4 □ APPLICATIONS OF DIFFERENTIATION

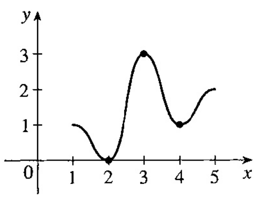
4.1 Maximum and Minimum Values

1. A function f has an **absolute minimum** at $x = c$ if $f(c)$ is the smallest function value on the entire domain of f, whereas f has a **local minimum** at c if $f(c)$ is the smallest function value when x is near c.

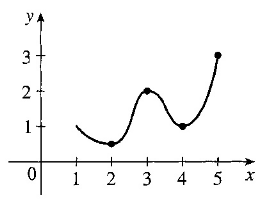
3. Absolute maximum at b; absolute minimum at d; local maxima at b and e; local minima at d and s; neither a maximum nor a minimum at a, c, r, and t.

5. Absolute maximum value is $f(4) = 4$; absolute minimum value is $f(7) = 0$; local maximum values are $f(4) = 4$ and $f(6) = 3$; local minimum values are $f(2) = 1$ and $f(5) = 2$.

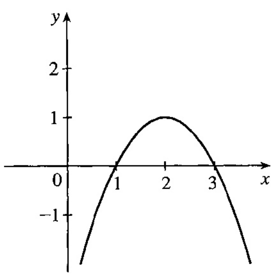
7. Absolute minimum at 2, absolute maximum at 3, local minimum at 4



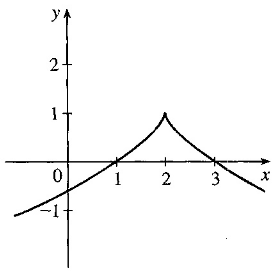
9. Absolute maximum at 5, absolute minimum at 2, local maximum at 3, local minima at 2 and 4



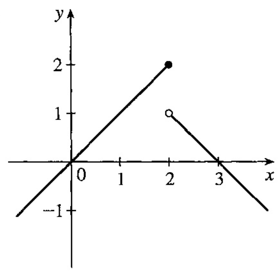
11. (a)



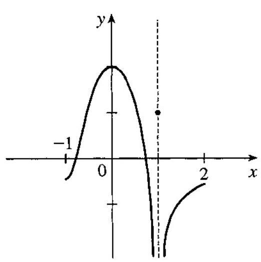
(b)



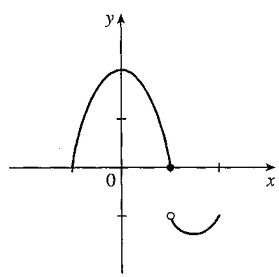
(c)



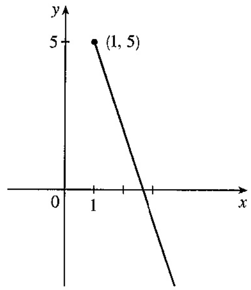
13. (a) *Note:* By the Extreme Value Theorem, f must *not* be continuous; because if it were, it would attain an absolute minimum.



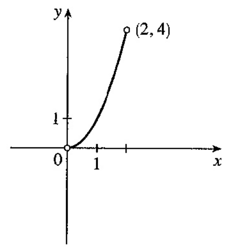
(b)



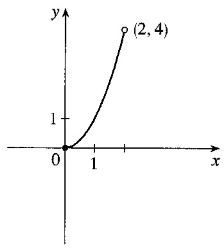
15. $f(x) = 8 - 3x$, $x \geq 1$. Absolute maximum $f(1) = 5$; no local maximum. No absolute or local minimum.



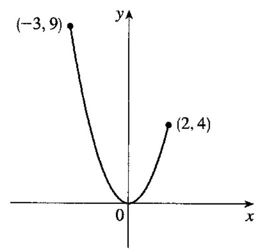
17. $f(x) = x^2$, $0 < x < 2$. No absolute or local maximum or minimum value.



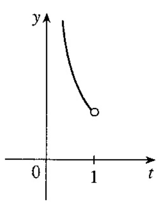
19. $f(x) = x^2$, $0 \leq x < 2$. Absolute minimum $f(0) = 0$; no local minimum. No absolute or local maximum.



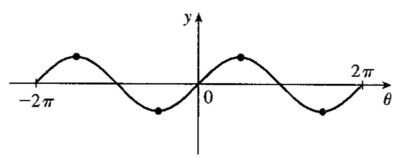
21. $f(x) = x^2$, $-3 \leq x \leq 2$. Absolute maximum $f(-3) = 9$. No local maximum. Absolute and local minimum $f(0) = 0$.



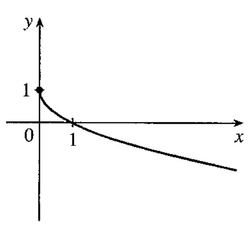
23. $f(t) = 1/t$, $0 < t < 1$. No maximum or minimum.



25. $f(\theta) = \sin\theta$, $-2\pi \leq \theta \leq 2\pi$. Absolute and local maxima $f\left(-\frac{3\pi}{2}\right) = f\left(\frac{\pi}{2}\right) = 1$. Absolute and local minima $f\left(-\frac{\pi}{2}\right) = f\left(\frac{3\pi}{2}\right) = -1$.

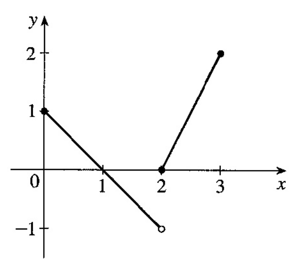


27. $f(x) = 1 - \sqrt{x}$. Absolute maximum $f(0) = 1$; no local maximum. No absolute or local minimum.



29. $f(x) = \begin{cases} 1 - x & \text{if } 0 \leq x < 2 \\ 2x - 4 & \text{if } 2 \leq x \leq 3 \end{cases}$

Absolute maximum $f(3) = 2$; no local maximum. No absolute or local minimum.



31. $f(x) = 5x^2 + 4x \Rightarrow f'(x) = 10x + 4$. $f'(x) = 0 \Rightarrow x = -\frac{2}{5}$, so $-\frac{2}{5}$ is the only critical number.

33. $f(x) = x^3 + 3x^2 - 24x \Rightarrow f'(x) = 3x^2 + 6x - 24 = 3(x^2 + 2x - 8)$.

$f'(x) = 0 \Rightarrow 3(x+4)(x-2) = 0 \Rightarrow x = -4, 2$. These are the only critical numbers.

35. $s(t) = 3t^4 + 4t^3 - 6t^2 \Rightarrow s'(t) = 12t^3 + 12t^2 - 12t$. $s'(t) = 0 \Rightarrow 12t(t^2 + t - 1) \Rightarrow t = 0$ or $t^2 + t - 1 = 0$. Using the quadratic formula to solve the latter equation gives us

$t = \dfrac{-1 \pm \sqrt{1^2 - 4(1)(-1)}}{2(1)} = \dfrac{-1 \pm \sqrt{5}}{2} \approx 0.618, -1.618$. The three critical numbers are $0, \dfrac{-1 \pm \sqrt{5}}{2}$.

37. $$g(x) = |2x + 3| = \begin{cases} 2x + 3 & \text{if } 2x + 3 \geq 0 \\ -(2x+3) & \text{if } 2x + 3 < 0 \end{cases} \Rightarrow g'(x) = \begin{cases} 2 & \text{if } x > -\frac{3}{2} \\ -2 & \text{if } x < -\frac{3}{2} \end{cases}$$

$g'(x)$ is never 0, but $g'(x)$ does not exist for $x = -\frac{3}{2}$, so $-\frac{3}{2}$ is the only critical number.

39. $g(t) = 5t^{2/3} + t^{5/3} \Rightarrow g'(t) = \frac{10}{3}t^{-1/3} + \frac{5}{3}t^{2/3}$. $g'(0)$ does not exist, so $t = 0$ is a critical number.

$g'(t) = \frac{5}{3}t^{-1/3}(2 + t) = 0 \Leftrightarrow t = -2$, so $t = -2$ is also a critical number.

41. $F(x) = x^{4/5}(x-4)^2 \Rightarrow$

$$F'(x) = x^{4/5} \cdot 2(x-4) + (x-4)^2 \cdot \tfrac{4}{5}x^{-1/5} = \tfrac{1}{5}x^{-1/5}(x-4)\,[5 \cdot x \cdot 2 + (x-4) \cdot 4]$$
$$= \frac{(x-4)(14x-16)}{5x^{1/5}} = \frac{2(x-4)(7x-8)}{5x^{1/5}} = 0 \text{ when } x = 4, \tfrac{8}{7}\text{; and } F'(0) \text{ does not exist.}$$

Critical numbers are $0, \frac{8}{7}, 4$.

43. $f(\theta) = 2\cos\theta + \sin^2\theta \Rightarrow f'(\theta) = -2\sin\theta + 2\sin\theta\cos\theta$. $f'(\theta) = 0 \Rightarrow 2\sin\theta(\cos\theta - 1) = 0 \Rightarrow$ $\sin\theta = 0$ or $\cos\theta = 1 \Rightarrow \theta = n\pi$ (n an integer) or $\theta = 2n\pi$. The solutions $\theta = n\pi$ include the solutions $\theta = 2n\pi$, so the critical numbers are $\theta = n\pi$.

45. $f(x) = x\ln x \Rightarrow f'(x) = x(1/x) + (\ln x) \cdot 1 = \ln x + 1$. $f'(x) = 0 \Leftrightarrow \ln x = -1 \Leftrightarrow$ $x = e^{-1} = 1/e$. Therefore, the only critical number is $x = 1/e$.

47. $f(x) = 3x^2 - 12x + 5$, $[0, 3]$. $f'(x) = 6x - 12 = 0 \Leftrightarrow x = 2$. Applying the Closed Interval Method, we find that $f(0) = 5$, $f(2) = -7$, and $f(3) = -4$. So $f(0) = 5$ is the absolute maximum value and $f(2) = -7$ is the absolute minimum value.

49. $f(x) = 2x^3 - 3x^2 - 12x + 1$, $[-2, 3]$. $f'(x) = 6x^2 - 6x - 12 = 6(x^2 - x - 2) = 6(x-2)(x+1) = 0 \Leftrightarrow$ $x = 2, -1$. $f(-2) = -3$, $f(-1) = 8$, $f(2) = -19$, and $f(3) = -8$. So $f(-1) = 8$ is the absolute maximum value and $f(2) = -19$ is the absolute minimum value.

51. $f(x) = x^4 - 2x^2 + 3$, $[-2, 3]$. $f'(x) = 4x^3 - 4x = 4x(x^2 - 1) = 4x(x+1)(x-1) = 0 \Leftrightarrow x = -1, 0, 1$. $f(-2) = 11$, $f(-1) = 2$, $f(0) = 3$, $f(1) = 2$, $f(3) = 66$. So $f(3) = 66$ is the absolute maximum value and $f(\pm 1) = 2$ is the absolute minimum value.

53. $f(x) = \dfrac{x}{x^2+1}$, $[0,2]$. $f'(x) = \dfrac{(x^2+1) - x(2x)}{(x^2+1)^2} = \dfrac{1-x^2}{(x^2+1)^2} = 0 \quad \Leftrightarrow \quad x = \pm 1$, but -1 is not in $[0,2]$.

$f(0) = 0$, $f(1) = \frac{1}{2}$, $f(2) = \frac{2}{5}$. So $f(1) = \frac{1}{2}$ is the absolute maximum value and $f(0) = 0$ is the absolute minimum value.

55. $f(t) = t\sqrt{4-t^2}$, $[-1,2]$.

$f'(t) = t \cdot \frac{1}{2}(4-t^2)^{-1/2}(-2t) + (4-t^2)^{1/2} \cdot 1 = \dfrac{-t^2}{\sqrt{4-t^2}} + \sqrt{4-t^2} = \dfrac{-t^2 + (4-t^2)}{\sqrt{4-t^2}} = \dfrac{4-2t^2}{\sqrt{4-t^2}}$.

$f'(t) = 0 \quad \Rightarrow \quad 4 - 2t^2 = 0 \quad \Rightarrow \quad t^2 = 2 \quad \Rightarrow \quad t = \pm\sqrt{2}$, but $t = -\sqrt{2}$ is not in the given interval, $[-1,2]$. $f'(t)$ does not exist if $4 - t^2 = 0 \quad \Rightarrow \quad t = \pm 2$, but -2 is not in the given interval. $f(-1) = -\sqrt{3}$, $f(\sqrt{2}) = 2$, and $f(2) = 0$. So $f(\sqrt{2}) = 2$ is the absolute maximum value and $f(-1) = -\sqrt{3}$ is the absolute minimum value.

57. $f(x) = \sin x + \cos x$, $\left[0, \frac{\pi}{3}\right]$. $f'(x) = \cos x - \sin x = 0 \quad \Leftrightarrow \quad \sin x = \cos x \quad \Rightarrow \quad \dfrac{\sin x}{\cos x} = 1 \quad \Rightarrow$

$\tan x = 1 \quad \Rightarrow \quad x = \frac{\pi}{4}$. $f(0) = 1$, $f\left(\frac{\pi}{4}\right) = \sqrt{2} \approx 1.41$, $f\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}+1}{2} \approx 1.37$. So $f\left(\frac{\pi}{4}\right) = \sqrt{2}$ is the absolute maximum value and $f(0) = 1$ is the absolute minimum value.

59. $f(x) = xe^{-x}$, $[0,2]$. $f'(x) = x(-e^{-x}) + e^{-x} = e^{-x}(1-x) = 0 \quad \Leftrightarrow \quad x = 1$.

$f(0) = 0$, $f(1) = e^{-1} = 1/e \approx 0.37$, $f(2) = 2/e^2 \approx 0.27$. So $f(1) = 1/e$ is the absolute maximum value and $f(0) = 0$ is the absolute minimum value.

61. $f(x) = x - 3\ln x$, $[1,4]$. $f'(x) = 1 - \dfrac{3}{x} = \dfrac{x-3}{x} = 0 \quad \Leftrightarrow \quad x = 3$. f' does not exist for $x = 0$, but 0 is not in the domain of f. $f(1) = 1$, $f(3) = 3 - 3\ln 3 \approx -0.296$, $f(4) = 4 - 3\ln 4 \approx -0.159$. So $f(1) = 1$ is the absolute maximum value and $f(3) = 3 - 3\ln 3 \approx -0.296$ is the absolute minimum value.

63. $f(x) = x^a(1-x)^b$, $0 \le x \le 1$, $a > 0$, $b > 0$.

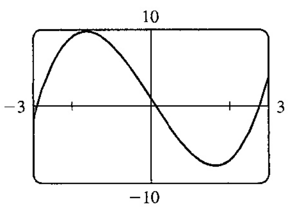
$$\begin{aligned} f'(x) &= x^a \cdot b(1-x)^{b-1}(-1) + (1-x)^b \cdot ax^{a-1} = x^{a-1}(1-x)^{b-1}\,[x \cdot b(-1) + (1-x) \cdot a] \\ &= x^{a-1}(1-x)^{b-1}(a - ax - bx) \end{aligned}$$

At the endpoints, we have $f(0) = f(1) = 0$ [the minimum value of f]. In the interval $(0,1)$, $f'(x) = 0 \quad \Leftrightarrow$ $x = \dfrac{a}{a+b}$.

$$f\left(\frac{a}{a+b}\right) = \left(\frac{a}{a+b}\right)^a \left(1 - \frac{a}{a+b}\right)^b = \frac{a^a}{(a+b)^a}\left(\frac{a+b-a}{a+b}\right)^b = \frac{a^a}{(a+b)^a} \cdot \frac{b^b}{(a+b)^b} = \frac{a^a b^b}{(a+b)^{a+b}}.$$

So $f\left(\dfrac{a}{a+b}\right) = \dfrac{a^a b^b}{(a+b)^{a+b}}$ is the absolute maximum value.

65. (a)



From the graph, it appears that the absolute maximum value is about $f(-1.63) = 9.71$, and the absolute minimum value is about $f(1.63) = -7.71$. These values make sense because the graph is symmetric about the point $(0,1)$. ($y = x^3 - 8x$ is symmetric about the origin.)

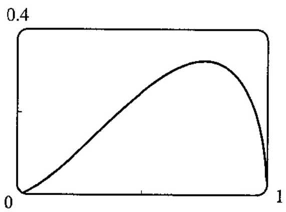
(b) $f(x) = x^3 - 8x + 1 \quad \Rightarrow \quad f'(x) = 3x^2 - 8$. So $f'(x) = 0 \quad \Rightarrow \quad x = \pm\sqrt{\frac{8}{3}}$.

$$f\left(\pm\sqrt{\tfrac{8}{3}}\right) = \left(\pm\sqrt{\tfrac{8}{3}}\right)^3 - 8\left(\pm\sqrt{\tfrac{8}{3}}\right) + 1 = \pm\tfrac{8}{3}\sqrt{\tfrac{8}{3}} \mp 8\sqrt{\tfrac{8}{3}} + 1$$

$$= -\tfrac{16}{3}\sqrt{\tfrac{8}{3}} + 1 = 1 - \tfrac{32\sqrt{6}}{9} \text{ [minimum]} \quad \text{or} \quad \tfrac{16}{3}\sqrt{\tfrac{8}{3}} + 1 = 1 + \tfrac{32\sqrt{6}}{9} \text{ [maximum]}$$

(From the graph, we see that the extreme values do not occur at the endpoints.)

67. (a)



From the graph, it appears that the absolute maximum value is about $f(0.75) = 0.32$, and the absolute minimum value is $f(0) = f(1) = 0$; that is, at both endpoints.

(b) $f(x) = x\sqrt{x - x^2} \quad \Rightarrow \quad f'(x) = x \cdot \dfrac{1 - 2x}{2\sqrt{x - x^2}} + \sqrt{x - x^2} = \dfrac{(x - 2x^2) + (2x - 2x^2)}{2\sqrt{x - x^2}} = \dfrac{3x - 4x^2}{2\sqrt{x - x^2}}$.

So $f'(x) = 0 \quad \Rightarrow \quad 3x - 4x^2 = 0 \quad \Rightarrow \quad x(3 - 4x) = 0 \quad \Rightarrow \quad x = 0$ or $\frac{3}{4}$. $f(0) = f(1) = 0$ [minimum], and $f\left(\frac{3}{4}\right) = \frac{3}{4}\sqrt{\frac{3}{4} - \left(\frac{3}{4}\right)^2} = \frac{3\sqrt{3}}{16}$ [maximum].

69. The density is defined as $\rho = \dfrac{\text{mass}}{\text{volume}} = \dfrac{1000}{V(T)}$ (in g/cm^3). But a critical point of ρ will also be a critical point of V $\left[\text{since } \dfrac{d\rho}{dT} = -1000V^{-2}\dfrac{dV}{dT} \text{ and } V \text{ is never } 0\right]$, and V is easier to differentiate than ρ.

$V(T) = 999.87 - 0.06426T + 0.0085043T^2 - 0.0000679T^3 \quad \Rightarrow$

$V'(T) = -0.06426 + 0.0170086T - 0.0002037T^2$. Setting this equal to 0 and using the quadratic formula to find T, we get $T = \dfrac{-0.0170086 \pm \sqrt{0.0170086^2 - 4 \cdot 0.0002037 \cdot 0.06426}}{2(-0.0002037)} \approx 3.9665°\text{C}$ or $79.5318°\text{C}$. Since we are only interested in the region $0°\text{C} \le T \le 30°\text{C}$, we check the density ρ at the endpoints and at $3.9665°\text{C}$: $\rho(0) \approx \dfrac{1000}{999.87} \approx 1.00013$; $\rho(30) \approx \dfrac{1000}{1003.7628} \approx 0.99625$; $\rho(3.9665) \approx \dfrac{1000}{999.7447} \approx 1.000255$. So water has its maximum density at about $3.9665°\text{C}$.

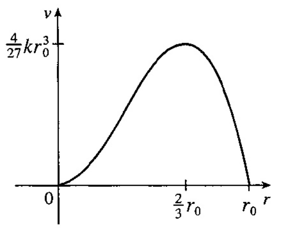
71. We apply the Closed Interval Method to the continuous function

$I(t) = 0.00009045t^5 + 0.001438t^4 - 0.06561t^3 + 0.4598t^2 - 0.6270t + 99.33$ on $[0, 10]$. Its derivative is $I'(t) = 0.00045225t^4 + 0.005752t^3 - 0.19683t^2 + 0.9196t - 0.6270$. Since I' exists for all t, the only critical numbers of I occur when $I'(t) = 0$. We use a root-finder on a computer algebra system (or a graphing device) to find that $I'(t) = 0$ when $t \approx -29.7186, 0.8231, 5.1309$, or 11.0459, but only the second and third roots lie in the interval $[0, 10]$. The values of I at these critical numbers are $I(0.8231) \approx 99.09$ and $I(5.1309) \approx 100.67$. The values of I at the endpoints of the interval are $I(0) = 99.33$ and $I(10) \approx 96.86$. Comparing these four numbers, we see that food was most expensive at $t \approx 5.1309$ (corresponding roughly to August, 1989) and cheapest at $t = 10$ (midyear 1994).

73. (a) $v(r) = k(r_0 - r)r^2 = kr_0r^2 - kr^3 \;\Rightarrow\; v'(r) = 2kr_0r - 3kr^2$. $v'(r) = 0 \;\Rightarrow\; kr(2r_0 - 3r) = 0$ $\Rightarrow\; r = 0$ or $\frac{2}{3}r_0$ (but 0 is not in the interval). Evaluating v at $\frac{1}{2}r_0$, $\frac{2}{3}r_0$, and r_0, we get $v(\frac{1}{2}r_0) = \frac{1}{8}kr_0^3$, $v(\frac{2}{3}r_0) = \frac{4}{27}kr_0^3$, and $v(r_0) = 0$. Since $\frac{4}{27} > \frac{1}{8}$, v attains its maximum value at $r = \frac{2}{3}r_0$. This supports the statement in the text.

