y - 1)^2 = r^2$ and $y - 1 = -\frac{5}{2}(x - 3)$. As above, we get $x = 3 - \frac{2}{\sqrt{29}}r$, $y = 1 + \frac{5}{\sqrt{29}}r$. Now the slope of the line PQ is $\frac{2}{5}$, so

$$m_{PQ} = \frac{1 + \frac{5}{\sqrt{29}}r - \left(-\frac{5}{\sqrt{29}}r\right)}{3 - \frac{2}{\sqrt{29}}r - \frac{2}{\sqrt{29}}r} = \frac{1 + \frac{10}{\sqrt{29}}r}{3 - \frac{4}{\sqrt{29}}r} = \frac{\sqrt{29} + 10r}{3\sqrt{29} - 4r} = \frac{2}{5} \Rightarrow 5\sqrt{29} + 50r = 6\sqrt{29} - 8r \Leftrightarrow$$

$58r = \sqrt{29} \Leftrightarrow r = \frac{\sqrt{29}}{58}$. So the minimum value of r for which any line with slope $\frac{2}{5}$ intersects circles with radius r centered at the lattice points on the plane is $r = \frac{\sqrt{29}}{58} \approx 0.093$.

29.



By similar triangles, $\dfrac{r}{5} = \dfrac{h}{16} \Rightarrow r = \dfrac{5h}{16}$. The volume of the cone is $V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi\left(\dfrac{5h}{16}\right)^2 h = \dfrac{25\pi}{768}h^3$, so $\dfrac{dV}{dt} = \dfrac{25\pi}{256}h^2\dfrac{dh}{dt}$. Now the rate of change of the volume is also equal to the difference of what is being added ($2\text{ cm}^3/\text{min}$) and what is oozing out ($k\pi rl$, where πrl is the area of the cone and k is a proportionality constant). Thus, $\dfrac{dV}{dt} = 2 - k\pi rl$.

Equating the two expressions for $\dfrac{dV}{dt}$ and substituting $h = 10$, $\dfrac{dh}{dt} = -0.3$, $r = \dfrac{5(10)}{16} = \dfrac{25}{8}$, and $\dfrac{l}{\sqrt{281}} = \dfrac{10}{16}$ $\Leftrightarrow l = \frac{5}{8}\sqrt{281}$, we get $\frac{25\pi}{256}(10)^2(-0.3) = 2 - k\pi\frac{25}{8} \cdot \frac{5}{8}\sqrt{281} \Leftrightarrow \dfrac{125k\pi\sqrt{281}}{64} = 2 + \dfrac{750\pi}{256}$. Solving for k gives us $k = \dfrac{256 + 375\pi}{250\pi\sqrt{281}}$. To maintain a certain height, the rate of oozing, $k\pi rl$, must equal the rate of the liquid being poured in; that is, $\dfrac{dV}{dt} = 0$. $k\pi rl = \dfrac{256 + 375\pi}{250\pi\sqrt{281}} \cdot \pi \cdot \dfrac{25}{8} \cdot \dfrac{5\sqrt{281}}{8} = \dfrac{256 + 375\pi}{128} \approx 11.204\text{ cm}^3/\text{min}$.