(b) From part (a), the maximum value of v is $\frac{4}{27}kr_0^3$.

(c)



75. $f(x) = x^{101} + x^{51} + x + 1 \;\Rightarrow\; f'(x) = 101x^{100} + 51x^{50} + 1 \geq 1$ for all x, so $f'(x) = 0$ has no solution. Thus, $f(x)$ has no critical number, so $f(x)$ can have no local maximum or minimum.

77. If f has a local minimum at c, then $g(x) = -f(x)$ has a local maximum at c, so $g'(c) = 0$ by the case of Fermat's Theorem proved in the text. Thus, $f'(c) = -g'(c) = 0$.

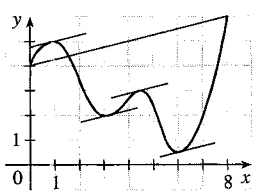
4.2 The Mean Value Theorem

1. $f(x) = x^2 - 4x + 1$, $[0, 4]$. Since f is a polynomial, it is continuous and differentiable on $\mathbb{R}$, so it is continuous on $[0, 4]$ and differentiable on $(0, 4)$. Also, $f(0) = 1 = f(4)$. $f'(c) = 0 \;\Leftrightarrow\; 2c - 4 = 0 \;\Leftrightarrow\; c = 2$, which is in the open interval $(0, 4)$, so $c = 2$ satisfies the conclusion of Rolle's Theorem.

3. $f(x) = \sin 2\pi x$, $[-1, 1]$. f, being the composite of the sine function and the polynomial $2\pi x$, is continuous and differentiable on $\mathbb{R}$, so it is continuous on $[-1, 1]$ and differentiable on $(-1, 1)$. Also, $f(-1) = 0 = f(1)$. $f'(c) = 0 \;\Leftrightarrow\; 2\pi\cos 2\pi c = 0 \;\Leftrightarrow\; \cos 2\pi c = 0 \;\Leftrightarrow\; 2\pi c = \pm\frac{\pi}{2} + 2\pi n \;\Leftrightarrow\; c = \pm\frac{1}{4} + n$. If $n = 0$ or ± 1, then $c = \pm\frac{1}{4}, \pm\frac{3}{4}$ is in $(-1, 1)$.

5. $f(x) = 1 - x^{2/3}$. $f(-1) = 1 - (-1)^{2/3} = 1 - 1 = 0 = f(1)$. $f'(x) = -\frac{2}{3}x^{-1/3}$, so $f'(c) = 0$ has no solution. This does not contradict Rolle's Theorem, since $f'(0)$ does not exist, and so f is not differentiable on $(-1, 1)$.

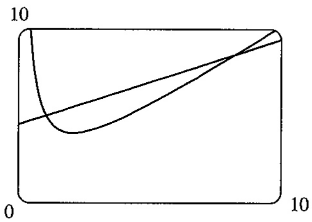
7. $\dfrac{f(8) - f(0)}{8 - 0} = \dfrac{6 - 4}{8} = \dfrac{1}{4}$. The values of c which satisfy $f'(c) = \frac{1}{4}$ seem to be about $c = 0.8, 3.2, 4.4$, and 6.1.



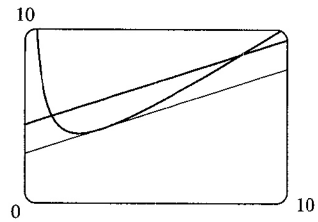
9. (a), (b) The equation of the secant line is

$$y - 5 = \frac{8.5 - 5}{8 - 1}(x - 1) \quad \Leftrightarrow$$

$$y = \tfrac{1}{2}x + \tfrac{9}{2}.$$



(c) $f(x) = x + 4/x \quad \Rightarrow \quad f'(x) = 1 - 4/x^2$.
So $f'(c) = \frac{1}{2} \quad \Rightarrow \quad c^2 = 8 \quad \Rightarrow \quad c = 2\sqrt{2}$, and $f(c) = 2\sqrt{2} + \frac{4}{2\sqrt{2}} = 3\sqrt{2}$. Thus, an equation of the tangent line is $y - 3\sqrt{2} = \frac{1}{2}(x - 2\sqrt{2}) \quad \Leftrightarrow$
$y = \frac{1}{2}x + 2\sqrt{2}$.



11. $f(x) = 3x^2 + 2x + 5$, $[-1, 1]$. f is continuous on $[-1, 1]$ and differentiable on $(-1, 1)$ since polynomials are continuous and differentiable on $\mathbb{R}$. $f'(c) = \dfrac{f(b) - f(a)}{b - a} \quad \Leftrightarrow \quad 6c + 2 = \dfrac{f(1) - f(-1)}{1 - (-1)} = \dfrac{10 - 6}{2} = 2 \quad \Leftrightarrow$ $6c = 0 \quad \Leftrightarrow \quad c = 0$, which is in $(-1, 1)$.

13. $f(x) = e^{-2x}$, $[0, 3]$. f is continuous and differentiable on $\mathbb{R}$, so it is continuous on $[0, 3]$ and differentiable on $(0, 3)$. $f'(c) = \dfrac{f(b) - f(a)}{b - a} \quad \Leftrightarrow \quad -2e^{-2c} = \dfrac{e^{-6} - e^0}{3 - 0} \quad \Leftrightarrow \quad e^{-2c} = \dfrac{1 - e^{-6}}{6} \quad \Leftrightarrow \quad -2c = \ln\left(\dfrac{1 - e^{-6}}{6}\right)$ $\Leftrightarrow \quad c = -\dfrac{1}{2}\ln\left(\dfrac{1 - e^{-6}}{6}\right) \approx 0.897$, which is in $(0, 3)$.

15. $f(x) = |x - 1|$. $f(3) - f(0) = |3 - 1| - |0 - 1| = 1$. Since $f'(c) = -1$ if $c < 1$ and $f'(c) = 1$ if $c > 1$, $f'(c)(3 - 0) = \pm 3$ and so is never equal to 1. This does not contradict the Mean Value Theorem since $f'(1)$ does not exist.

17. Let $f(x) = 1 + 2x + x^3 + 4x^5$. Then $f(-1) = -6 < 0$ and $f(0) = 1 > 0$. Since f is a polynomial, it is continuous, so the Intermediate Value Theorem says that there is a number c between -1 and 0 such that $f(c) = 0$. Thus, the given equation has a real root. Suppose the equation has distinct real roots a and b with $a < b$. Then $f(a) = f(b) = 0$. Since f is a polynomial, it is differentiable on (a, b) and continuous on $[a, b]$. By Rolle's Theorem, there is a number r in (a, b) such that $f'(r) = 0$. But $f'(x) = 2 + 3x^2 + 20x^4 \geq 2$ for all x, so $f'(x)$ can never be 0. This contradiction shows that the equation can't have two distinct real roots. Hence, it has exactly one real root.

19. Let $f(x) = x^3 - 15x + c$ for x in $[-2, 2]$. If f has two real roots a and b in $[-2, 2]$, with $a < b$, then $f(a) = f(b) = 0$. Since the polynomial f is continuous on $[a, b]$ and differentiable on (a, b), Rolle's Theorem implies that there is a number r in (a, b) such that $f'(r) = 0$. Now $f'(r) = 3r^2 - 15$. Since r is in (a, b), which is contained in $[-2, 2]$, we have $|r| < 2$, so $r^2 < 4$. It follows that $3r^2 - 15 < 3 \cdot 4 - 15 = -3 < 0$. This contradicts $f'(r) = 0$, so the given equation can't have two real roots in $[-2, 2]$. Hence, it has at most one real root in $[-2, 2]$.

21. (a) Suppose that a cubic polynomial $P(x)$ has roots $a_1 < a_2 < a_3 < a_4$, so $P(a_1) = P(a_2) = P(a_3) = P(a_4)$. By Rolle's Theorem there are numbers c_1, c_2, c_3 with $a_1 < c_1 < a_2$, $a_2 < c_2 < a_3$ and $a_3 < c_3 < a_4$ and $P'(c_1) = P'(c_2) = P'(c_3) = 0$. Thus, the second-degree polynomial $P'(x)$ has three distinct real roots, which is impossible.

(b) We prove by induction that a polynomial of degree n has at most n real roots. This is certainly true for $n = 1$. Suppose that the result is true for all polynomials of degree n and let $P(x)$ be a polynomial of degree $n + 1$. Suppose that $P(x)$ has more than $n + 1$ real roots, say $a_1 < a_2 < a_3 < \cdots < a_{n+1} < a_{n+2}$. Then $P(a_1) = P(a_2) = \cdots = P(a_{n+2}) = 0$. By Rolle's Theorem there are real numbers $c_1, \ldots, c_{n+1}$ with $a_1 < c_1 < a_2, \ldots, a_{n+1} < c_{n+1} < a_{n+2}$ and $P'(c_1) = \cdots = P'(c_{n+1}) = 0$. Thus, the nth degree polynomial $P'(x)$ has at least $n + 1$ roots. This contradiction shows that $P(x)$ has at most $n + 1$ real roots.

23. By the Mean Value Theorem, $f(4) - f(1) = f'(c)(4 - 1)$ for some $c \in (1, 4)$. But for every $c \in (1, 4)$ we have $f'(c) \geq 2$. Putting $f'(c) \geq 2$ into the above equation and substituting $f(1) = 10$, we get $f(4) = f(1) + f'(c)(4 - 1) = 10 + 3f'(c) \geq 10 + 3 \cdot 2 = 16$. So the smallest possible value of $f(4)$ is 16.

25. Suppose that such a function f exists. By the Mean Value Theorem there is a number $0 < c < 2$ with $f'(c) = \dfrac{f(2) - f(0)}{2 - 0} = \dfrac{5}{2}$. But this is impossible since $f'(x) \leq 2 < \frac{5}{2}$ for all x, so no such function can exist.

27. We use Exercise 26 with $f(x) = \sqrt{1 + x}$, $g(x) = 1 + \frac{1}{2}x$, and $a = 0$. Notice that $f(0) = 1 = g(0)$ and $f'(x) = \dfrac{1}{2\sqrt{1 + x}} < \dfrac{1}{2} = g'(x)$ for $x > 0$. So by Exercise 26, $f(b) < g(b) \Rightarrow \sqrt{1 + b} < 1 + \frac{1}{2}b$ for $b > 0$.

Another method: Apply the Mean Value Theorem directly to either $f(x) = 1 + \frac{1}{2}x - \sqrt{1 + x}$ or $g(x) = \sqrt{1 + x}$ on $[0, b]$.

29. Let $f(x) = \sin x$ and let $b < a$. Then $f(x)$ is continuous on $[b, a]$ and differentiable on (b, a). By the Mean Value Theorem, there is a number $c \in (b, a)$ with $\sin a - \sin b = f(a) - f(b) = f'(c)(a - b) = (\cos c)(a - b)$. Thus, $|\sin a - \sin b| \leq |\cos c|\,|b - a| \leq |a - b|$. If $a < b$, then $|\sin a - \sin b| = |\sin b - \sin a| \leq |b - a| = |a - b|$. If $a = b$, both sides of the inequality are 0.

31. For $x > 0$, $f(x) = g(x)$, so $f'(x) = g'(x)$. For $x < 0$, $f'(x) = (1/x)' = -1/x^2$ and $g'(x) = (1 + 1/x)' = -1/x^2$, so again $f'(x) = g'(x)$. However, the domain of $g(x)$ is not an interval [it is $(-\infty, 0) \cup (0, \infty)$] so we cannot conclude that $f - g$ is constant (in fact it is not).

33. Let $f(x) = \arcsin\left(\dfrac{x - 1}{x + 1}\right) - 2\arctan\sqrt{x} + \frac{\pi}{2}$. Note that the domain of f is $[0, \infty)$. Thus,

$$f'(x) = \frac{1}{\sqrt{1 - \left(\dfrac{x - 1}{x + 1}\right)^2}}\,\frac{(x + 1) - (x - 1)}{(x + 1)^2} - \frac{2}{1 + x} \cdot \frac{1}{2\sqrt{x}} = \frac{1}{\sqrt{x}\,(x + 1)} - \frac{1}{\sqrt{x}\,(x + 1)} = 0.$$ Then

$f(x) = C$ on $(0, \infty)$ by Theorem 5. By continuity of f, $f(x) = C$ on $[0, \infty)$. To find C, we let $x = 0 \Rightarrow$ $\arcsin(-1) - 2\arctan(0) + \frac{\pi}{2} = C \Rightarrow -\frac{\pi}{2} - 0 + \frac{\pi}{2} = 0 = C$. Thus, $f(x) = 0 \Rightarrow$ $\arcsin\left(\frac{x-1}{x+1}\right) = 2\arctan\sqrt{x} - \frac{\pi}{2}$.

35. Let $g(t)$ and $h(t)$ be the position functions of the two runners and let $f(t) = g(t) - h(t)$. By hypothesis, $f(0) = g(0) - h(0) = 0$ and $f(b) = g(b) - h(b) = 0$, where b is the finishing time. Then by the Mean Value Theorem, there is a time c, with $0 < c < b$, such that $f'(c) = \frac{f(b) - f(0)}{b - 0}$. But $f(b) = f(0) = 0$, so $f'(c) = 0$. Since $f'(c) = g'(c) - h'(c) = 0$, we have $g'(c) = h'(c)$. So at time c, both runners have the same speed $g'(c) = h'(c)$.

4.3 How Derivatives Affect the Shape of a Graph

1. (a) f is increasing on $(0, 6)$ and $(8, 9)$.

(b) f is decreasing on $(6, 8)$.

(c) f is concave upward on $(2, 4)$ and $(7, 9)$.

(d) f is concave downward on $(0, 2)$ and $(4, 7)$.

(e) The points of inflection are $(2, 3)$, $(4, 4.5)$ and $(7, 4)$ (where the concavity changes).

3. (a) Use the Increasing/Decreasing (I/D) Test.

(b) Use the Concavity Test.

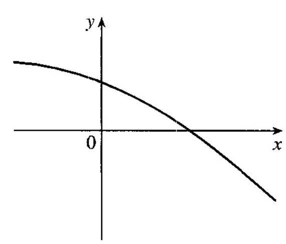
(c) At any value of x where the concavity changes, we have an inflection point at $(x, f(x))$.

5. (a) Since $f'(x) > 0$ on $(1, 5)$, f is increasing on this interval. Since $f'(x) < 0$ on $(0, 1)$ and $(5, 6)$, f is decreasing on these intervals.

(b) Since $f'(x) = 0$ at $x = 1$ and f' changes from negative to positive there, f changes from decreasing to increasing and has a local minimum at $x = 1$. Since $f'(x) = 0$ at $x = 5$ and f' changes from positive to negative there, f changes from increasing to decreasing and has a local maximum at $x = 5$.

7. There is an inflection point at $x = 1$ because $f''(x)$ changes from negative to positive there, and so the graph of f changes from concave downward to concave upward. There is an inflection point at $x = 7$ because $f''(x)$ changes from positive to negative there, and so the graph of f changes from concave upward to concave downward.

9. The function must be always decreasing and concave downward.



11. (a) $f(x) = x^3 - 12x + 1 \Rightarrow f'(x) = 3x^2 - 12 = 3(x+2)(x-2)$.
We don't need to include "3" in the chart to determine the sign of $f'(x)$.

Interval	$x+2$	$x-2$	$f'(x)$	f
$x < -2$	$-$	$-$	$+$	increasing on $(-\infty, -2)$
$-2 < x < 2$	$+$	$-$	$-$	decreasing on $(-2, 2)$
$x > 2$	$+$	$+$	$+$	increasing on $(2, \infty)$

So f is increasing on $(-\infty, -2)$ and $(2, \infty)$ and f is decreasing on $(-2, 2)$.

(b) f changes from increasing to decreasing at $x = -2$ and from decreasing to increasing at $x = 2$. Thus, $f(-2) = 17$ is a local maximum value and $f(2) = -15$ is a local minimum value.

(c) $f''(x) = 6x$. $f''(x) > 0 \Leftrightarrow x > 0$ and $f''(x) < 0 \Leftrightarrow x < 0$. Thus, f is concave upward on $(0, \infty)$ and concave downward on $(-\infty, 0)$. There is an inflection point where the concavity changes, at $(0, f(0)) = (0, 1)$.

13. (a) $f(x) = x^4 - 2x^2 + 3 \Rightarrow f'(x) = 4x^3 - 4x = 4x(x^2 - 1) = 4x(x+1)(x-1)$.

Interval	$x+1$	x	$x-1$	$f'(x)$	f
$x < -1$	$-$	$-$	$-$	$-$	decreasing on $(-\infty, -1)$
$-1 < x < 0$	$+$	$-$	$-$	$+$	increasing on $(-1, 0)$
$0 < x < 1$	$+$	$+$	$-$	$-$	decreasing on $(0, 1)$
$x > 1$	$+$	$+$	$+$	$+$	increasing on $(1, \infty)$

So f is increasing on $(-1, 0)$ and $(1, \infty)$ and f is decreasing on $(-\infty, -1)$ and $(0, 1)$.

(b) f changes from increasing to decreasing at $x = 0$ and from decreasing to increasing at $x = -1$ and $x = 1$. Thus, $f(0) = 3$ is a local maximum value and $f(\pm 1) = 2$ are local minimum values.

(c) $f''(x) = 12x^2 - 4 = 12\left(x^2 - \frac{1}{3}\right) = 12\left(x + 1/\sqrt{3}\right)\left(x - 1/\sqrt{3}\right)$. $f''(x) > 0 \Leftrightarrow x < -1/\sqrt{3}$ or $x > 1/\sqrt{3}$ and $f''(x) < 0 \Leftrightarrow -1/\sqrt{3} < x < 1/\sqrt{3}$. Thus, f is concave upward on $\left(-\infty, -\sqrt{3}/3\right)$ and $\left(\sqrt{3}/3, \infty\right)$ and concave downward on $\left(-\sqrt{3}/3, \sqrt{3}/3\right)$. There are inflection points at $\left(\pm\sqrt{3}/3, \frac{22}{9}\right)$.

15. (a) $f(x) = x - 2\sin x$ on $(0, 3\pi) \Rightarrow f'(x) = 1 - 2\cos x$. $f'(x) > 0 \Leftrightarrow 1 - 2\cos x > 0 \Leftrightarrow \cos x < \frac{1}{2}$ $\Leftrightarrow \frac{\pi}{3} < x < \frac{5\pi}{3}$ or $\frac{7\pi}{3} < x < 3\pi$. $f'(x) < 0 \Leftrightarrow \cos x > \frac{1}{2} \Leftrightarrow 0 < x < \frac{\pi}{3}$ or $\frac{5\pi}{3} < x < \frac{7\pi}{3}$. So f is increasing on $\left(\frac{\pi}{3}, \frac{5\pi}{3}\right)$ and $\left(\frac{7\pi}{3}, 3\pi\right)$, and f is decreasing on $\left(0, \frac{\pi}{3}\right)$ and $\left(\frac{5\pi}{3}, \frac{7\pi}{3}\right)$.

(b) f changes from increasing to decreasing at $x = \frac{5\pi}{3}$, and from decreasing to increasing at $x = \frac{\pi}{3}$ and at $x = \frac{7\pi}{3}$. Thus, $f\left(\frac{5\pi}{3}\right) = \frac{5\pi}{3} + \sqrt{3} \approx 6.97$ is a local maximum value and $f\left(\frac{\pi}{3}\right) = \frac{\pi}{3} - \sqrt{3} \approx -0.68$ and $f\left(\frac{7\pi}{3}\right) = \frac{7\pi}{3} - \sqrt{3} \approx 5.60$ are local minimum values.

(c) $f''(x) = 2\sin x > 0 \Leftrightarrow 0 < x < \pi$ and $2\pi < x < 3\pi$, $f''(x) < 0 \Leftrightarrow \pi < x < 2\pi$. Thus, f is concave upward on $(0, \pi)$ and $(2\pi, 3\pi)$, and f is concave downward on $(\pi, 2\pi)$. There are inflection points at (π, π) and $(2\pi, 2\pi)$.

17. (a) $y = f(x) = xe^x \Rightarrow f'(x) = xe^x + e^x = e^x(x+1)$. So $f'(x) > 0 \Leftrightarrow x+1>0 \Leftrightarrow x > -1$. Thus, f is increasing on $(-1, \infty)$ and decreasing on $(-\infty, -1)$.

(b) f changes from decreasing to increasing at its only critical number, $x = -1$. Thus, $f(-1) = -e^{-1}$ is a local minimum value.

(c) $f'(x) = e^x(x+1) \Rightarrow f''(x) = e^x(1) + (x+1)e^x = e^x(x+2)$. So $f''(x) > 0 \Leftrightarrow x+2>0 \Leftrightarrow x > -2$. Thus, f is concave upward on $(-2, \infty)$ and concave downward on $(-\infty, -2)$. Since the concavity changes direction at $x = -2$, the point $\left(-2, -2e^{-2}\right)$ is an inflection point.

19. (a) $y = f(x) = \dfrac{\ln x}{\sqrt{x}}$. (Note that f is only defined for $x > 0$.)

$$f'(x) = \frac{\sqrt{x}\,(1/x) - \ln x\left(\frac{1}{2}x^{-1/2}\right)}{x} = \frac{\dfrac{1}{\sqrt{x}} - \dfrac{\ln x}{2\sqrt{x}}}{x} \cdot \frac{2\sqrt{x}}{2\sqrt{x}} = \frac{2 - \ln x}{2x^{3/2}} > 0 \Leftrightarrow$$

$2 - \ln x > 0 \Leftrightarrow \ln x < 2 \Leftrightarrow x < e^2$. Therefore f is increasing on $\left(0, e^2\right)$ and decreasing on $\left(e^2, \infty\right)$.

(b) f changes from increasing to decreasing at $x = e^2$, so $f\left(e^2\right) = \dfrac{\ln e^2}{\sqrt{e^2}} = \dfrac{2}{e}$ is a local maximum value.

(c) $$f''(x) = \frac{2x^{3/2}(-1/x) - (2 - \ln x)(3x^{1/2})}{(2x^{3/2})^2} = \frac{-2x^{1/2} + 3x^{1/2}(\ln x - 2)}{4x^3}$$

$$= \frac{x^{1/2}(-2 + 3\ln x - 6)}{4x^3} = \frac{3\ln x - 8}{4x^{5/2}}$$

$f''(x) = 0 \Leftrightarrow \ln x = \frac{8}{3} \Leftrightarrow x = e^{8/3}$. $f''(x) > 0 \Leftrightarrow x > e^{8/3}$, so f is concave upward on $(e^{8/3}, \infty)$ and concave downward on $(0, e^{8/3})$. There is an inflection point at $\left(e^{8/3}, \frac{8}{3}e^{-4/3}\right) \approx (14.39, 0.70)$.

21. $f(x) = x^5 - 5x + 3 \Rightarrow f'(x) = 5x^4 - 5 = 5(x^2+1)(x+1)(x-1)$.

First Derivative Test: $f'(x) < 0 \Rightarrow -1 < x < 1$ and $f'(x) > 0 \Rightarrow x > 1$ or $x < -1$. Since f' changes from positive to negative at $x = -1$, $f(-1) = 7$ is a local maximum value; and since f' changes from negative to positive at $x = 1$, $f(1) = -1$ is a local minimum value.

Second Derivative Test: $f''(x) = 20x^3$. $f'(x) = 0 \Leftrightarrow x = \pm 1$. $f''(-1) = -20 < 0 \Rightarrow f(-1) = 7$ is a local maximum value. $f''(1) = 20 > 0 \Rightarrow f(1) = -1$ is a local minimum value.

Preference: For this function, the two tests are equally easy.

23. $f(x) = x + \sqrt{1-x} \Rightarrow f'(x) = 1 + \frac{1}{2}(1-x)^{-1/2}(-1) = 1 - \dfrac{1}{2\sqrt{1-x}}$. Note that f is defined for $1 - x \geq 0$; that is, for $x \leq 1$. $f'(x) = 0 \Rightarrow 2\sqrt{1-x} = 1 \Rightarrow \sqrt{1-x} = \frac{1}{2} \Rightarrow 1 - x = \frac{1}{4} \Rightarrow x = \frac{3}{4}$. f' does not exist at $x = 1$, but we can't have a local maximum or minimum at an endpoint.

First Derivative Test: $f'(x) > 0 \Rightarrow x < \frac{3}{4}$ and $f'(x) < 0 \Rightarrow \frac{3}{4} < x < 1$. Since f' changes from positive to negative at $x = \frac{3}{4}$, $f\left(\frac{3}{4}\right) = \frac{5}{4}$ is a local maximum value.

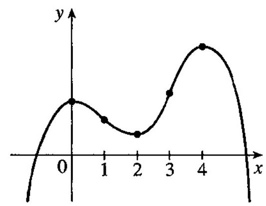
Second Derivative Test: $f''(x) = -\frac{1}{2}\left(-\frac{1}{2}\right)(1-x)^{-3/2}(-1) = -\dfrac{1}{4\left(\sqrt{1-x}\right)^3}$. $f''\left(\frac{3}{4}\right) = -2 < 0 \Rightarrow$ $f\left(\frac{3}{4}\right) = \frac{5}{4}$ is a local maximum value.

Preference: The First Derivative Test may be slightly easier to apply in this case.

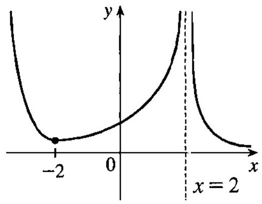
25. (a) By the Second Derivative Test, if $f'(2) = 0$ and $f''(2) = -5 < 0$, f has a local maximum at $x = 2$.

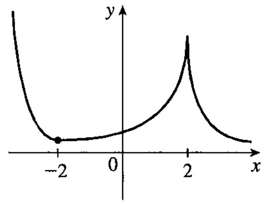
(b) If $f'(6) = 0$, we know that f has a horizontal tangent at $x = 6$. Knowing that $f''(6) = 0$ does not provide any additional information since the Second Derivative Test fails. For example, the first and second derivatives of $y = (x-6)^4$, $y = -(x-6)^4$, and $y = (x-6)^3$ all equal zero for $x = 6$, but the first has a local minimum at $x = 6$, the second has a local maximum at $x = 6$, and the third has an inflection point at $x = 6$.

27. $f'(0) = f'(2) = f'(4) = 0 \;\Leftrightarrow\;$ horizontal tangents at $x = 0, 2, 4$. $f'(x) > 0$ if $x < 0$ or $2 < x < 4 \;\Rightarrow\; f$ is increasing on $(-\infty, 0)$ and $(2, 4)$. $f'(x) < 0$ if $0 < x < 2$ or $x > 4 \;\Rightarrow\; f$ is decreasing on $(0, 2)$ and $(4, \infty)$. $f''(x) > 0$ if $1 < x < 3 \;\Rightarrow\; f$ is concave upward on $(1, 3)$. $f''(x) < 0$ if $x < 1$ or $x > 3 \;\Rightarrow\; f$ is concave downward on $(-\infty, 1)$ and $(3, \infty)$. There are inflection points when $x = 1$ and 3.



29. $f'(x) > 0$ if $|x| < 2 \;\Rightarrow\; f$ is increasing on $(-2, 2)$. $f'(x) < 0$ if $|x| > 2 \;\Rightarrow\; f$ is decreasing on $(-\infty, -2)$ and $(2, \infty)$. $f'(-2) = 0 \;\Rightarrow\;$ horizontal tangent at $x = -2$. $\lim_{x \to 2} |f'(x)| = \infty \;\Rightarrow\;$ there is a vertical asymptote or vertical tangent (cusp) at $x = 2$. $f''(x) > 0$ if $x \neq 2 \;\Rightarrow\; f$ is concave upward on $(-\infty, 2)$ and $(2, \infty)$.





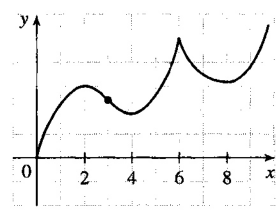
31. (a) f is increasing where f' is positive, that is, on $(0, 2)$, $(4, 6)$, and $(8, \infty)$; and decreasing where f' is negative, that is, on $(2, 4)$ and $(6, 8)$.

(b) f has local maxima where f' changes from positive to negative, at $x = 2$ and at $x = 6$, and local minima where f' changes from negative to positive, at $x = 4$ and at $x = 8$.

(c) f is concave upward (CU) where f' is increasing, that is, on $(3, 6)$ and $(6, \infty)$, and concave downward (CD) where f' is decreasing, that is, on $(0, 3)$.

(d) There is a point of inflection where f changes from being CD to being CU, that is, at $x = 3$.

(e)

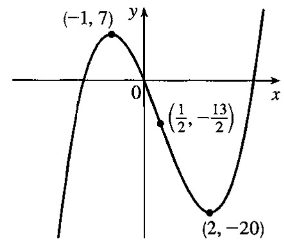


33. (a) $f(x) = 2x^3 - 3x^2 - 12x \;\Rightarrow\; f'(x) = 6x^2 - 6x - 12 = 6(x^2 - x - 2) = 6(x-2)(x+1)$. $f'(x) > 0$ $\Leftrightarrow\; x < -1$ or $x > 2$ and $f'(x) < 0 \;\Leftrightarrow\; -1 < x < 2$. So f is increasing on $(-\infty, -1)$ and $(2, \infty)$, and f is decreasing on $(-1, 2)$.

(b) Since f changes from increasing to decreasing at $x = -1$, $f(-1) = 7$ is a local maximum value. Since f changes from decreasing to increasing at $x = 2$, $f(2) = -20$ is a local minimum value.

(c) $f''(x) = 6(2x-1) \quad\Rightarrow\quad f''(x) > 0$ on $\left(\frac{1}{2}, \infty\right)$ and $f''(x) < 0$ on $\left(-\infty, \frac{1}{2}\right)$. So f is concave upward on $\left(\frac{1}{2}, \infty\right)$ and concave downward on $\left(-\infty, \frac{1}{2}\right)$. There is a change in concavity at $x = \frac{1}{2}$, and we have an inflection point at $\left(\frac{1}{2}, -\frac{13}{2}\right)$.

(d)



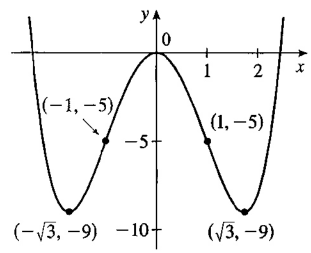
35. (a) $f(x) = x^4 - 6x^2 \quad\Rightarrow\quad f'(x) = 4x^3 - 12x = 4x(x^2-3) = 0$ when $x = 0, \pm\sqrt{3}$.

Interval	$4x$	x^2-3	$f'(x)$	f
$x < -\sqrt{3}$	$-$	$+$	$-$	decreasing on $(-\infty, -\sqrt{3})$
$-\sqrt{3} < x < 0$	$-$	$-$	$+$	increasing on $(-\sqrt{3}, 0)$
$0 < x < \sqrt{3}$	$+$	$-$	$-$	decreasing on $(0, \sqrt{3})$
$x > \sqrt{3}$	$+$	$+$	$+$	increasing on $(\sqrt{3}, \infty)$

(b) Local minimum values $f(\pm\sqrt{3}) = -9$, local maximum value $f(0) = 0$

(c) $f''(x) = 12x^2 - 12 = 12(x^2-1) > 0 \quad\Leftrightarrow\quad x^2 > 1 \quad\Leftrightarrow\quad |x| > 1 \quad\Leftrightarrow\quad x > 1$ or $x < -1$, so f is CU on $(-\infty, -1)$, $(1, \infty)$ and CD on $(-1, 1)$. Inflection points at $(\pm 1, -5)$

(d)

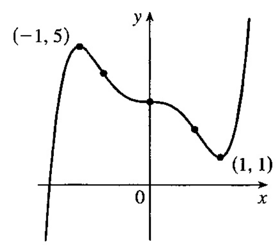


37. (a) $h(x) = 3x^5 - 5x^3 + 3 \quad\Rightarrow\quad h'(x) = 15x^4 - 15x^2 = 15x^2(x^2-1) = 0$ when $x = 0, \pm 1$. Since $15x^2$ is nonnegative, $h'(x) > 0 \quad\Leftrightarrow\quad x^2 > 1 \quad\Leftrightarrow\quad |x| > 1 \quad\Leftrightarrow\quad x > 1$ or $x < -1$, so h is increasing on $(-\infty, -1)$ and $(1, \infty)$ and decreasing on $(-1, 1)$, with a horizontal tangent at $x = 0$.

(b) Local maximum value $h(-1) = 5$, local minimum value $h(1) = 1$

(c) $h''(x) = 60x^3 - 30x = 30x(2x^2-1)$
$$= 60x\left(x + \frac{1}{\sqrt{2}}\right)\left(x - \frac{1}{\sqrt{2}}\right) \quad\Rightarrow$$
$h''(x) > 0$ when $x > \frac{1}{\sqrt{2}}$ or $-\frac{1}{\sqrt{2}} < x < 0$, so h is CU on $\left(-\frac{1}{\sqrt{2}}, 0\right)$ and $\left(\frac{1}{\sqrt{2}}, \infty\right)$ and CD on $\left(-\infty, -\frac{1}{\sqrt{2}}\right)$ and $\left(0, \frac{1}{\sqrt{2}}\right)$. Inflection points at $(0, 3)$ and $\left(\pm\frac{1}{\sqrt{2}}, 3 \mp \frac{7}{8}\sqrt{2}\right)$ [about $(-0.71, 4.24)$ and $(0.71, 1.76)$].

(d)



39. (a) $A(x) = x\sqrt{x+3} \;\Rightarrow$

$$A'(x) = x \cdot \tfrac{1}{2}(x+3)^{-1/2} + \sqrt{x+3} \cdot 1 = \frac{x}{2\sqrt{x+3}} + \sqrt{x+3} = \frac{x+2(x+3)}{2\sqrt{x+3}} = \frac{3x+6}{2\sqrt{x+3}}.$$

The domain of A is $[-3, \infty)$. $A'(x) > 0$ for $x > -2$ and $A'(x) < 0$ for $-3 < x < -2$, so A is increasing on $(-2, \infty)$ and decreasing on $(-3, -2)$.

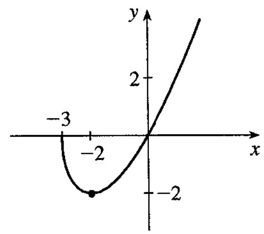
(b) $A(-2) = -2$ is a local minimum value.

(c) $$A''(x) = \frac{2\sqrt{x+3} \cdot 3 - (3x+6) \cdot \dfrac{1}{\sqrt{x+3}}}{\left(2\sqrt{x+3}\right)^2}$$

$$= \frac{6(x+3) - (3x+6)}{4(x+3)^{3/2}} = \frac{3x+12}{4(x+3)^{3/2}} = \frac{3(x+4)}{4(x+3)^{3/2}}$$

$A''(x) > 0$ for all $x > -3$, so A is concave upward on $(-3, \infty)$. There is no inflection point.

(d)



41. (a) $C(x) = x^{1/3}(x+4) = x^{4/3} + 4x^{1/3} \;\Rightarrow\; C'(x) = \frac{4}{3}x^{1/3} + \frac{4}{3}x^{-2/3} = \frac{4}{3}x^{-2/3}(x+1) = \dfrac{4(x+1)}{3\sqrt[3]{x^2}}.$

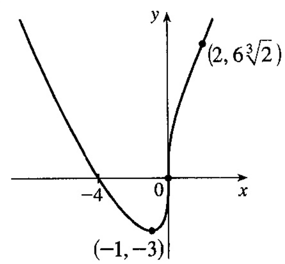
$C'(x) > 0$ if $-1 < x < 0$ or $x > 0$ and $C'(x) < 0$ for $x < -1$, so C is increasing on $(-1, \infty)$ and C is decreasing on $(-\infty, -1)$.

(b) $C(-1) = -3$ is a local minimum value.

(c) $C''(x) = \frac{4}{9}x^{-2/3} - \frac{8}{9}x^{-5/3} = \frac{4}{9}x^{-5/3}(x-2) = \dfrac{4(x-2)}{9\sqrt[3]{x^5}}.$

$C''(x) < 0$ for $0 < x < 2$ and $C''(x) > 0$ for $x < 0$ and $x > 2$, so C is concave downward on $(0, 2)$ and concave upward on $(-\infty, 0)$ and $(2, \infty)$. There are inflection points at $(0, 0)$ and $(2, 6\sqrt[3]{2}) \approx (2, 7.56)$.

(d)



43. (a) $f(\theta) = 2\cos\theta - \cos 2\theta,\; 0 \le \theta \le 2\pi.$
$f'(\theta) = -2\sin\theta + 2\sin 2\theta = -2\sin\theta + 2(2\sin\theta\cos\theta) = 2\sin\theta\,(2\cos\theta - 1).$

Interval	$\sin\theta$	$2\cos\theta - 1$	$f'(\theta)$	f
$0 < \theta < \frac{\pi}{3}$	$+$	$+$	$+$	increasing on $(0, \frac{\pi}{3})$
$\frac{\pi}{3} < \theta < \pi$	$+$	$-$	$-$	decreasing on $(\frac{\pi}{3}, \pi)$
$\pi < \theta < \frac{5\pi}{3}$	$-$	$-$	$+$	increasing on $(\pi, \frac{5\pi}{3})$
$\frac{5\pi}{3} < \theta < 2\pi$	$-$	$+$	$-$	decreasing on $(\frac{5\pi}{3}, 2\pi)$

(b) $f(\frac{\pi}{3}) = \frac{3}{2}$ and $f(\frac{5\pi}{3}) = \frac{3}{2}$ are local maximum values and $f(\pi) = -3$ is a local minimum value.

(c) $f'(\theta) = -2\sin\theta + 2\sin 2\theta \quad \Rightarrow$

$$f''(\theta) = -2\cos\theta + 4\cos 2\theta = -2\cos\theta + 4(2\cos^2\theta - 1)$$
$$= 2(4\cos^2\theta - \cos\theta - 2)$$

$$f''(\theta) = 0 \quad \Leftrightarrow \quad \cos\theta = \frac{1 \pm \sqrt{33}}{8} \quad \Leftrightarrow \quad \theta = \cos^{-1}\left(\frac{1 \pm \sqrt{33}}{8}\right)$$

$$\Leftrightarrow \quad \theta = \cos^{-1}\left(\frac{1 + \sqrt{33}}{8}\right) \approx 0.5678,$$

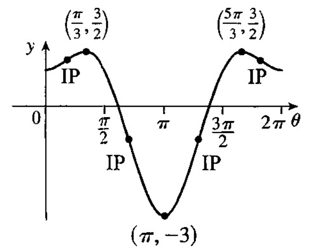
$$2\pi - \cos^{-1}\left(\frac{1+\sqrt{33}}{8}\right) \approx 5.7154, \cos^{-1}\left(\frac{1-\sqrt{33}}{8}\right) \approx 2.2057, \text{ or } 2\pi - \cos^{-1}\left(\frac{1-\sqrt{33}}{8}\right) \approx 4.0775.$$

Denote these four values of θ by θ_1, θ_4, θ_2, and θ_3, respectively. Then f is CU on $(0, \theta_1)$, CD on (θ_1, θ_2), CU on (θ_2, θ_3), CD on (θ_3, θ_4), and CU on $(\theta_4, 2\pi)$. To find the *exact* y-coordinate for $\theta = \theta_1$, we have

$$f(\theta_1) = 2\cos\theta_1 - \cos 2\theta_1 = 2\cos\theta_1 - \left(2\cos^2\theta_1 - 1\right) = 2\left(\frac{1+\sqrt{33}}{8}\right) - 2\left(\frac{1+\sqrt{33}}{8}\right)^2 + 1$$
$$= \tfrac{1}{4} + \tfrac{1}{4}\sqrt{33} - \tfrac{1}{32} - \tfrac{1}{16}\sqrt{33} - \tfrac{33}{32} + 1 = \tfrac{3}{16} + \tfrac{3}{16}\sqrt{33} = \tfrac{3}{16}\left(1 + \sqrt{33}\right) = y_1 \approx 1.26.$$

Similarly, $f(\theta_2) = \frac{3}{16}\left(1 - \sqrt{33}\right) = y_2 \approx -0.89$. So f has inflection points at (θ_1, y_1), (θ_2, y_2), (θ_3, y_2), and (θ_4, y_1).

(d)



45. $f(x) = \dfrac{x^2}{x^2 - 1} = \dfrac{x^2}{(x+1)(x-1)}$ has domain $(-\infty, -1) \cup (-1, 1) \cup (1, \infty)$.

(a) $\displaystyle\lim_{x\to\pm\infty} f(x) = \lim_{x\to\pm\infty} \frac{x^2/x^2}{(x^2-1)/x^2} = \lim_{x\to\pm\infty} \frac{1}{1 - 1/x^2} = \frac{1}{1-0} = 1$, so $y = 1$ is a HA.

$\displaystyle\lim_{x\to -1^-} \frac{x^2}{x^2-1} = \infty$ since $x^2 \to 1$ and $(x^2 - 1) \to 0^+$ as $x \to -1^-$, so $x = -1$ is a VA.

$\displaystyle\lim_{x\to 1^+} \frac{x^2}{x^2-1} = \infty$ since $x^2 \to 1$ and $(x^2 - 1) \to 0^+$ as $x \to 1^+$, so $x = 1$ is a VA.

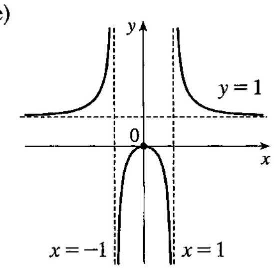
(b) $f(x) = \dfrac{x^2}{x^2-1} \quad \Rightarrow \quad f'(x) = \dfrac{(x^2-1)(2x) - x^2(2x)}{(x^2-1)^2} = \dfrac{2x[(x^2-1) - x^2]}{(x^2-1)^2} = \dfrac{-2x}{(x^2-1)^2}$. Since $(x^2 - 1)^2$ is positive for all x in the domain of f, the sign of the derivative is determined by the sign of $-2x$. Thus, $f'(x) > 0$ if $x < 0$ ($x \neq -1$) and $f'(x) < 0$ if $x > 0$ ($x \neq 1$). So f is increasing on $(-\infty, -1)$ and $(-1, 0)$, and f is decreasing on $(0, 1)$ and $(1, \infty)$.

(c) $f'(x) = 0 \quad \Rightarrow \quad x = 0$ and $f(0) = 0$ is a local maximum value.

(d) $$f''(x) = \frac{(x^2-1)^2(-2) - (-2x)\cdot 2(x^2-1)(2x)}{[(x^2-1)^2]^2}$$
$$= \frac{2(x^2-1)[-(x^2-1) + 4x^2]}{(x^2-1)^4} = \frac{2(3x^2+1)}{(x^2-1)^3}.$$

The sign of $f''(x)$ is determined by the denominator; that is, $f''(x) > 0$ if $|x| > 1$ and $f''(x) < 0$ if $|x| < 1$. Thus, f is CU on $(-\infty, -1)$ and $(1, \infty)$, and f is CD on $(-1, 1)$. There are no inflection points.

(e)



47. (a) $\lim\limits_{x\to-\infty}\left(\sqrt{x^2+1}-x\right)=\infty$ and

$$\lim_{x\to\infty}\left(\sqrt{x^2+1}-x\right)=\lim_{x\to\infty}\left(\sqrt{x^2+1}-x\right)\frac{\sqrt{x^2+1}+x}{\sqrt{x^2+1}+x}=\lim_{x\to\infty}\frac{1}{\sqrt{x^2+1}+x}=0\text{, so } y=0 \text{ is a HA.}$$

(b) $f(x)=\sqrt{x^2+1}-x \;\Rightarrow\; f'(x)=\dfrac{x}{\sqrt{x^2+1}}-1$. Since $\dfrac{x}{\sqrt{x^2+1}}<1$ for all x, $f'(x)<0$, so f is decreasing on $\mathbb{R}$.

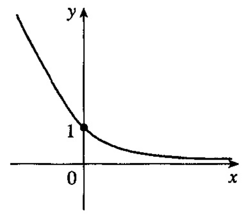
(c) No minimum or maximum

(d) $$f''(x)=\frac{\left(x^2+1\right)^{1/2}(1)-x\cdot\frac{1}{2}\left(x^2+1\right)^{-1/2}(2x)}{\left(\sqrt{x^2+1}\right)^2}$$

$$=\frac{\left(x^2+1\right)^{1/2}-\dfrac{x^2}{\left(x^2+1\right)^{1/2}}}{x^2+1}=\frac{\left(x^2+1\right)-x^2}{\left(x^2+1\right)^{3/2}}$$

$$=\frac{1}{\left(x^2+1\right)^{3/2}}>0\text{, so } f \text{ is CU on } \mathbb{R}\text{. No IP}$$

(e)



49. $f(x)=\ln(1-\ln x)$ is defined when $x>0$ (so that $\ln x$ is defined) and $1-\ln x>0$ [so that $\ln(1-\ln x)$ is defined]. The second condition is equivalent to $1>\ln x \;\Leftrightarrow\; x<e$, so f has domain $(0,e)$.

(a) As $x\to0^+$, $\ln x\to-\infty$, so $1-\ln x\to\infty$ and $f(x)\to\infty$. As $x\to e^-$, $\ln x\to1^-$, so $1-\ln x\to0^+$ and $f(x)\to-\infty$. Thus, $x=0$ and $x=e$ are vertical asymptotes. There is no horizontal asymptote.

(b) $f'(x)=\dfrac{1}{1-\ln x}\left(-\dfrac{1}{x}\right)=-\dfrac{1}{x(1-\ln x)}<0$ on $(0,e)$. Thus, f is decreasing on its domain, $(0,e)$.

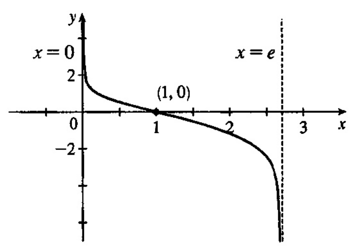
(c) $f'(x)\neq0$ on $(0,e)$, so f has no local maximum or minimum value.

(d) $$f''(x)=-\frac{-\left[x(1-\ln x)\right]'}{\left[x(1-\ln x)\right]^2}=\frac{x(-1/x)+(1-\ln x)}{x^2(1-\ln x)^2}$$

$$=-\frac{\ln x}{x^2(1-\ln x)^2}$$

so $f''(x)>0 \;\Leftrightarrow\; \ln x<0 \;\Leftrightarrow\; 0<x<1$. Thus, f is CU on $(0,1)$ and CD on $(1,e)$. There is an inflection point at $(1,0)$.

(e)



51. (a) $\lim\limits_{x\to\pm\infty}e^{-1/(x+1)}=1$ since $-1/(x+1)\to0$, so $y=1$ is a HA. $\lim\limits_{x\to-1^+}e^{-1/(x+1)}=0$ since $-1/(x+1)\to-\infty$, $\lim\limits_{x\to-1^-}e^{-1/(x+1)}=\infty$ since $-1/(x+1)\to\infty$, so $x=-1$ is a VA.

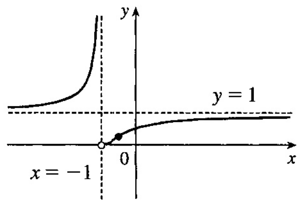
(b) $f(x)=e^{-1/(x+1)} \;\Rightarrow\; f'(x)=e^{-1/(x+1)}\left[-(-1)\dfrac{1}{(x+1)^2}\right]$ [Reciprocal Rule] $=e^{-1/(x+1)}/(x+1)^2$

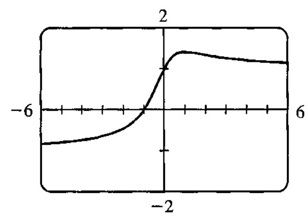
$\Rightarrow\; f'(x)>0$ for all x except -1, so f is increasing on $(-\infty,-1)$ and $(-1,\infty)$.

(c) No local maximum or minimum

(d) $$f''(x) = \frac{(x+1)^2 e^{-1/(x+1)}\left[1/(x+1)^2\right] - e^{-1/(x+1)}\left[2(x+1)\right]}{[(x+1)^2]^2}$$
$$= \frac{e^{-1/(x+1)}\left[1-(2x+2)\right]}{(x+1)^4} = -\frac{e^{-1/(x+1)}(2x+1)}{(x+1)^4} \Rightarrow$$

$f''(x) > 0 \Leftrightarrow 2x+1 < 0 \Leftrightarrow x < -\frac{1}{2}$, so f is CU on $(-\infty,-1)$ and $\left(-1,-\frac{1}{2}\right)$, and CD on $\left(-\frac{1}{2},\infty\right)$. f has an IP at $\left(-\frac{1}{2},e^{-2}\right)$.

(e) 

53. (a) 

From the graph, we get an estimate of $f(1) \approx 1.41$ as a local maximum value, and no local minimum value.

$$f(x) = \frac{x+1}{\sqrt{x^2+1}} \Rightarrow f'(x) = \frac{1-x}{(x^2+1)^{3/2}}.$$

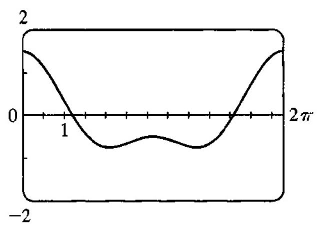
$f'(x) = 0 \Leftrightarrow x = 1$. $f(1) = \frac{2}{\sqrt{2}} = \sqrt{2}$ is the exact value.

(b) From the graph in part (a), f increases most rapidly somewhere between $x = -\frac{1}{2}$ and $x = -\frac{1}{4}$. To find the exact value, we need to find the maximum value of f', which we can do by finding the critical numbers of f'.

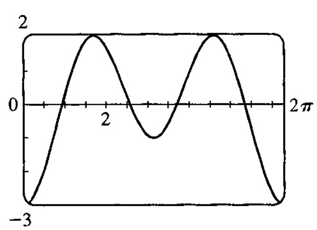
$f''(x) = \dfrac{2x^2-3x-1}{(x^2+1)^{5/2}} = 0 \Leftrightarrow x = \dfrac{3 \pm \sqrt{17}}{4}$. $x = \dfrac{3+\sqrt{17}}{4}$ corresponds to the *minimum* value of f'.

The maximum value of f' is at $\left(\frac{3-\sqrt{17}}{4}, \sqrt{\frac{7}{6}-\frac{\sqrt{17}}{6}}\right) \approx (-0.28, 0.69)$.

55. $f(x) = \cos x + \frac{1}{2}\cos 2x \Rightarrow f'(x) = -\sin x - \sin 2x \Rightarrow f''(x) = -\cos x - 2\cos 2x$

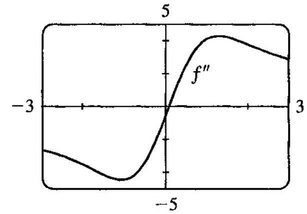
(a) 

From the graph of f, it seems that f is CD on $(0,1)$, CU on $(1,2.5)$, CD on $(2.5,3.7)$, CU on $(3.7,5.3)$, and CD on $(5.3,2\pi)$. The points of inflection appear to be at $(1,0.4)$, $(2.5,-0.6)$, $(3.7,-0.6)$, and $(5.3,0.4)$.

(b) 

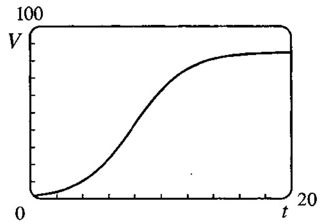
From the graph of f'' (and zooming in near the zeros), it seems that f is CD on $(0,0.94)$, CU on $(0.94,2.57)$, CD on $(2.57,3.71)$, CU on $(3.71,5.35)$, and CD on $(5.35,2\pi)$. Refined estimates of the inflection points are $(0.94,0.44)$, $(2.57,-0.63)$, $(3.71,-0.63)$, and $(5.35,0.44)$.

57. In Maple, we define f and then use the command `plot(diff(diff(f,x),x),x=-3..3);`. In Mathematica, we define f and then use `Plot[Dt[Dt[f,x],x],{x,-3,3}]`.We see that $f'' > 0$ for $x > 0.1$ and $f'' < 0$ for $x < 0.1$. So f is concave up on $(0.1, \infty)$ and concave down on $(-\infty, 0.1)$.



59. Most students learn more in the third hour of studying than in the eighth hour, so $K(3) - K(2)$ is larger than $K(8) - K(7)$. In other words, as you begin studying for a test, the rate of knowledge gain is large and then starts to taper off, so $K'(t)$ decreases and the graph of K is concave downward.

61.



From the graph, we estimate that the most rapid increase in the percentage of households in the United States with at least one VCR occurs at about $t = 8$. To maximize the first derivative, we need to determine the values for which the second derivative is 0. We'll use $V(t) = \dfrac{a}{1 + be^{ct}}$, and substitute $a = 85$, $b = 53$, and $c = -0.5$ later.

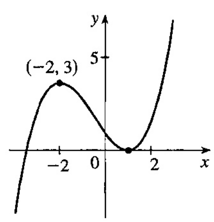
$V'(t) = -\dfrac{a(bce^{ct})}{(1 + be^{ct})^2}$ [by the Reciprocal Rule] and

$$V''(t) = -abc \cdot \frac{(1 + be^{ct})^2 \cdot ce^{ct} - e^{ct} \cdot 2(1 + be^{ct}) \cdot bce^{ct}}{\left[(1 + be^{ct})^2\right]^2}$$

$$= \frac{-abc \cdot ce^{ct}(1 + be^{ct})[(1 + be^{ct}) - 2be^{ct}]}{(1 + be^{ct})^4} = \frac{-abc^2e^{ct}(1 - be^{ct})}{(1 + be^{ct})^3}$$

So $V''(t) = 0 \iff 1 = be^{ct} \iff e^{ct} = 1/b$. Now graph $y = e^{-0.5t}$ and $y = \frac{1}{53}$. These graphs intersect at $t \approx 7.94$ years, which corresponds to roughly midyear 1988. [Alternatively, we could use the rootfinder on a calculator to solve $e^{-0.5t} = \frac{1}{53}$. Or, if you have already studied logarithms, you can solve $e^{ct} = 1/b$ as follows: $ct = \ln(1/b) \iff t = (1/c)\ln(1/b) = -2\ln\frac{1}{53} \approx 7.94$ years.

63. $f(x) = ax^3 + bx^2 + cx + d \Rightarrow f'(x) = 3ax^2 + 2bx + c$. We are given that $f(1) = 0$ and $f(-2) = 3$, so $f(1) = a + b + c + d = 0$ and $f(-2) = -8a + 4b - 2c + d = 3$. Also $f'(1) = 3a + 2b + c = 0$ and $f'(-2) = 12a - 4b + c = 0$ by Fermat's Theorem. Solving these four equations, we get $a = \frac{2}{9}$, $b = \frac{1}{3}$, $c = -\frac{4}{3}$, $d = \frac{7}{9}$, so the function is $f(x) = \frac{1}{9}(2x^3 + 3x^2 - 12x + 7)$.



65. Suppose that f is differentiable on an interval I and $f'(x) > 0$ for all x in I except $x = c$. To show that f is increasing on I, let x_1, x_2 be two numbers in I with $x_1 < x_2$.

Case 1 $x_1 < x_2 < c$. Let J be the interval $\{x \in I \mid x < c\}$. By applying the Increasing/Decreasing Test to f on J, we see that f is increasing on J, so $f(x_1) < f(x_2)$.

Case 2 $c < x_1 < x_2$. Apply the Increasing/Decreasing Test to f on $K = \{x \in I \mid x > c\}$.

Case 3 $x_1 < x_2 = c$. Apply the proof of the Increasing/Decreasing Test, using the Mean Value Theorem (MVT) on the interval $[x_1, x_2]$ and noting that the MVT does not require f to be differentiable at the endpoints of $[x_1, x_2]$.

Case 4 $c = x_1 < x_2$. Same proof as in Case 3.

Case 5 $x_1 < c < x_2$. By Cases 3 and 4, f is increasing on $[x_1, c]$ and on $[c, x_2]$, so $f(x_1) < f(c) < f(x_2)$.

In all cases, we have shown that $f(x_1) < f(x_2)$. Since x_1, x_2 were any numbers in I with $x_1 < x_2$, we have shown that f is increasing on I.

67. (a) Since f and g are positive, increasing, and CU on I with f'' and g'' never equal to 0, we have $f > 0$, $f' \geq 0$, $f'' > 0$, $g > 0$, $g' \geq 0$, $g'' > 0$ on I. Then $(fg)' = f'g + fg' \Rightarrow$ $(fg)'' = f''g + 2f'g' + fg'' \geq f''g + fg'' > 0$ on $I \Rightarrow fg$ is CU on I.

(b) In part (a), if f and g are both decreasing instead of increasing, then $f' \leq 0$ and $g' \leq 0$ on I, so we still have $2f'g' \geq 0$ on I. Thus, $(fg)'' = f''g + 2f'g' + fg'' \geq f''g + fg'' > 0$ on $I \Rightarrow fg$ is CU on I as in part (a).

(c) Suppose f is increasing and g is decreasing [with f and g positive and CU]. Then $f' \geq 0$ and $g' \leq 0$ on I, so $2f'g' \leq 0$ on I and the argument in parts (a) and (b) fails.

Example 1. $I = (0, \infty)$, $f(x) = x^3$, $g(x) = 1/x$. Then $(fg)(x) = x^2$, so $(fg)'(x) = 2x$ and $(fg)''(x) = 2 > 0$ on I. Thus, fg is CU on I.

Example 2. $I = (0, \infty)$, $f(x) = 4x\sqrt{x}$, $g(x) = 1/x$. Then $(fg)(x) = 4\sqrt{x}$, so $(fg)'(x) = 2/\sqrt{x}$ and $(fg)''(x) = -1/\sqrt{x^3} < 0$ on I. Thus, fg is CD on I.

Example 3. $I = (0, \infty)$, $f(x) = x^2$, $g(x) = 1/x$. Thus, $(fg)(x) = x$, so fg is linear on I.

69. $f(x) = \tan x - x \Rightarrow f'(x) = \sec^2 x - 1 > 0$ for $0 < x < \frac{\pi}{2}$ since $\sec^2 x > 1$ for $0 < x < \frac{\pi}{2}$. So f is increasing on $\left(0, \frac{\pi}{2}\right)$. Thus, $f(x) > f(0) = 0$ for $0 < x < \frac{\pi}{2} \Rightarrow \tan x - x > 0 \Rightarrow \tan x > x$ for $0 < x < \frac{\pi}{2}$.

71. Let the cubic function be $f(x) = ax^3 + bx^2 + cx + d \Rightarrow f'(x) = 3ax^2 + 2bx + c \Rightarrow f''(x) = 6ax + 2b$. So f is CU when $6ax + 2b > 0 \Leftrightarrow x > -b/(3a)$, CD when $x < -b/(3a)$, and so the only point of inflection occurs when $x = -b/(3a)$. If the graph has three x-intercepts x_1, x_2 and x_3, then the expression for $f(x)$ must factor as $f(x) = a(x - x_1)(x - x_2)(x - x_3)$. Multiplying these factors together gives us
$f(x) = a\left[x^3 - (x_1 + x_2 + x_3)x^2 + (x_1x_2 + x_1x_3 + x_2x_3)x - x_1x_2x_3\right]$. Equating the coefficients of the x^2-terms for the two forms of f gives us $b = -a(x_1 + x_2 + x_3)$. Hence, the x-coordinate of the point of inflection is $-\dfrac{b}{3a} = -\dfrac{-a(x_1 + x_2 + x_3)}{3a} = \dfrac{x_1 + x_2 + x_3}{3}$.

73. By hypothesis $g = f'$ is differentiable on an open interval containing c. Since $(c, f(c))$ is a point of inflection, the concavity changes at $x = c$, so $f''(x)$ changes signs at $x = c$. Hence, by the First Derivative Test, f' has a local extremum at $x = c$. Thus, by Fermat's Theorem $f''(c) = 0$.

75. Using the fact that $|x| = \sqrt{x^2}$, we have that $g(x) = x\sqrt{x^2} \Rightarrow g'(x) = \sqrt{x^2} + \sqrt{x^2} = 2\sqrt{x^2} = 2|x| \Rightarrow$
$g''(x) = 2x\left(x^2\right)^{-1/2} = \dfrac{2x}{|x|} < 0$ for $x < 0$ and $g''(x) > 0$ for $x > 0$, so $(0, 0)$ is an inflection point. But $g''(0)$ does not exist.

4.4 Indeterminate Forms and L'Hospital's Rule

The use of l'Hospital's Rule is indicated by an H above the equal sign: $\overset{\text{H}}{=}$.

1. (a) $\lim\limits_{x \to a} \dfrac{f(x)}{g(x)}$ is an indeterminate form of type $\dfrac{0}{0}$.

(b) $\lim\limits_{x \to a} \dfrac{f(x)}{p(x)} = 0$ because the numerator approaches 0 while the denominator becomes large.

(c) $\lim\limits_{x \to a} \dfrac{h(x)}{p(x)} = 0$ because the numerator approaches a finite number while the denominator becomes large.

(d) If $\lim\limits_{x \to a} p(x) = \infty$ and $f(x) \to 0$ through positive values, then $\lim\limits_{x \to a} \dfrac{p(x)}{f(x)} = \infty$. [For example, take $a = 0$, $p(x) = 1/x^2$, and $f(x) = x^2$.] If $f(x) \to 0$ through negative values, then $\lim\limits_{x \to a} \dfrac{p(x)}{f(x)} = -\infty$. [For example, take $a = 0$, $p(x) = 1/x^2$, and $f(x) = -x^2$.] If $f(x) \to 0$ through both positive and negative values, then the limit might not exist. [For example, take $a = 0$, $p(x) = 1/x^2$, and $f(x) = x$.]

(e) $\lim\limits_{x \to a} \dfrac{p(x)}{q(x)}$ is an indeterminate form of type $\dfrac{\infty}{\infty}$.

3. (a) When x is near a, $f(x)$ is near 0 and $p(x)$ is large, so $f(x) - p(x)$ is large negative. Thus, $\lim\limits_{x \to a} [f(x) - p(x)] = -\infty$.

(b) $\lim\limits_{x \to a} [p(x) - q(x)]$ is an indeterminate form of type $\infty - \infty$.

(c) When x is near a, $p(x)$ and $q(x)$ are both large, so $p(x) + q(x)$ is large. Thus, $\lim\limits_{x \to a} [p(x) + q(x)] = \infty$.

5. This limit has the form $\frac{0}{0}$. We can simply factor the numerator to evaluate this limit.

$$\lim_{x\to -1}\frac{x^2-1}{x+1}=\lim_{x\to -1}\frac{(x+1)(x-1)}{x+1}=\lim_{x\to -1}(x-1)=-2$$

7. This limit has the form $\frac{0}{0}$. $\lim\limits_{x\to 1}\dfrac{x^9-1}{x^5-1}\overset{\text{H}}{=}\lim\limits_{x\to 1}\dfrac{9x^8}{5x^4}=\dfrac{9}{5}\lim\limits_{x\to 1}x^4=\dfrac{9}{5}(1)=\dfrac{9}{5}$

9. This limit has the form $\frac{0}{0}$. $\lim\limits_{x\to(\pi/2)^+}\dfrac{\cos x}{1-\sin x}\overset{\text{H}}{=}\lim\limits_{x\to(\pi/2)^+}\dfrac{-\sin x}{-\cos x}=\lim\limits_{x\to(\pi/2)^+}\tan x=-\infty$.

11. This limit has the form $\frac{0}{0}$. $\lim\limits_{t\to 0}\dfrac{e^t-1}{t^3}\overset{\text{H}}{=}\lim\limits_{t\to 0}\dfrac{e^t}{3t^2}=\infty$ since $e^t\to 1$ and $3t^2\to 0^+$ as $t\to 0$.

13. This limit has the form $\frac{0}{0}$. $\lim\limits_{x\to 0}\dfrac{\tan px}{\tan qx}\overset{\text{H}}{=}\lim\limits_{x\to 0}\dfrac{p\sec^2 px}{q\sec^2 qx}=\dfrac{p(1)^2}{q(1)^2}=\dfrac{p}{q}$

15. This limit has the form $\frac{\infty}{\infty}$. $\lim\limits_{x\to\infty}\dfrac{\ln x}{x}\overset{\text{H}}{=}\lim\limits_{x\to\infty}\dfrac{1/x}{1}=0$

17. $\lim\limits_{x\to 0^+}[(\ln x)/x]=-\infty$ since $\ln x\to -\infty$ as $x\to 0^+$ and dividing by small values of x just increases the magnitude of the quotient $(\ln x)/x$. L'Hospital's Rule does not apply.

19. This limit has the form $\frac{0}{0}$. $\lim\limits_{t\to 0}\dfrac{5^t-3^t}{t}\overset{\text{H}}{=}\lim\limits_{t\to 0}\dfrac{5^t\ln 5-3^t\ln 3}{1}=\ln 5-\ln 3=\ln\frac{5}{3}$

21. This limit has the form $\frac{0}{0}$. $\lim\limits_{x\to 0}\dfrac{e^x-1-x}{x^2}\overset{\text{H}}{=}\lim\limits_{x\to 0}\dfrac{e^x-1}{2x}\overset{\text{H}}{=}\lim\limits_{x\to 0}\dfrac{e^x}{2}=\dfrac{1}{2}$

23. This limit has the form $\frac{\infty}{\infty}$. $\lim\limits_{x\to\infty}\dfrac{e^x}{x^3}\overset{\text{H}}{=}\lim\limits_{x\to\infty}\dfrac{e^x}{3x^2}\overset{\text{H}}{=}\lim\limits_{x\to\infty}\dfrac{e^x}{6x}\overset{\text{H}}{=}\lim\limits_{x\to\infty}\dfrac{e^x}{6}=\infty$

25. This limit has the form $\frac{0}{0}$. $\lim\limits_{x\to 0}\dfrac{\sin^{-1}x}{x}\overset{\text{H}}{=}\lim\limits_{x\to 0}\dfrac{1/\sqrt{1-x^2}}{1}=\lim\limits_{x\to 0}\dfrac{1}{\sqrt{1-x^2}}=\dfrac{1}{1}=1$

27. This limit has the form $\frac{0}{0}$. $\lim\limits_{x\to 0}\dfrac{1-\cos x}{x^2}\overset{\text{H}}{=}\lim\limits_{x\to 0}\dfrac{\sin x}{2x}\overset{\text{H}}{=}\lim\limits_{x\to 0}\dfrac{\cos x}{2}=\dfrac{1}{2}$

29. $\lim\limits_{x\to 0}\dfrac{x+\sin x}{x+\cos x}=\dfrac{0+0}{0+1}=\dfrac{0}{1}=0$. L'Hospital's Rule does not apply.

31. This limit has the form $\frac{\infty}{\infty}$. $\lim\limits_{x\to\infty}\dfrac{x}{\ln(1+2e^x)}\overset{\text{H}}{=}\lim\limits_{x\to\infty}\dfrac{1}{\dfrac{1}{1+2e^x}\cdot 2e^x}=\lim\limits_{x\to\infty}\dfrac{1+2e^x}{2e^x}\overset{\text{H}}{=}\lim\limits_{x\to\infty}\dfrac{2e^x}{2e^x}=1$

33. This limit has the form $\frac{0}{0}$. $\lim\limits_{x\to 1}\dfrac{1-x+\ln x}{1+\cos\pi x}\overset{\text{H}}{=}\lim\limits_{x\to 1}\dfrac{-1+1/x}{-\pi\sin\pi x}\overset{\text{H}}{=}\lim\limits_{x\to 1}\dfrac{-1/x^2}{-\pi^2\cos\pi x}=\dfrac{-1}{-\pi^2(-1)}=-\dfrac{1}{\pi^2}$

35. This limit has the form $\frac{0}{0}$. $\lim\limits_{x\to 1}\dfrac{x^a-ax+a-1}{(x-1)^2}\overset{\text{H}}{=}\lim\limits_{x\to 1}\dfrac{ax^{a-1}-a}{2(x-1)}\overset{\text{H}}{=}\lim\limits_{x\to 1}\dfrac{a(a-1)x^{a-2}}{2}=\dfrac{a(a-1)}{2}$

37. This limit has the form $0\cdot(-\infty)$. We need to write this product as a quotient, but keep in mind that we will have to differentiate both the numerator and the denominator. If we differentiate $\dfrac{1}{\ln x}$, we get a complicated expression that results in a more difficult limit. Instead we write the quotient as $\dfrac{\ln x}{x^{-1/2}}$.

$$\lim_{x\to 0^+}\sqrt{x}\ln x=\lim_{x\to 0^+}\frac{\ln x}{x^{-1/2}}\overset{\text{H}}{=}\lim_{x\to 0^+}\frac{1/x}{-\frac{1}{2}x^{-3/2}}\cdot\frac{-2x^{3/2}}{-2x^{3/2}}=\lim_{x\to 0^+}(-2\sqrt{x})=0$$

39. This limit has the form $\infty \cdot 0$. We'll change it to the form $\frac{0}{0}$.

$$\lim_{x\to 0} \cot 2x \sin 6x = \lim_{x\to 0} \frac{\sin 6x}{\tan 2x} \overset{\text{H}}{=} \lim_{x\to 0} \frac{6\cos 6x}{2\sec^2 2x} = \frac{6(1)}{2(1)^2} = 3$$

41. This limit has the form $\infty \cdot 0$. $\lim\limits_{x\to\infty} x^3 e^{-x^2} = \lim\limits_{x\to\infty} \dfrac{x^3}{e^{x^2}} \overset{\text{H}}{=} \lim\limits_{x\to\infty} \dfrac{3x^2}{2xe^{x^2}} = \lim\limits_{x\to\infty} \dfrac{3x}{2e^{x^2}} \overset{\text{H}}{=} \lim\limits_{x\to\infty} \dfrac{3}{4xe^{x^2}} = 0$

43. This limit has the form $0 \cdot (-\infty)$.

$$\lim_{x\to 1^+} \ln x \, \tan(\pi x/2) = \lim_{x\to 1^+} \frac{\ln x}{\cot(\pi x/2)} \overset{\text{H}}{=} \lim_{x\to 1^+} \frac{1/x}{(-\pi/2)\csc^2(\pi x/2)} = \frac{1}{(-\pi/2)(1)^2} = -\frac{2}{\pi}$$

45. $\lim\limits_{x\to 0}\left(\dfrac{1}{x} - \csc x\right) = \lim\limits_{x\to 0}\left(\dfrac{1}{x} - \dfrac{1}{\sin x}\right) = \lim\limits_{x\to 0} \dfrac{\sin x - x}{x \sin x}$

$$\overset{\text{H}}{=} \lim_{x\to 0} \frac{\cos x - 1}{x\cos x + \sin x} \overset{\text{H}}{=} \lim_{x\to 0} \frac{-\sin x}{2\cos x - x\sin x} = \frac{0}{2} = 0$$

47. We will multiply and divide by the conjugate of the expression to change the form of the expression.

$$\lim_{x\to\infty} \left(\sqrt{x^2+x} - x\right) = \lim_{x\to\infty} \left(\frac{\sqrt{x^2+x} - x}{1} \cdot \frac{\sqrt{x^2+x}+x}{\sqrt{x^2+x}+x}\right) = \lim_{x\to\infty} \frac{(x^2+x) - x^2}{\sqrt{x^2+x}+x}$$

$$= \lim_{x\to\infty} \frac{x}{\sqrt{x^2+x}+x} = \lim_{x\to\infty} \frac{1}{\sqrt{1+1/x}+1} = \frac{1}{\sqrt{1}+1} = \frac{1}{2}.$$

As an alternate solution, write $\sqrt{x^2+x} - x$ as $\sqrt{x^2+x} - \sqrt{x^2}$, factor out $\sqrt{x^2}$, rewrite as $\left(\sqrt{1+1/x} - 1\right)/(1/x)$, and apply l'Hospital's Rule.

49. The limit has the form $\infty - \infty$ and we will change the form to a product by factoring out x.

$$\lim_{x\to\infty} (x - \ln x) = \lim_{x\to\infty} x\left(1 - \frac{\ln x}{x}\right) = \infty \text{ since } \lim_{x\to\infty} \frac{\ln x}{x} \overset{\text{H}}{=} \lim_{x\to\infty} \frac{1/x}{1} = 0.$$

51. $y = x^{x^2} \quad \Rightarrow \quad \ln y = x^2 \ln x$, so

$$\lim_{x\to 0^+} \ln y = \lim_{x\to 0^+} x^2 \ln x = \lim_{x\to 0^+} \frac{\ln x}{1/x^2} \overset{\text{H}}{=} \lim_{x\to 0^+} \frac{1/x}{-2/x^3} = \lim_{x\to 0^+}\left(-\frac{1}{2}x^2\right) = 0 \quad \Rightarrow$$

$$\lim_{x\to 0^+} x^{x^2} = \lim_{x\to 0^+} e^{\ln y} = e^0 = 1.$$

53. $y = (1-2x)^{1/x} \quad \Rightarrow \quad \ln y = \dfrac{1}{x}\ln(1-2x)$, so $\lim\limits_{x\to 0} \ln y = \lim\limits_{x\to 0} \dfrac{\ln(1-2x)}{x} \overset{\text{H}}{=} \lim\limits_{x\to 0} \dfrac{-2/(1-2x)}{1} = -2 \quad \Rightarrow$

$$\lim_{x\to 0} (1-2x)^{1/x} = \lim_{x\to 0} e^{\ln y} = e^{-2}.$$

55. $y = \left(1 + \dfrac{3}{x} + \dfrac{5}{x^2}\right)^x \quad \Rightarrow \quad \ln y = x \ln\left(1 + \dfrac{3}{x} + \dfrac{5}{x^2}\right) \quad \Rightarrow$

$$\lim_{x\to\infty} \ln y = \lim_{x\to\infty} \frac{\ln\left(1 + \frac{3}{x} + \frac{5}{x^2}\right)}{1/x} \overset{\text{H}}{=} \lim_{x\to\infty} \frac{\left(-\frac{3}{x^2} - \frac{10}{x^3}\right)\Big/\left(1 + \frac{3}{x} + \frac{5}{x^2}\right)}{-1/x^2} = \lim_{x\to\infty} \frac{3 + \frac{10}{x}}{1 + \frac{3}{x} + \frac{5}{x^2}} = 3,$$

so $\lim\limits_{x\to\infty}\left(1 + \dfrac{3}{x} + \dfrac{5}{x^2}\right)^x = \lim\limits_{x\to\infty} e^{\ln y} = e^3$.

57. $y = x^{1/x} \quad \Rightarrow \quad \ln y = (1/x)\ln x \quad \Rightarrow \quad \lim\limits_{x\to\infty} \ln y = \lim\limits_{x\to\infty} \dfrac{\ln x}{x} \overset{\text{H}}{=} \lim\limits_{x\to\infty} \dfrac{1/x}{1} = 0 \quad \Rightarrow$

$$\lim_{x\to\infty} x^{1/x} = \lim_{x\to\infty} e^{\ln y} = e^0 = 1$$

59. $y = \left(\dfrac{x}{x+1}\right)^x \quad \Rightarrow \quad \ln y = x\ln\left(\dfrac{x}{x+1}\right) \quad \Rightarrow$

$$\lim_{x\to\infty} \ln y = \lim_{x\to\infty} x\ln\left(\frac{x}{x+1}\right) = \lim_{x\to\infty} \frac{\ln x - \ln(x+1)}{1/x} \overset{\text{H}}{=} \lim_{x\to\infty} \frac{1/x - 1/(x+1)}{-1/x^2}$$

$$= \lim_{x\to\infty} \left(-x + \frac{x^2}{x+1}\right) = \lim_{x\to\infty} \frac{-x}{x+1} = -1$$

so $\displaystyle\lim_{x\to\infty} \left(\frac{x}{x+1}\right)^x = \lim_{x\to\infty} e^{\ln y} = e^{-1}$

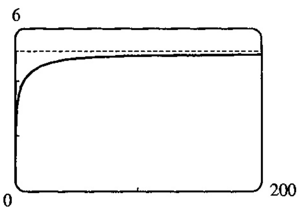
Or: $\displaystyle\lim_{x\to\infty} \left(\frac{x}{x+1}\right)^x = \lim_{x\to\infty} \left[\left(\frac{x+1}{x}\right)^{-1}\right]^x = \left[\lim_{x\to\infty} \left(1+\frac{1}{x}\right)^x\right]^{-1} = e^{-1}$

61. $y = (\cos x)^{1/x^2} \quad \Rightarrow \quad \ln y = \dfrac{1}{x^2}\ln\cos x \quad \Rightarrow$

$$\lim_{x\to0^+} \ln y = \lim_{x\to0^+} \frac{\ln\cos x}{x^2} \overset{\text{H}}{=} \lim_{x\to0^+} \frac{-\tan x}{2x} \overset{\text{H}}{=} \lim_{x\to0^+} \frac{-\sec^2 x}{2} = -\frac{1}{2} \quad \Rightarrow$$

$$\lim_{x\to0^+} (\cos x)^{1/x^2} = \lim_{x\to0^+} e^{\ln y} = e^{-1/2} = 1/\sqrt{e}$$

63.



From the graph, it appears that $\displaystyle\lim_{x\to\infty} x\,[\ln(x+5) - \ln x] = 5.$

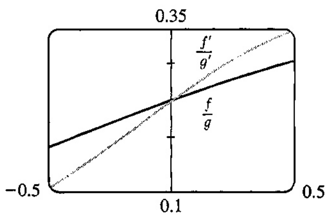
To prove this, we first note that

$\ln(x+5) - \ln x = \ln\dfrac{x+5}{x} = \ln\left(1+\dfrac{5}{x}\right) \to \ln 1 = 0$ as $x \to \infty$. Thus,

$$\lim_{x\to\infty} x\,[\ln(x+5) - \ln x] = \lim_{x\to\infty} \frac{\ln(x+5) - \ln x}{1/x} \overset{\text{H}}{=} \lim_{x\to\infty} \frac{\dfrac{1}{x+5} - \dfrac{1}{x}}{-1/x^2}$$

$$= \lim_{x\to\infty} \left[\frac{x-(x+5)}{x(x+5)} \cdot \frac{-x^2}{1}\right] = \lim_{x\to\infty} \frac{5x^2}{x^2+5x} = 5$$

65.



From the graph, it appears that

$\displaystyle\lim_{x\to0} \frac{f(x)}{g(x)} = \lim_{x\to0} \frac{f'(x)}{g'(x)} = 0.25.$ We calculate

$$\lim_{x\to0} \frac{f(x)}{g(x)} = \lim_{x\to0} \frac{e^x - 1}{x^3+4x} \overset{\text{H}}{=} \lim_{x\to0} \frac{e^x}{3x^2+4} = \frac{1}{4}.$$

67. $\displaystyle\lim_{x\to\infty} \frac{e^x}{x^n} \overset{\text{H}}{=} \lim_{x\to\infty} \frac{e^x}{nx^{n-1}} \overset{\text{H}}{=} \lim_{x\to\infty} \frac{e^x}{n(n-1)x^{n-2}} \overset{\text{H}}{=} \cdots \overset{\text{H}}{=} \lim_{x\to\infty} \frac{e^x}{n!} = \infty$

69. First we will find $\displaystyle\lim_{n\to\infty} \left(1+\frac{i}{n}\right)^{nt}$, which is of the form 1^∞. $y = \left(1+\dfrac{i}{n}\right)^{nt} \quad \Rightarrow \quad \ln y = nt\ln\left(1+\dfrac{i}{n}\right)$, so

$$\lim_{n\to\infty} \ln y = \lim_{n\to\infty} nt\ln\left(1+\frac{i}{n}\right) = t\lim_{n\to\infty} \frac{\ln(1+i/n)}{1/n} \overset{\text{H}}{=} t\lim_{n\to\infty} \frac{(-i/n^2)}{(1+i/n)(-1/n^2)} = t\lim_{n\to\infty} \frac{i}{1+i/n} = ti$$

$\Rightarrow \quad \displaystyle\lim_{n\to\infty} y = e^{it}$. Thus, as $n \to \infty$, $A = A_0\left(1+\dfrac{i}{n}\right)^{nt} \to A_0e^{it}$.

71. We see that both numerator and denominator approach 0, so we can use l'Hospital's Rule:

$$\lim_{x\to a}\frac{\sqrt{2a^3x-x^4}-a\sqrt[3]{aax}}{a-\sqrt[4]{ax^3}}\overset{\text{H}}{=}\lim_{x\to a}\frac{\frac{1}{2}\left(2a^3x-x^4\right)^{-1/2}\left(2a^3-4x^3\right)-a\left(\frac{1}{3}\right)(aax)^{-2/3}a^2}{-\frac{1}{4}(ax^3)^{-3/4}(3ax^2)}$$

$$=\frac{\frac{1}{2}\left(2a^3a-a^4\right)^{-1/2}\left(2a^3-4a^3\right)-\frac{1}{3}a^3\left(a^2a\right)^{-2/3}}{-\frac{1}{4}(aa^3)^{-3/4}(3aa^2)}$$

$$=\frac{\left(a^4\right)^{-1/2}\left(-a^3\right)-\frac{1}{3}a^3\left(a^3\right)^{-2/3}}{-\frac{3}{4}a^3\left(a^4\right)^{-3/4}}=\frac{-a-\frac{1}{3}a}{-\frac{3}{4}}=\frac{4}{3}\left(\frac{4}{3}a\right)=\frac{16}{9}a$$

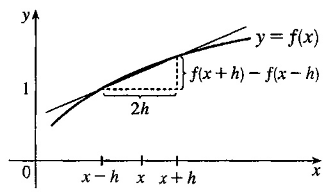
73. Since $f(2)=0$, the given limit has the form $\frac{0}{0}$.

$$\lim_{x\to 0}\frac{f(2+3x)+f(2+5x)}{x}\overset{\text{H}}{=}\lim_{x\to 0}\frac{f'(2+3x)\cdot 3+f'(2+5x)\cdot 5}{1}=f'(2)\cdot 3+f'(2)\cdot 5=8f'(2)=8\cdot 7=56$$

75. Since $\lim\limits_{h\to 0}[f(x+h)-f(x-h)]=f(x)-f(x)=0$ (f is differentiable and hence continuous) and $\lim\limits_{h\to 0}2h=0$, we use l'Hospital's Rule:

$$\lim_{h\to 0}\frac{f(x+h)-f(x-h)}{2h}\overset{\text{H}}{=}\lim_{h\to 0}\frac{f'(x+h)(1)-f'(x-h)(-1)}{2}=\frac{f'(x)+f'(x)}{2}=\frac{2f'(x)}{2}=f'(x)$$

$\dfrac{f(x+h)-f(x-h)}{2h}$ is the slope of the secant line between $(x-h, f(x-h))$ and $(x+h, f(x+h))$. As $h\to 0$, this line gets closer to the tangent line and its slope approaches $f'(x)$.



77. (a) We show that $\lim\limits_{x\to 0}\dfrac{f(x)}{x^n}=0$ for every integer $n\geq 0$. Let $y=\dfrac{1}{x^2}$. Then

$$\lim_{x\to 0}\frac{f(x)}{x^{2n}}=\lim_{x\to 0}\frac{e^{-1/x^2}}{\left(x^2\right)^n}=\lim_{y\to\infty}\frac{y^n}{e^y}\overset{\text{H}}{=}\lim_{y\to\infty}\frac{ny^{n-1}}{e^y}\overset{\text{H}}{=}\cdots\overset{\text{H}}{=}\lim_{y\to\infty}\frac{n!}{e^y}=0\;\Rightarrow$$

$$\lim_{x\to 0}\frac{f(x)}{x^n}=\lim_{x\to 0}x^n\frac{f(x)}{x^{2n}}=\lim_{x\to 0}x^n\lim_{x\to 0}\frac{f(x)}{x^{2n}}=0.\text{ Thus, }f'(0)=\lim_{x\to 0}\frac{f(x)-f(0)}{x-0}=\lim_{x\to 0}\frac{f(x)}{x}=0.$$

(b) Using the Chain Rule and the Quotient Rule we see that $f^{(n)}(x)$ exists for $x\neq 0$. In fact, we prove by induction that for each $n\geq 0$, there is a polynomial p_n and a non-negative integer k_n with $f^{(n)}(x)=p_n(x)f(x)/x^{k_n}$ for $x\neq 0$. This is true for $n=0$; suppose it is true for the nth derivative. Then $f'(x)=f(x)(2/x^3)$, so

$$f^{(n+1)}(x)=\left[x^{k_n}\left[p_n'(x)f(x)+p_n(x)f'(x)\right]-k_nx^{k_n-1}p_n(x)f(x)\right]x^{-2k_n}$$

$$=\left[x^{k_n}p_n'(x)+p_n(x)\left(2/x^3\right)-k_nx^{k_n-1}p_n(x)\right]f(x)x^{-2k_n}$$

$$=\left[x^{k_n+3}p_n'(x)+2p_n(x)-k_nx^{k_n+2}p_n(x)\right]f(x)x^{-(2k_n+3)}$$

which has the desired form.

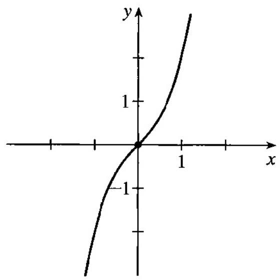
Now we show by induction that $f^{(n)}(0) = 0$ for all n. By part (a), $f'(0) = 0$. Suppose that $f^{(n)}(0) = 0$. Then

$$f^{(n+1)}(0) = \lim_{x \to 0} \frac{f^{(n)}(x) - f^{(n)}(0)}{x - 0} = \lim_{x \to 0} \frac{f^{(n)}(x)}{x} = \lim_{x \to 0} \frac{p_n(x) f(x)/x^{k_n}}{x} = \lim_{x \to 0} \frac{p_n(x) f(x)}{x^{k_n + 1}}$$
$$= \lim_{x \to 0} p_n(x) \lim_{x \to 0} \frac{f(x)}{x^{k_n+1}} = p_n(0) \cdot 0 = 0$$

4.5 Summary of Curve Sketching

1. $y = f(x) = x^3 + x = x(x^2 + 1)$ **A.** f is a polynomial, so $D = \mathbb{R}$. **B.** x-intercept $= 0$, y-intercept $= f(0) = 0$ **C.** $f(-x) = -f(x)$, so f is odd; the curve is symmetric about the origin. **D.** f is a polynomial, so there is no asymptote. **E.** $f'(x) = 3x^2 + 1 > 0$, so f is increasing on $(-\infty, \infty)$. **F.** There is no critical number and hence, no local maximum or minimum value. **G.** $f''(x) = 6x > 0$ on $(0, \infty)$ and $f''(x) < 0$ on $(-\infty, 0)$, so f is CU on $(0, \infty)$ and CD on $(-\infty, 0)$. Since the concavity changes at $x = 0$, there is an inflection point at $(0, 0)$.

H.



3. $y = f(x) = 2 - 15x + 9x^2 - x^3 = -(x - 2)(x^2 - 7x + 1)$ **A.** $D = \mathbb{R}$ **B.** y-intercept: $f(0) = 2$; x-intercepts: $f(x) = 0 \Rightarrow x = 2$ or (by the quadratic formula) $x = \frac{7 \pm \sqrt{45}}{2} \approx 0.15, 6.85$

C. No symmetry **D.** No asymptote

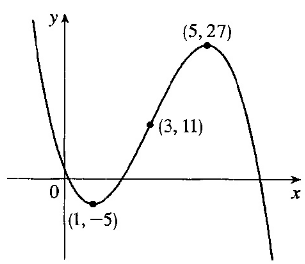
E. $f'(x) = -15 + 18x - 3x^2 = -3(x^2 - 6x + 5)$
$= -3(x - 1)(x - 5) > 0 \quad \Leftrightarrow \quad 1 < x < 5$

so f is increasing on $(1, 5)$ and decreasing on $(-\infty, 1)$ and $(5, \infty)$.

F. Local maximum value $f(5) = 27$, local minimum value $f(1) = -5$

G. $f''(x) = 18 - 6x = -6(x - 3) > 0 \quad \Leftrightarrow \quad x < 3$, so f is CU on $(-\infty, 3)$ and CD on $(3, \infty)$. IP at $(3, 11)$

H.



5. $y = f(x) = x^4 + 4x^3 = x^3(x + 4)$ **A.** $D = \mathbb{R}$ **B.** y-intercept: $f(0) = 0$; x-intercepts: $f(x) = 0 \quad \Leftrightarrow \quad x = -4, 0$ **C.** No symmetry

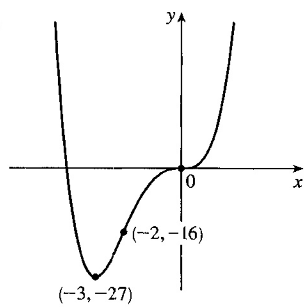
D. No asymptote **E.** $f'(x) = 4x^3 + 12x^2 = 4x^2(x + 3) > 0 \quad \Leftrightarrow$ $x > -3$, so f is increasing on $(-3, \infty)$ and decreasing on $(-\infty, -3)$.

F. Local minimum value $f(-3) = -27$, no local maximum

G. $f''(x) = 12x^2 + 24x = 12x(x + 2) < 0 \quad \Leftrightarrow \quad -2 < x < 0$, so f is CD on $(-2, 0)$ and CU on $(-\infty, -2)$ and $(0, \infty)$.

IP at $(0, 0)$ and $(-2, -16)$

H.



7. $y = f(x) = 2x^5 - 5x^2 + 1$ **A.** $D = \mathbb{R}$ **B.** y-intercept: $f(0) = 1$ **C.** No symmetry **D.** No asymptote

E. $f'(x) = 10x^4 - 10x = 10x(x^3 - 1) = 10x(x-1)(x^2+x+1)$, so $f'(x) < 0 \Leftrightarrow 0 < x < 1$ and $f'(x) > 0 \Leftrightarrow x < 0$ or $x > 1$. Thus, f is increasing on $(-\infty, 0)$ and $(1, \infty)$ and decreasing on $(0, 1)$.

F. Local maximum value $f(0) = 1$, local minimum value $f(1) = -2$

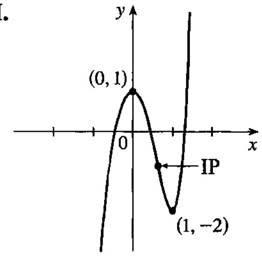
G. $f''(x) = 40x^3 - 10 = 10(4x^3 - 1)$ so $f''(x) = 0 \Leftrightarrow x = 1/\sqrt[3]{4}$.

$f''(x) > 0 \Leftrightarrow x > 1/\sqrt[3]{4}$ and $f''(x) < 0 \Leftrightarrow x < 1/\sqrt[3]{4}$,

so f is CD on $\left(-\infty, 1/\sqrt[3]{4}\right)$ and CU on $\left(1/\sqrt[3]{4}, \infty\right)$.

IP at $\left(\dfrac{1}{\sqrt[3]{4}}, 1 - \dfrac{9}{2\left(\sqrt[3]{4}\right)^2}\right) \approx (0.630, -0.786)$

H.



9. $y = f(x) = x/(x-1)$ **A.** $D = \{x \mid x \neq 1\} = (-\infty, 1) \cup (1, \infty)$ **B.** x-intercept $= 0$,

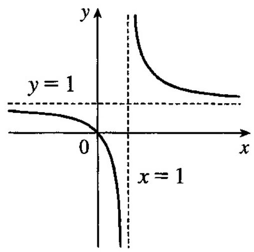
y-intercept $= f(0) = 0$ **C.** No symmetry **D.** $\displaystyle\lim_{x \to \pm\infty} \frac{x}{x-1} = 1$, so $y = 1$ is a HA.

$\displaystyle\lim_{x \to 1^-} \frac{x}{x-1} = -\infty$, $\displaystyle\lim_{x \to 1^+} \frac{x}{x-1} = \infty$, so $x = 1$ is a VA.

E. $f'(x) = \dfrac{(x-1) - x}{(x-1)^2} = \dfrac{-1}{(x-1)^2} < 0$ for $x \neq 1$, so f is decreasing on $(-\infty, 1)$ and $(1, \infty)$. **F.** No extreme values

G. $f''(x) = \dfrac{2}{(x-1)^3} > 0 \Leftrightarrow x > 1$, so f is CU on $(1, \infty)$ and CD on $(-\infty, 1)$. No IP

H.



11. $y = f(x) = 1/(x^2 - 9)$ **A.** $D = \{x \mid x \neq \pm 3\} = (-\infty, -3) \cup (-3, 3) \cup (3, \infty)$

B. y-intercept $= f(0) = -\frac{1}{9}$, no x-intercept **C.** $f(-x) = f(x) \Rightarrow f$ is even; the curve is symmetric about the y-axis. **D.** $\displaystyle\lim_{x \to \pm\infty} \frac{1}{x^2 - 9} = 0$, so $y = 0$ is a HA. $\displaystyle\lim_{x \to 3^-} \frac{1}{x^2 - 9} = -\infty$, $\displaystyle\lim_{x \to 3^+} \frac{1}{x^2 - 9} = \infty$,

$\displaystyle\lim_{x \to -3^-} \frac{1}{x^2 - 9} = \infty$, $\displaystyle\lim_{x \to -3^+} \frac{1}{x^2 - 9} = -\infty$, so $x = 3$ and $x = -3$ are VA.

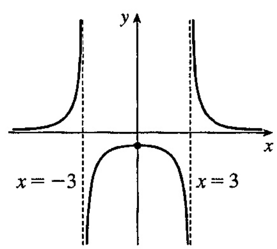
E. $f'(x) = -\dfrac{2x}{(x^2 - 9)^2} > 0 \Leftrightarrow x < 0$ $(x \neq -3)$ so f is increasing on $(-\infty, -3)$ and $(-3, 0)$ and decreasing on $(0, 3)$ and $(3, \infty)$.

F. Local maximum value $f(0) = -\frac{1}{9}$.

G. $y'' = \dfrac{-2(x^2-9)^2 + (2x)2(x^2-9)(2x)}{(x^2-9)^4} = \dfrac{6(x^2+3)}{(x^2-9)^3} > 0 \Leftrightarrow$

$x^2 > 9 \Leftrightarrow x > 3$ or $x < -3$, so f is CU on $(-\infty, -3)$ and $(3, \infty)$ and CD on $(-3, 3)$. No IP

H.



13. $y = f(x) = x/(x^2 + 9)$ **A.** $D = \mathbb{R}$ **B.** y-intercept: $f(0) = 0$; x-intercept: $f(x) = 0 \Leftrightarrow x = 0$

C. $f(-x) = -f(x)$, so f is odd and the curve is symmetric about the origin. **D.** $\lim_{x \to \pm\infty} [x/(x^2 + 9)] = 0$, so

$y = 0$ is a HA; no VA **E.** $f'(x) = \dfrac{(x^2 + 9)(1) - x(2x)}{(x^2 + 9)^2} = \dfrac{9 - x^2}{(x^2 + 9)^2} = \dfrac{(3 + x)(3 - x)}{(x^2 + 9)^2} > 0 \Leftrightarrow$

$-3 < x < 3$, so f is increasing on $(-3, 3)$ and decreasing on $(-\infty, -3)$ and $(3, \infty)$.

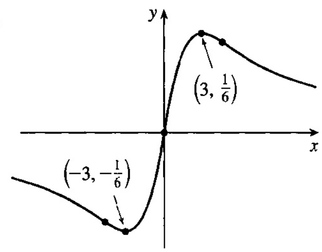
F. Local minimum value $f(-3) = -\frac{1}{6}$, local maximum value $f(3) = \frac{1}{6}$

G.
$$f''(x) = \frac{(x^2 + 9)^2(-2x) - (9 - x^2) \cdot 2(x^2 + 9)(2x)}{[(x^2 + 9)^2]^2}$$
$$= \frac{(2x)(x^2 + 9)[-(x^2 + 9) - 2(9 - x^2)]}{(x^2 + 9)^4}$$
$$= \frac{2x(x^2 - 27)}{(x^2 + 9)^3} = 0 \Leftrightarrow x = 0, \pm\sqrt{27} = \pm 3\sqrt{3}$$

$f''(x) > 0 \Leftrightarrow -3\sqrt{3} < x < 0$ or $x > 3\sqrt{3}$, so f is CU on $(-3\sqrt{3}, 0)$ and $(3\sqrt{3}, \infty)$, and CD on $(-\infty, -3\sqrt{3})$ and $(0, 3\sqrt{3})$.

There are three inflection points: $(0, 0)$ and $(\pm 3\sqrt{3}, \pm\frac{1}{12}\sqrt{3})$.

H.



15. $y = f(x) = \dfrac{x - 1}{x^2}$ **A.** $D = \{x \mid x \neq 0\} = (-\infty, 0) \cup (0, \infty)$ **B.** No y-intercept; x-intercept: $f(x) = 0 \Leftrightarrow$

$x = 1$ **C.** No symmetry **D.** $\lim_{x \to \pm\infty} \dfrac{x - 1}{x^2} = 0$, so $y = 0$ is a HA. $\lim_{x \to 0} \dfrac{x - 1}{x^2} = -\infty$, so $x = 0$ is a VA.

E. $f'(x) = \dfrac{x^2 \cdot 1 - (x - 1) \cdot 2x}{(x^2)^2} = \dfrac{-x^2 + 2x}{x^4} = \dfrac{-(x - 2)}{x^3}$, so $f'(x) > 0 \Leftrightarrow 0 < x < 2$ and

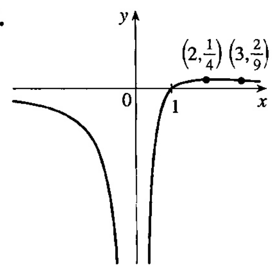
$f'(x) < 0 \Leftrightarrow x < 0$ or $x > 2$. Thus, f is increasing on $(0, 2)$ and decreasing on $(-\infty, 0)$ and $(2, \infty)$.

F. No local minimum, local maximum value $f(2) = \frac{1}{4}$.

G. $f''(x) = \dfrac{x^3 \cdot (-1) - [-(x - 2)] \cdot 3x^2}{(x^3)^2} = \dfrac{2x^3 - 6x^2}{x^6} = \dfrac{2(x - 3)}{x^4}$.

$f''(x)$ is negative on $(-\infty, 0)$ and $(0, 3)$ and positive on $(3, \infty)$, so f is CD on $(-\infty, 0)$ and $(0, 3)$ and CU on $(3, \infty)$. IP at $(3, \frac{2}{9})$

H.



17. $y = f(x) = \dfrac{x^2}{x^2 + 3} = \dfrac{(x^2 + 3) - 3}{x^2 + 3} = 1 - \dfrac{3}{x^2 + 3}$ **A.** $D = \mathbb{R}$ **B.** y-intercept: $f(0) = 0$; x-intercepts:

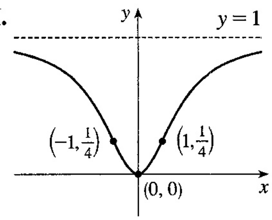
$f(x) = 0 \Leftrightarrow x = 0$ **C.** $f(-x) = f(x)$, so f is even; the graph is symmetric about the y-axis.

D. $\lim_{x \to \pm\infty} \dfrac{x^2}{x^2 + 3} = 1$, so $y = 1$ is a HA. No VA. **E.** Using the Reciprocal Rule,

$f'(x) = -3 \cdot \dfrac{-2x}{(x^2+3)^2} = \dfrac{6x}{(x^2+3)^2}$. $f'(x) > 0 \Leftrightarrow x > 0$ and $f'(x) < 0 \Leftrightarrow x < 0$, so f is decreasing on $(-\infty, 0)$ and increasing on $(0, \infty)$. **F.** Local minimum value $f(0) = 0$, no local maximum.

G.
$$f''(x) = \frac{(x^2+3)^2 \cdot 6 - 6x \cdot 2(x^2+3) \cdot 2x}{[(x^2+3)^2]^2} = \frac{6(x^2+3)\left[(x^2+3) - 4x^2\right]}{(x^2+3)^4} = \frac{6(3-3x^2)}{(x^2+3)^3} = \frac{-18(x+1)(x-1)}{(x^2+3)^3}$$

H.



$f''(x)$ is negative on $(-\infty, -1)$ and $(1, \infty)$ and positive on $(-1, 1)$, so f is CD on $(-\infty, -1)$ and $(1, \infty)$ and CU on $(-1, 1)$. IP at $\left(\pm 1, \frac{1}{4}\right)$

19. $y = f(x) = x\sqrt{5-x}$ **A.** The domain is $\{x \mid 5 - x \geq 0\} = (-\infty, 5]$ **B.** y-intercept: $f(0) = 0$; x-intercepts: $f(x) = 0 \Leftrightarrow x = 0, 5$ **C.** No symmetry **D.** No asymptote

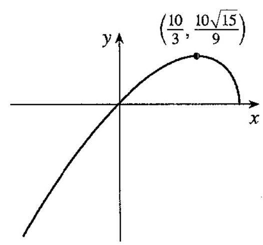
E. $f'(x) = x \cdot \frac{1}{2}(5-x)^{-1/2}(-1) + (5-x)^{1/2} \cdot 1 = \frac{1}{2}(5-x)^{-1/2}\left[-x + 2(5-x)\right] = \dfrac{10-3x}{2\sqrt{5-x}} > 0 \Leftrightarrow$ $x < \frac{10}{3}$, so f is increasing on $\left(-\infty, \frac{10}{3}\right)$ and decreasing on $\left(\frac{10}{3}, 5\right)$.

F. Local maximum value $f\left(\frac{10}{3}\right) = \frac{10}{9}\sqrt{15} \approx 4.3$; no local minimum

G.
$$f''(x) = \frac{2(5-x)^{1/2}(-3) - (10-3x) \cdot 2\left(\frac{1}{2}\right)(5-x)^{-1/2}(-1)}{\left(2\sqrt{5-x}\right)^2} = \frac{(5-x)^{-1/2}\left[-6(5-x) + (10-3x)\right]}{4(5-x)} = \frac{3x-20}{4(5-x)^{3/2}}$$

$f''(x) < 0$ for $x < 5$, so f is CD on $(-\infty, 5)$. No IP

H.



21. $y = f(x) = \sqrt{x^2+1} - x$ **A.** $D = \mathbb{R}$ **B.** No x-intercept, y-intercept $= 1$ **C.** No symmetry

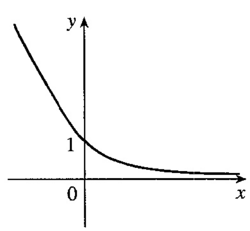
D. $\lim\limits_{x \to -\infty} \left(\sqrt{x^2+1} - x\right) = \infty$ and

$$\lim_{x \to \infty} \left(\sqrt{x^2+1} - x\right) = \lim_{x \to \infty} \left(\sqrt{x^2+1} - x\right) \frac{\sqrt{x^2+1} + x}{\sqrt{x^2+1} + x} = \lim_{x \to \infty} \frac{1}{\sqrt{x^2+1} + x} = 0,$$

so $y = 0$ is a HA. **E.** $f'(x) = \dfrac{x}{\sqrt{x^2+1}} - 1 = \dfrac{x - \sqrt{x^2+1}}{\sqrt{x^2+1}} \Rightarrow$ $f'(x) < 0$, so f is decreasing on $\mathbb{R}$. **F.** No extreme values

G. $f''(x) = \dfrac{1}{(x^2+1)^{3/2}} > 0$, so f is CU on $\mathbb{R}$. No IP

H.



23. $y = f(x) = x/\sqrt{x^2+1}$ **A.** $D = \mathbb{R}$ **B.** y-intercept: $f(0) = 0$; x-intercepts: $f(x) = 0 \Rightarrow x = 0$

C. $f(-x) = -f(x)$, so f is odd; the graph is symmetric about the origin.

D. $$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{x}{\sqrt{x^2+1}} = \lim_{x\to\infty} \frac{x/x}{\sqrt{x^2+1}/x} = \lim_{x\to\infty} \frac{x/x}{\sqrt{x^2+1}/\sqrt{x^2}} = \lim_{x\to\infty} \frac{1}{\sqrt{1+1/x^2}} = \frac{1}{\sqrt{1+0}} = 1$$

and

$$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} \frac{x}{\sqrt{x^2+1}} = \lim_{x\to-\infty} \frac{x/x}{\sqrt{x^2+1}/x} = \lim_{x\to-\infty} \frac{x/x}{\sqrt{x^2+1}\big/\left(-\sqrt{x^2}\right)} = \lim_{x\to-\infty} \frac{1}{-\sqrt{1+1/x^2}} = \frac{1}{-\sqrt{1+0}} = -1$$

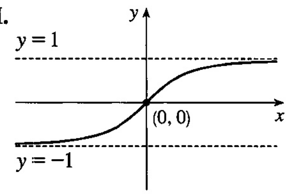
so $y = \pm 1$ are HA. No VA.

E. $$f'(x) = \frac{\sqrt{x^2+1} - x \cdot \dfrac{2x}{2\sqrt{x^2+1}}}{[(x^2+1)^{1/2}]^2} = \frac{x^2+1-x^2}{(x^2+1)^{3/2}} = \frac{1}{(x^2+1)^{3/2}} > 0$$ for all x, so f is increasing on $\mathbb{R}$.

F. No extreme values

G. $f''(x) = -\frac{3}{2}\left(x^2+1\right)^{-5/2} \cdot 2x = \dfrac{-3x}{(x^2+1)^{5/2}}$, so $f''(x) > 0$ for $x < 0$ and $f''(x) < 0$ for $x > 0$. Thus, f is CU on $(-\infty, 0)$ and CD on $(0, \infty)$. IP at $(0, 0)$

H.



25. $y = f(x) = \sqrt{1-x^2}/x$ **A.** $D = \left\{x \,\middle|\, |x| \le 1, x \ne 0\right\} = [-1, 0) \cup (0, 1]$ **B.** x-intercepts ± 1, no y-intercept

C. $f(-x) = -f(x)$, so the curve is symmetric about $(0, 0)$. **D.** $\lim\limits_{x\to 0^+} \dfrac{\sqrt{1-x^2}}{x} = \infty$, $\lim\limits_{x\to 0^-} \dfrac{\sqrt{1-x^2}}{x} = -\infty$,

so $x = 0$ is a VA. **E.** $f'(x) = \dfrac{\left(-x^2/\sqrt{1-x^2}\right) - \sqrt{1-x^2}}{x^2} = -\dfrac{1}{x^2\sqrt{1-x^2}} < 0$, so f is decreasing on $(-1, 0)$

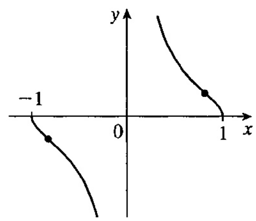
and $(0, 1)$. **F.** No extreme values

G. $f''(x) = \dfrac{2-3x^2}{x^3(1-x^2)^{3/2}} > 0 \Leftrightarrow -1 < x < -\sqrt{\frac{2}{3}}$ or

$0 < x < \sqrt{\frac{2}{3}}$, so f is CU on $\left(-1, -\sqrt{\frac{2}{3}}\right)$ and $\left(0, \sqrt{\frac{2}{3}}\right)$ and CD on

$\left(-\sqrt{\frac{2}{3}}, 0\right)$ and $\left(\sqrt{\frac{2}{3}}, 1\right)$. IP at $\left(\pm\sqrt{\frac{2}{3}}, \pm\frac{1}{\sqrt{2}}\right)$

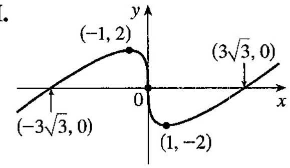
H.



27. $y = f(x) = x - 3x^{1/3}$ **A.** $D = \mathbb{R}$ **B.** y-intercept: $f(0) = 0$; x-intercepts: $f(x) = 0 \;\Rightarrow\; x = 3x^{1/3} \;\Rightarrow$ $x^3 = 27x \;\Rightarrow\; x^3 - 27x = 0 \;\Rightarrow\; x(x^2 - 27) = 0 \;\Rightarrow\; x = 0, \pm 3\sqrt{3}$ **C.** $f(-x) = -f(x)$, so f is odd; the graph is symmetric about the origin. **D.** No asymptote **E.** $f'(x) = 1 - x^{-2/3} = 1 - \dfrac{1}{x^{2/3}} = \dfrac{x^{2/3} - 1}{x^{2/3}}$.

$f'(x) > 0$ when $|x| > 1$ and $f'(x) < 0$ when $0 < |x| < 1$, so f is increasing on $(-\infty, -1)$ and $(1, \infty)$, and decreasing on $(-1, 0)$ and $(0, 1)$ [hence decreasing on $(-1, 1)$ since f is continuous on $(-1, 1)$]. **F.** Local maximum value $f(-1) = 2$, local minimum value $f(1) = -2$ **G.** $f''(x) = \frac{2}{3}x^{-5/3} < 0$ when $x < 0$ and $f''(x) > 0$ when $x > 0$, so f is CD on $(-\infty, 0)$ and CU on $(0, \infty)$. IP at $(0, 0)$

H.



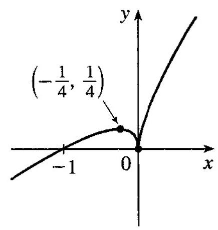
29. $y = f(x) = x + \sqrt{|x|}$ **A.** $D = \mathbb{R}$ **B.** x-intercepts $0, -1$; y-intercept 0 **C.** No symmetry

D. $\lim\limits_{x\to\infty}\left(x + \sqrt{|x|}\right) = \infty$, $\lim\limits_{x\to-\infty}\left(x + \sqrt{|x|}\right) = -\infty$. No asymptote **E.** For $x > 0$, $f(x) = x + \sqrt{x} \;\Rightarrow$ $f'(x) = 1 + \dfrac{1}{2\sqrt{x}} > 0$, so f increases on $(0, \infty)$. For $x < 0$, $f(x) = x + \sqrt{-x} \;\Rightarrow\; f'(x) = 1 - \dfrac{1}{2\sqrt{-x}} > 0$ $\Leftrightarrow\; 2\sqrt{-x} > 1 \;\Leftrightarrow\; -x > \frac{1}{4} \;\Leftrightarrow\; x < -\frac{1}{4}$, so f increases on $\left(-\infty, -\frac{1}{4}\right)$ and decreases on $\left(-\frac{1}{4}, 0\right)$.

F. Local maximum value $f\left(-\frac{1}{4}\right) = \frac{1}{4}$, local minimum value $f(0) = 0$

G. For $x > 0$, $f''(x) = -\frac{1}{4}x^{-3/2} \;\Rightarrow\; f''(x) < 0$, so f is CD on $(0, \infty)$. For $x < 0$, $f''(x) = -\frac{1}{4}(-x)^{-3/2} \;\Rightarrow\; f''(x) < 0$, so f is CD on $(-\infty, 0)$. No IP

H.



31. $y = f(x) = 3\sin x - \sin^3 x$ **A.** $D = \mathbb{R}$ **B.** y-intercept: $f(0) = 0$; x-intercepts: $f(x) = 0 \;\Rightarrow$ $\sin x\,(3 - \sin^2 x) = 0 \;\Rightarrow\; \sin x = 0$ [since $\sin^2 x \le 1 < 3$] $\Rightarrow\; x = n\pi$, n an integer.

C. $f(-x) = -f(x)$, so f is odd; the graph (shown for $-2\pi \le x \le 2\pi$) is symmetric about the origin and periodic with period 2π. **D.** No asymptote **E.** $f'(x) = 3\cos x - 3\sin^2 x\cos x = 3\cos x\,(1 - \sin^2 x) = 3\cos^3 x$.

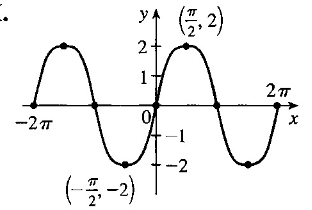
$f'(x) > 0 \;\Leftrightarrow\; \cos x > 0 \;\Leftrightarrow\; x \in \left(2n\pi - \frac{\pi}{2}, 2n\pi + \frac{\pi}{2}\right)$ for each integer n, and $f'(x) < 0 \;\Leftrightarrow$ $\cos x < 0 \;\Leftrightarrow\; x \in \left(2n\pi + \frac{\pi}{2}, 2n\pi + \frac{3\pi}{2}\right)$ for each integer n. Thus, f is increasing on $\left(2n\pi - \frac{\pi}{2}, 2n\pi + \frac{\pi}{2}\right)$ for each integer n, and f is decreasing on $\left(2n\pi + \frac{\pi}{2}, 2n\pi + \frac{3\pi}{2}\right)$ for each integer n.

F. f has local maximum values $f(2n\pi + \frac{\pi}{2}) = 2$ and local minimum values $f(2n\pi + \frac{3\pi}{2}) = -2$.

G. $f''(x) = -9\sin x\cos^2 x = -9\sin x\,(1 - \sin^2 x) = -9\sin x\,(1 - \sin x)(1 + \sin x)$. $f''(x) < 0 \;\Leftrightarrow$ $\sin x > 0$ and $\sin x \neq \pm 1 \;\Leftrightarrow\; x \in \left(2n\pi, 2n\pi + \frac{\pi}{2}\right) \cup \left(2n\pi + \frac{\pi}{2}, 2n\pi + \pi\right)$ for some integer n.

$f''(x) > 0 \;\Leftrightarrow\; \sin x < 0$ and $\sin x \neq \pm 1 \;\Leftrightarrow\; x \in \left((2n-1)\pi, (2n-1)\pi + \frac{\pi}{2}\right) \cup \left((2n-1)\pi + \frac{\pi}{2}, 2n\pi\right)$ for some integer n. Thus, f is CD on the intervals $\left(2n\pi, \left(2n + \frac{1}{2}\right)\pi\right)$ and $\left(\left(2n + \frac{1}{2}\right)\pi, (2n+1)\,\pi\right)$ [hence CD on the intervals $(2n\pi, (2n+1)\,\pi)$] for each integer n, and f is CU on the intervals $\left((2n-1)\pi, \left(2n - \frac{1}{2}\right)\pi\right)$ and $\left(\left(2n - \frac{1}{2}\right)\pi, 2n\pi\right)$ [hence CU on the intervals $((2n-1)\pi, 2n\pi)$] for each integer n. f has inflection points at $(n\pi, 0)$ for each integer n.

H.

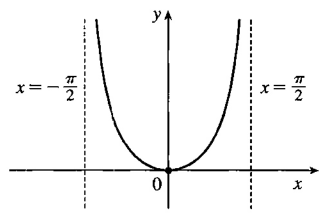


33. $y = f(x) = x\tan x$, $-\frac{\pi}{2} < x < \frac{\pi}{2}$ **A.** $D = \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ **B.** Intercepts are 0 **C.** $f(-x) = f(x)$, so the curve is symmetric about the y-axis. **D.** $\lim\limits_{x\to(\pi/2)^-} x\tan x = \infty$ and $\lim\limits_{x\to-(\pi/2)^+} x\tan x = \infty$, so $x = \frac{\pi}{2}$ and $x = -\frac{\pi}{2}$ are VA. **E.** $f'(x) = \tan x + x\sec^2 x > 0 \;\Leftrightarrow\; 0 < x < \frac{\pi}{2}$, so f increases on $\left(0, \frac{\pi}{2}\right)$ and decreases on $\left(-\frac{\pi}{2}, 0\right)$.

F. Absolute and local minimum value $f(0) = 0$.

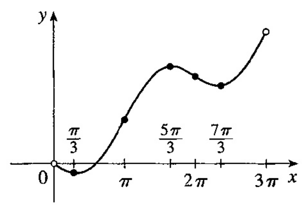
G. $y'' = 2\sec^2 x + 2x\tan x\sec^2 x > 0$ for $-\frac{\pi}{2} < x < \frac{\pi}{2}$, so f is CU on $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. No IP

H.



35. $y = f(x) = \frac{1}{2}x - \sin x$, $0 < x < 3\pi$ **A.** $D = (0, 3\pi)$ **B.** No y-intercept. The x-intercept, approximately 1.9, can be found using Newton's Method. **C.** No symmetry **D.** No asymptote **E.** $f'(x) = \frac{1}{2} - \cos x > 0 \;\Leftrightarrow$ $\cos x < \frac{1}{2} \;\Leftrightarrow\; \frac{\pi}{3} < x < \frac{5\pi}{3}$ or $\frac{7\pi}{3} < x < 3\pi$, so f is increasing on $\left(\frac{\pi}{3}, \frac{5\pi}{3}\right)$ and $\left(\frac{7\pi}{3}, 3\pi\right)$ and decreasing on $\left(0, \frac{\pi}{3}\right)$ and $\left(\frac{5\pi}{3}, \frac{7\pi}{3}\right)$. **F.** Local minimum value $f\left(\frac{\pi}{3}\right) = \frac{\pi}{6} - \frac{\sqrt{3}}{2}$, local maximum value $f\left(\frac{5\pi}{3}\right) = \frac{5\pi}{6} + \frac{\sqrt{3}}{2}$, local minimum value $f\left(\frac{7\pi}{3}\right) = \frac{7\pi}{6} - \frac{\sqrt{3}}{2}$ **G.** $f''(x) = \sin x > 0 \;\Leftrightarrow\; 0 < x < \pi$ or $2\pi < x < 3\pi$, so f is CU on $(0, \pi)$ and $(2\pi, 3\pi)$ and CD on $(\pi, 2\pi)$. IPs at $\left(\pi, \frac{\pi}{2}\right)$ and $(2\pi, \pi)$.

H.



37. $y = f(x) = \sin 2x - 2\sin x$ **A.** $D = \mathbb{R}$ **B.** y-intercept $= f(0) = 0$. $y = 0 \Leftrightarrow$

$2\sin x = \sin 2x = 2\sin x\cos x \Leftrightarrow \sin x = 0$ or $\cos x = 1 \Leftrightarrow x = n\pi$ (x-intercepts)

C. $f(-x) = -f(x)$, so the curve is symmetric about $(0,0)$.

Note: f is periodic with period 2π, so we determine E–G for $-\pi \le x \le \pi$. **D.** No asymptotes

E. $f'(x) = 2\cos 2x - 2\cos x = 2\left(2\cos^2 x - 1 - \cos x\right) = 2(2\cos x + 1)(\cos x - 1) > 0 \Leftrightarrow \cos x < -\frac{1}{2}$

$\Leftrightarrow -\pi < x < -\frac{2\pi}{3}$ or $\frac{2\pi}{3} < x < \pi$, so f is increasing on $\left(-\pi, -\frac{2\pi}{3}\right)$, $\left(\frac{2\pi}{3}, \pi\right)$ and decreasing on $\left(-\frac{2\pi}{3}, \frac{2\pi}{3}\right)$.

F. Local maximum value $f\left(-\frac{2\pi}{3}\right) = \frac{3\sqrt{3}}{2}$,

local minimum value $f\left(\frac{2\pi}{3}\right) = -\frac{3\sqrt{3}}{2}$

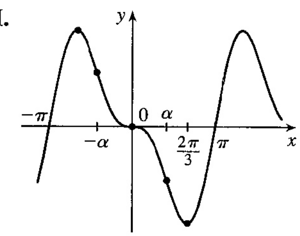
G. $f''(x) = -4\sin 2x + 2\sin x = 2\sin x\,(1 - 4\cos x) = 0$ when

$x = 0, \pm\pi$ or $\cos x = \frac{1}{4}$. If $\alpha = \cos^{-1}\frac{1}{4}$, then f is CU on $(-\alpha, 0)$ and

(α, π) and CD on $(-\pi, -\alpha)$ and $(0, \alpha)$.

IPs at $(0,0)$, $(\pm\pi, 0)$, $\left(\alpha, -\frac{3\sqrt{15}}{8}\right)$, $\left(-\alpha, \frac{3\sqrt{15}}{8}\right)$.

H.



39. $y = f(x) = \dfrac{\sin x}{1 + \cos x}$ $\left[\overset{\text{when } \cos x \ne 1}{=} \dfrac{\sin x}{1+\cos x}\cdot\dfrac{1-\cos x}{1-\cos x} = \dfrac{\sin x\,(1-\cos x)}{\sin^2 x} = \dfrac{1-\cos x}{\sin x} = \csc x - \cot x\right]$

A. The domain of f is the set of all real numbers except odd integer multiples of π. **B.** y-intercept: $f(0) = 0$; x-intercepts: $x = n\pi$, n an even integer. **C.** $f(-x) = -f(x)$, so f is an odd function; the graph is symmetric about the origin and has period 2π. **D.** When n is an odd integer, $\lim\limits_{x\to(n\pi)^-} f(x) = \infty$ and $\lim\limits_{x\to(n\pi)^+} f(x) = -\infty$,

so $x = n\pi$ is a VA for each odd integer n. No HA.

E. $f'(x) = \dfrac{(1+\cos x)\cdot\cos x - \sin x(-\sin x)}{(1+\cos x)^2} = \dfrac{1+\cos x}{(1+\cos x)^2} = \dfrac{1}{1+\cos x}$. $f'(x) > 0$ for all x except odd multiples of π, so f is increasing on $((2k-1)\pi, (2k+1)\pi)$ for each integer k. **F.** No extreme values

G. $f''(x) = \dfrac{\sin x}{(1+\cos x)^2} > 0 \Rightarrow \sin x > 0 \Rightarrow$

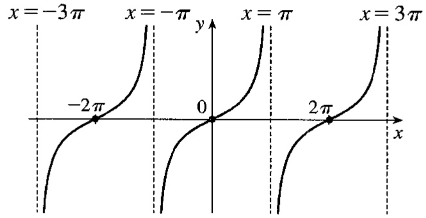
$x \in (2k\pi, (2k+1)\pi)$ and $f''(x) < 0$ on $((2k-1)\pi, 2k\pi)$

for each integer k. f is CU on $(2k\pi, (2k+1)\pi)$ and CD on

$((2k-1)\pi, 2k\pi)$ for each integer k. f has IPs at $(2k\pi, 0)$

for each integer k.

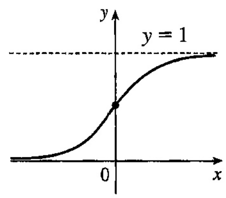
H.



41. $y = 1/(1+e^{-x})$ **A.** $D = \mathbb{R}$ **B.** No x-intercept; y-intercept $= f(0) = \frac{1}{2}$. **C.** No symmetry

D. $\lim_{x\to\infty} 1/(1+e^{-x}) = \frac{1}{1+0} = 1$ and $\lim_{x\to-\infty} 1/(1+e^{-x}) = 0$ (since $\lim_{x\to-\infty} e^{-x} = \infty$), so f has horizontal asymptotes $y=0$ and $y=1$. **E.** $f'(x) = -(1+e^{-x})^{-2}(-e^{-x}) = e^{-x}/(1+e^{-x})^2$. This is positive for all x, so f is increasing on $\mathbb{R}$. **F.** No extreme values

G. $$f''(x) = \frac{(1+e^{-x})^2(-e^{-x}) - e^{-x}(2)(1+e^{-x})(-e^{-x})}{(1+e^{-x})^4} = \frac{e^{-x}(e^{-x}-1)}{(1+e^{-x})^3}$$

H. 

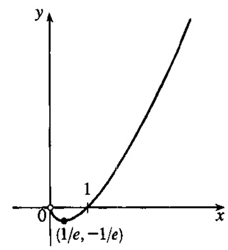
The second factor in the numerator is negative for $x > 0$ and positive for $x < 0$, and the other factors are always positive, so f is CU on $(-\infty, 0)$ and CD on $(0, \infty)$. f has an inflection point at $(0, \frac{1}{2})$.

43. $y = f(x) = x\ln x$ **A.** $D = (0,\infty)$ **B.** x-intercept when $\ln x = 0 \Leftrightarrow x = 1$, no y-intercept

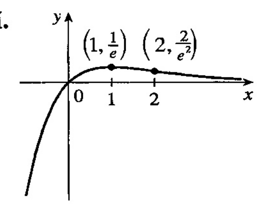
C. No symmetry **D.** $\lim_{x\to\infty} x\ln x = \infty$,

$$\lim_{x\to 0^+} x\ln x = \lim_{x\to 0^+}\frac{\ln x}{1/x} \overset{\text{H}}{=} \lim_{x\to 0^+}\frac{1/x}{-1/x^2} = \lim_{x\to 0^+}(-x) = 0, \text{ no}$$

asymptote. **E.** $f'(x) = \ln x + 1 = 0$ when $\ln x = -1 \Leftrightarrow x = e^{-1}$. $f'(x) > 0 \Leftrightarrow \ln x > -1 \Leftrightarrow x > e^{-1}$, so f is increasing on $(1/e, \infty)$ and decreasing on $(0, 1/e)$. **F.** $f(1/e) = -1/e$ is an absolute and local minimum value. **G.** $f''(x) = 1/x > 0$, so f is CU on $(0,\infty)$. No IP

H. 

45. $y = f(x) = xe^{-x}$ **A.** $D = \mathbb{R}$ **B.** Intercepts are 0 **C.** No symmetry

H. 

D. $\lim_{x\to\infty} xe^{-x} = \lim_{x\to\infty}\frac{x}{e^x} \overset{\text{H}}{=} \lim_{x\to\infty}\frac{1}{e^x} = 0$, so $y = 0$ is a HA.

$\lim_{x\to-\infty} xe^{-x} = -\infty$ **E.** $f'(x) = e^{-x} - xe^{-x} = e^{-x}(1-x) > 0 \Leftrightarrow x < 1$, so f is increasing on $(-\infty, 1)$ and decreasing on $(1,\infty)$.

F. Absolute and local maximum value $f(1) = 1/e$.

G. $f''(x) = e^{-x}(x-2) > 0 \Leftrightarrow x > 2$, so f is CU on $(2,\infty)$ and CD on $(-\infty, 2)$. IP at $(2, 2/e^2)$

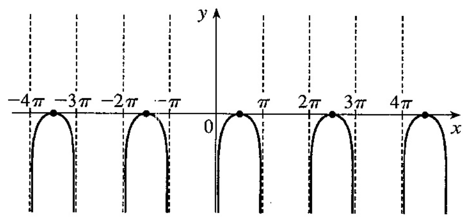
47. $y = f(x) = \ln(\sin x)$

A. $D = \{x \text{ in } \mathbb{R} \mid \sin x > 0\} = \bigcup\limits_{n=-\infty}^{\infty} (2n\pi, (2n+1)\pi)$

$= \cdots \cup (-4\pi, -3\pi) \cup (-2\pi, -\pi) \cup (0, \pi) \cup (2\pi, 3\pi) \cup \cdots$

B. No y-intercept; x-intercepts: $f(x) = 0 \;\Leftrightarrow\; \ln(\sin x) = 0 \;\Leftrightarrow\; \sin x = e^0 = 1 \;\Leftrightarrow\; x = 2n\pi + \frac{\pi}{2}$ for each integer n. **C.** f is periodic with period 2π. **D.** $\lim\limits_{x\to(2n\pi)^+} f(x) = -\infty$ and $\lim\limits_{x\to[(2n+1)\pi]^-} f(x) = -\infty$, so the lines $x = n\pi$ are VAs for all integers n. **E.** $f'(x) = \frac{\cos x}{\sin x} = \cot x$, so $f'(x) > 0$ when $2n\pi < x < 2n\pi + \frac{\pi}{2}$ for each integer n, and $f'(x) < 0$ when $2n\pi + \frac{\pi}{2} < x < (2n+1)\pi$. Thus, f is increasing on $\left(2n\pi, 2n\pi + \frac{\pi}{2}\right)$ and decreasing on $\left(2n\pi + \frac{\pi}{2}, (2n+1)\pi\right)$ for each integer n. **F.** Local maximum values $f\left(2n\pi + \frac{\pi}{2}\right) = 0$, no local minimum. **G.** $f''(x) = -\csc^2 x < 0$, so f is CD on $(2n\pi, (2n+1)\pi)$ for each integer n. No IP

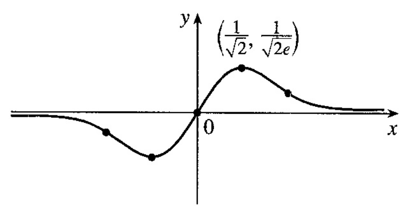
H.



49. $y = f(x) = xe^{-x^2}$ **A.** $D = \mathbb{R}$ **B.** Intercepts are 0 **C.** $f(-x) = -f(x)$, so the curve is symmetric about the origin. **D.** $\lim\limits_{x\to\pm\infty} xe^{-x^2} = \lim\limits_{x\to\pm\infty} \frac{x}{e^{x^2}} \stackrel{H}{=} \lim\limits_{x\to\pm\infty} \frac{1}{2xe^{x^2}} = 0$, so $y = 0$ is a HA.

E. $f'(x) = e^{-x^2} - 2x^2e^{-x^2} = e^{-x^2}(1 - 2x^2) > 0 \;\Leftrightarrow\; x^2 < \frac{1}{2} \;\Leftrightarrow\; |x| < \frac{1}{\sqrt{2}}$, so f is increasing on $\left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$ and decreasing on $\left(-\infty, -\frac{1}{\sqrt{2}}\right)$ and $\left(\frac{1}{\sqrt{2}}, \infty\right)$. **F.** Local maximum value $f\left(\frac{1}{\sqrt{2}}\right) = 1/\sqrt{2e}$, local minimum value $f\left(-\frac{1}{\sqrt{2}}\right) = -1/\sqrt{2e}$ **G.** $f''(x) = -2xe^{-x^2}(1 - 2x^2) - 4xe^{-x^2} = 2xe^{-x^2}(2x^2 - 3) > 0$ $\Leftrightarrow\; x > \sqrt{\frac{3}{2}}$ or $-\sqrt{\frac{3}{2}} < x < 0$, so f is CU on $\left(\sqrt{\frac{3}{2}}, \infty\right)$ and $\left(-\sqrt{\frac{3}{2}}, 0\right)$ and CD on $\left(-\infty, -\sqrt{\frac{3}{2}}\right)$ and $\left(0, \sqrt{\frac{3}{2}}\right)$. IP are $(0, 0)$ and $\left(\pm\sqrt{\frac{3}{2}}, \pm\sqrt{\frac{3}{2}}\,e^{-3/2}\right)$.

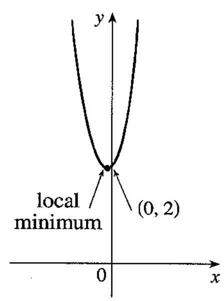
H.



51. $y = f(x) = e^{3x} + e^{-2x}$ **A.** $D = \mathbb{R}$ **B.** y-intercept $= f(0) = 2$; no x-intercept **C.** No symmetry **D.** No asymptotes

E. $f'(x) = 3e^{3x} - 2e^{-2x}$, so $f'(x) > 0 \;\Leftrightarrow\; 3e^{3x} > 2e^{-2x}$ [multiply by e^{2x}] $\;\Leftrightarrow\; e^{5x} > \frac{2}{3} \;\Leftrightarrow\; 5x > \ln\frac{2}{3} \;\Leftrightarrow$ $x > \frac{1}{5}\ln\frac{2}{3} \approx -0.081$. Similarly, $f'(x) < 0 \;\Leftrightarrow\; x < \frac{1}{5}\ln\frac{2}{3}$. f is decreasing on $\left(-\infty, \frac{1}{5}\ln\frac{2}{3}\right)$ and increasing on $\left(\frac{1}{5}\ln\frac{2}{3}, \infty\right)$.

H.

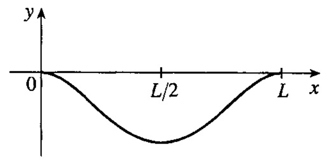


F. Local minimum value $f\left(\frac{1}{5}\ln\frac{2}{3}\right) = \left(\frac{2}{3}\right)^{3/5} + \left(\frac{2}{3}\right)^{-2/5} \approx 1.96$; no local maximum.

G. $f''(x) = 9e^{3x} + 4e^{-2x}$, so $f''(x) > 0$ for all x, and f is CU on $(-\infty, \infty)$. No IP

53. $y = -\dfrac{W}{24EI}x^4 + \dfrac{WL}{12EI}x^3 - \dfrac{WL^2}{24EI}x^2 = -\dfrac{W}{24EI}x^2(x^2 - 2Lx + L^2)$

$= \dfrac{-W}{24EI}x^2(x-L)^2 = cx^2(x-L)^2$



where $c = -\dfrac{W}{24EI}$ is a negative constant and $0 \le x \le L$. We sketch

$f(x) = cx^2(x-L)^2$ for $c = -1$. $f(0) = f(L) = 0$.

$f'(x) = cx^2[2(x-L)] + (x-L)^2(2cx) = 2cx(x-L)\,[x + (x-L)] = 2cx(x-L)(2x-L)$. So for $0 < x < L$, $f'(x) > 0 \Leftrightarrow x(x-L)(2x-L) < 0$ (since $c < 0$) $\Leftrightarrow L/2 < x < L$ and $f'(x) < 0 \Leftrightarrow 0 < x < L/2$. So f is increasing on $(L/2, L)$ and decreasing on $(0, L/2)$, and there is a local and absolute minimum at $(L/2, f(L/2)) = \left(L/2, cL^4/16\right)$.

$f'(x) = 2c\,[x(x-L)(2x-L)] \Rightarrow$

$f''(x) = 2c\,[1(x-L)(2x-L) + x(1)(2x-L) + x(x-L)(2)] = 2c\left(6x^2 - 6Lx + L^2\right) = 0 \Leftrightarrow$

$x = \dfrac{6L \pm \sqrt{12L^2}}{12} = \frac{1}{2}L \pm \frac{\sqrt{3}}{6}L$, and these are the x-coordinates of the two inflection points.

55. $y = \dfrac{x^2+1}{x+1}$. Long division gives us:

$$\begin{array}{r|l} & \quad x - 1 \\ \hline x+1 & x^2 \qquad + 1 \\ & x^2 + x \\ \hline & \quad -x + 1 \\ & \quad -x - 1 \\ \hline & \qquad\quad 2 \end{array}$$

Thus, $y = f(x) = \dfrac{x^2+1}{x+1} = x - 1 + \dfrac{2}{x+1}$ and $f(x) - (x-1) = \dfrac{2}{x+1} = \dfrac{\frac{2}{x}}{1 + \frac{1}{x}}$ [for $x \ne 0$] $\to 0$ as $x \to \pm\infty$. So the line $y = x - 1$ is a slant asymptote (SA).

57. $y = \dfrac{4x^3 - 2x^2 + 5}{2x^2 + x - 3}$. Long division gives us:

$$\begin{array}{r|l} & \qquad\qquad 2x - 2 \\ \hline 2x^2 + x - 3 & 4x^3 - 2x^2 \qquad + 5 \\ & 4x^3 + 2x^2 - 6x \\ \hline & \quad -4x^2 + 6x + 5 \\ & \quad -4x^2 - 2x + 6 \\ \hline & \qquad\qquad 8x - 1 \end{array}$$

Thus, $y = f(x) = \dfrac{4x^3 - 2x^2 + 5}{2x^2 + x - 3} = 2x - 2 + \dfrac{8x - 1}{2x^2 + x - 3}$ and

$f(x) - (2x - 2) = \dfrac{8x-1}{2x^2 + x - 3} = \dfrac{\frac{8}{x} - \frac{1}{x^2}}{2 + \frac{1}{x} - \frac{3}{x^2}}$ [for $x \ne 0$] $\to 0$ as $x \to \pm\infty$. So the line $y = 2x - 2$ is a SA.

59. $y = f(x) = \dfrac{-2x^2 + 5x - 1}{2x - 1} = -x + 2 + \dfrac{1}{2x - 1}$ **A.** $D = \{x \in \mathbb{R} \mid x \neq \frac{1}{2}\} = \left(-\infty, \frac{1}{2}\right) \cup \left(\frac{1}{2}, \infty\right)$

B. y-intercept: $f(0) = 1$; x-intercepts: $f(x) = 0 \;\Rightarrow\; -2x^2 + 5x - 1 = 0 \;\Rightarrow\; x = \dfrac{-5 \pm \sqrt{17}}{-4} \;\Rightarrow$

$x \approx 0.22, 2.28$. **C.** No symmetry

D. $\displaystyle\lim_{x \to (1/2)^-} f(x) = -\infty$ and $\displaystyle\lim_{x \to (1/2)^+} f(x) = \infty$, so $x = \frac{1}{2}$ is a VA.

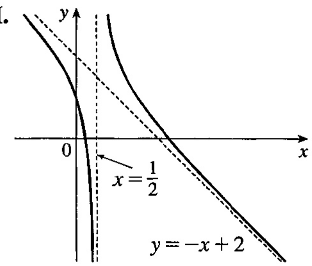
$\displaystyle\lim_{x \to \pm\infty} [f(x) - (-x + 2)] = \lim_{x \to \pm\infty} \frac{1}{2x - 1} = 0$, so the line $y = -x + 2$ is a SA.

E. $f'(x) = -1 - \dfrac{2}{(2x - 1)^2} < 0$ for $x \neq \frac{1}{2}$, so f is decreasing on $\left(-\infty, \frac{1}{2}\right)$ and $\left(\frac{1}{2}, \infty\right)$. **F.** No extreme values

G. $f'(x) = -1 - 2(2x - 1)^{-2} \;\Rightarrow$

$f''(x) = -2(-2)(2x - 1)^{-3}(2) = \dfrac{8}{(2x - 1)^3}$, so $f''(x) > 0$ when $x > \frac{1}{2}$ and $f''(x) < 0$ when $x < \frac{1}{2}$. Thus, f is CU on $\left(\frac{1}{2}, \infty\right)$ and CD on $\left(-\infty, \frac{1}{2}\right)$. No IP

H.



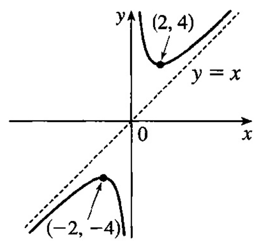
61. $y = f(x) = (x^2 + 4)/x = x + 4/x$ **A.** $D = \{x \mid x \neq 0\} = (-\infty, 0) \cup (0, \infty)$ **B.** No intercept

C. $f(-x) = -f(x) \;\Rightarrow\;$ symmetry about the origin **D.** $\displaystyle\lim_{x \to \infty} (x + 4/x) = \infty$ but $f(x) - x = 4/x \to 0$ as $x \to \pm\infty$, so $y = x$ is a slant asymptote. $\displaystyle\lim_{x \to 0^+} (x + 4/x) = \infty$ and $\displaystyle\lim_{x \to 0^-} (x + 4/x) = -\infty$, so $x = 0$ is a VA. **E.** $f'(x) = 1 - 4/x^2 > 0$ $\Leftrightarrow\; x^2 > 4 \;\Leftrightarrow\; x > 2$ or $x < -2$, so f is increasing on $(-\infty, -2)$ and $(2, \infty)$ and decreasing on $(-2, 0)$ and $(0, 2)$.

F. Local maximum value $f(-2) = -4$, local minimum value $f(2) = 4$

G. $f''(x) = 8/x^3 > 0 \;\Leftrightarrow\; x > 0$ so f is CU on $(0, \infty)$ and CD on $(-\infty, 0)$. No IP

H.



63. $y = f(x) = \dfrac{2x^3 + x^2 + 1}{x^2 + 1} = 2x + 1 + \dfrac{-2x}{x^2 + 1}$ **A.** $D = \mathbb{R}$ **B.** y-intercept: $f(0) = 1$; x-intercept: $f(x) = 0$ $\Rightarrow\; 0 = 2x^3 + x^2 + 1 = (x + 1)(2x^2 - x + 1) \;\Rightarrow\; x = -1$ **C.** No symmetry **D.** No VA

$\displaystyle\lim_{x \to \pm\infty} [f(x) - (2x + 1)] = \lim_{x \to \pm\infty} \frac{-2x}{x^2 + 1} = \lim_{x \to \pm\infty} \frac{-2/x}{1 + 1/x^2} = 0$, so the line $y = 2x + 1$ is a slant asymptote.

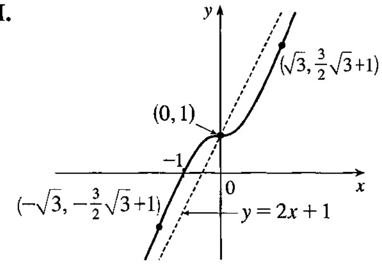
E. $f'(x) = 2 + \dfrac{(x^2 + 1)(-2) - (-2x)(2x)}{(x^2 + 1)^2} = \dfrac{2(x^4 + 2x^2 + 1) - 2x^2 - 2 + 4x^2}{(x^2 + 1)^2}$

$$= \frac{2x^4 + 6x^2}{(x^2 + 1)^2} = \frac{2x^2(x^2 + 3)}{(x^2 + 1)^2}$$

so $f'(x) > 0$ if $x \neq 0$. Thus, f is increasing on $(-\infty, 0)$ and $(0, \infty)$. Since f is continuous at 0, f is increasing on $\mathbb{R}$. **F.** No extreme values

G. $$f''(x) = \frac{(x^2+1)^2 \cdot (8x^3+12x) - (2x^4+6x^2) \cdot 2(x^2+1)(2x)}{[(x^2+1)^2]^2}$$
$$= \frac{4x(x^2+1)[(x^2+1)(2x^2+3) - 2x^4 - 6x^2]}{(x^2+1)^4}$$
$$= \frac{4x(-x^2+3)}{(x^2+1)^3}$$

H.



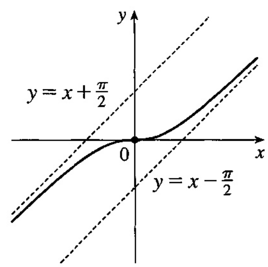
so $f''(x) > 0$ for $x < -\sqrt{3}$ and $0 < x < \sqrt{3}$, and $f''(x) < 0$ for $-\sqrt{3} < x < 0$ and $x > \sqrt{3}$. f is CU on $(-\infty, -\sqrt{3})$ and $(0, \sqrt{3})$, and CD on $(-\sqrt{3}, 0)$ and $(\sqrt{3}, \infty)$. There are three IPs: $(0, 1)$, $\left(-\sqrt{3}, -\frac{3}{2}\sqrt{3}+1\right) \approx (-1.73, -1.60)$, and $\left(\sqrt{3}, \frac{3}{2}\sqrt{3}+1\right) \approx (1.73, 3.60)$.

65. $y = f(x) = x - \tan^{-1} x$, $f'(x) = 1 - \dfrac{1}{1+x^2} = \dfrac{1+x^2-1}{1+x^2} = \dfrac{x^2}{1+x^2}$,

$$f''(x) = \frac{(1+x^2)(2x) - x^2(2x)}{(1+x^2)^2} = \frac{2x(1+x^2-x^2)}{(1+x^2)^2} = \frac{2x}{(1+x^2)^2}.$$

$\lim\limits_{x\to\infty} \left[f(x) - \left(x - \frac{\pi}{2}\right)\right] = \lim\limits_{x\to\infty} \left(\frac{\pi}{2} - \tan^{-1} x\right) = \frac{\pi}{2} - \frac{\pi}{2} = 0$, so $y = x - \frac{\pi}{2}$ is a SA. Also,

$$\lim_{x\to-\infty} \left[f(x) - \left(x + \tfrac{\pi}{2}\right)\right] = \lim_{x\to-\infty} \left(-\tfrac{\pi}{2} - \tan^{-1} x\right)$$
$$= -\tfrac{\pi}{2} - \left(-\tfrac{\pi}{2}\right) = 0$$



so $y = x + \frac{\pi}{2}$ is also a SA. $f'(x) \geq 0$ for all x, with equality $\Leftrightarrow$ $x = 0$, so f is increasing on $\mathbb{R}$. $f''(x)$ has the same sign as x, so f is CD on $(-\infty, 0)$ and CU on $(0, \infty)$. $f(-x) = -f(x)$, so f is an odd function; its graph is symmetric about the origin. f has no local extreme values. Its only IP is at $(0, 0)$.

67. $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1 \quad \Rightarrow \quad y = \pm\dfrac{b}{a}\sqrt{x^2 - a^2}$. Now

$$\lim_{x\to\infty} \left[\frac{b}{a}\sqrt{x^2-a^2} - \frac{b}{a}x\right] = \frac{b}{a} \cdot \lim_{x\to\infty} \left(\sqrt{x^2-a^2} - x\right) \frac{\sqrt{x^2-a^2}+x}{\sqrt{x^2-a^2}+x} = \frac{b}{a} \cdot \lim_{x\to\infty} \frac{-a^2}{\sqrt{x^2-a^2}+x} = 0,$$

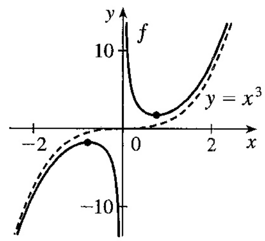
which shows that $y = \dfrac{b}{a}x$ is a slant asymptote. Similarly,

$$\lim_{x\to\infty} \left[-\frac{b}{a}\sqrt{x^2-a^2} - \left(-\frac{b}{a}x\right)\right] = -\frac{b}{a} \cdot \lim_{x\to\infty} \frac{-a^2}{\sqrt{x^2-a^2}+x} = 0, \text{ so } y = -\frac{b}{a}x \text{ is a slant asymptote.}$$

69. $\lim\limits_{x\to\pm\infty}\left[f(x)-x^3\right]=\lim\limits_{x\to\pm\infty}\dfrac{x^4+1}{x}-\dfrac{x^4}{x}=\lim\limits_{x\to\pm\infty}\dfrac{1}{x}=0$, so the graph of f is asymptotic to that of $y=x^3$.

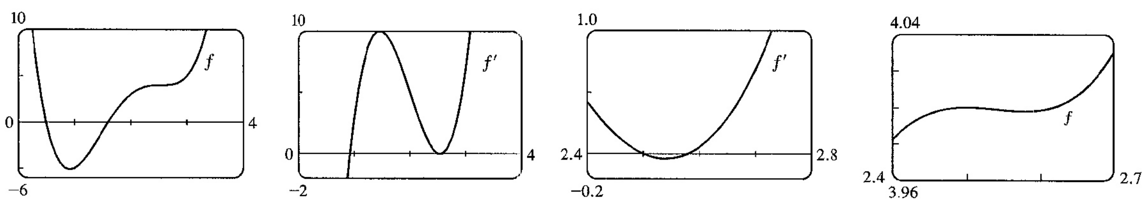
A. $D=\{x\mid x\neq 0\}$ **B.** No intercept **C.** f is symmetric about the origin. **D.** $\lim\limits_{x\to 0^-}\left(x^3+\dfrac{1}{x}\right)=-\infty$ and $\lim\limits_{x\to 0^+}\left(x^3+\dfrac{1}{x}\right)=\infty$, so $x=0$ is a vertical asymptote, and as shown above, the graph of f is asymptotic to that of $y=x^3$. **E.** $f'(x)=3x^2-1/x^2>0 \iff x^4>\frac{1}{3} \iff |x|>\frac{1}{\sqrt[4]{3}}$, so f is increasing on $\left(-\infty,-\frac{1}{\sqrt[4]{3}}\right)$ and $\left(\frac{1}{\sqrt[4]{3}},\infty\right)$ and decreasing on $\left(-\frac{1}{\sqrt[4]{3}},0\right)$ and $\left(0,\frac{1}{\sqrt[4]{3}}\right)$. **F.** Local maximum value $f\left(-\frac{1}{\sqrt[4]{3}}\right)=-4\cdot 3^{-5/4}$, local minimum value $f\left(\frac{1}{\sqrt[4]{3}}\right)=4\cdot 3^{-5/4}$
G. $f''(x)=6x+2/x^3>0 \iff x>0$, so f is CU on $(0,\infty)$ and CD on $(-\infty,0)$. No IP

H.



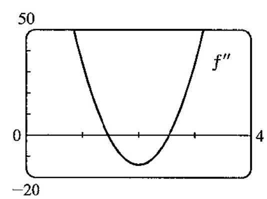
4.6 Graphing with Calculus and Calculators

1. $f(x)=4x^4-32x^3+89x^2-95x+29 \Rightarrow f'(x)=16x^3-96x^2+178x-95 \Rightarrow$
$f''(x)=48x^2-192x+178$. $f(x)=0 \iff x\approx 0.5,\ 1.60$; $f'(x)=0 \iff x\approx 0.92,\ 2.5,\ 2.58$ and
$f''(x)=0 \iff x\approx 1.46,\ 2.54$.

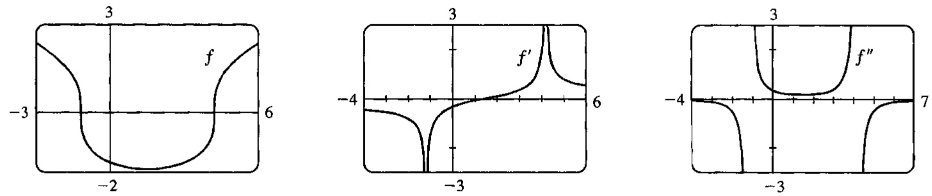


From the graphs of f', we estimate that $f'<0$ and that f is decreasing on $(-\infty,0.92)$ and $(2.5,2.58)$, and that $f'>0$ and f is increasing on $(0.92,2.5)$ and $(2.58,\infty)$ with local minimum values $f(0.92)\approx -5.12$ and $f(2.58)\approx 3.998$ and local maximum value $f(2.5)=4$. The graphs of f' make it clear that f has a maximum and a minimum near $x=2.5$, shown more clearly in the fourth graph.

From the graph of f'', we estimate that $f''>0$ and that f is CU on $(-\infty,1.46)$ and $(2.54,\infty)$, and that $f''<0$ and f is CD on $(1.46,2.54)$. There are inflection points at about $(1.46,-1.40)$ and $(2.54,3.999)$.



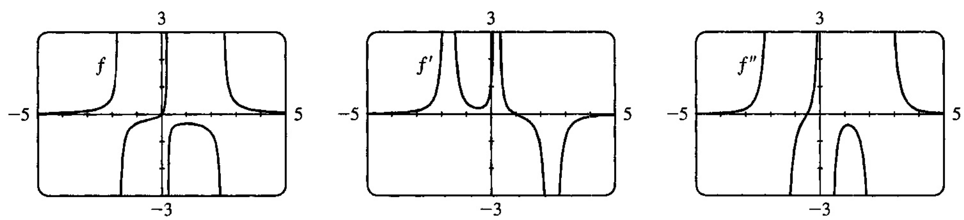
3. $f(x) = \sqrt[3]{x^2 - 3x - 5} \Rightarrow f'(x) = \dfrac{1}{3}\dfrac{2x-3}{(x^2-3x-5)^{2/3}} \Rightarrow f''(x) = -\dfrac{2}{9}\dfrac{x^2-3x+24}{(x^2-3x-5)^{5/3}}$



Note: With some CAS's, including Maple, it is necessary to define $f(x) = \dfrac{x^2 - 3x - 5}{|x^2 - 3x - 5|}\,|x^2 - 3x - 5|^{1/3}$, since the CAS does not compute real cube roots of negative numbers. We estimate from the graph of f' that f is increasing on $(1.5, \infty)$, and decreasing on $(-\infty, 1.5)$. f has no maximum. Minimum value: $f(1.5) \approx -1.9$. From the graph of f'', we estimate that f is CU on $(-1.2, 4.2)$ and CD on $(-\infty, -1.2)$ and $(4.2, \infty)$. IP at $(-1.2, 0)$ and $(4.2, 0)$.

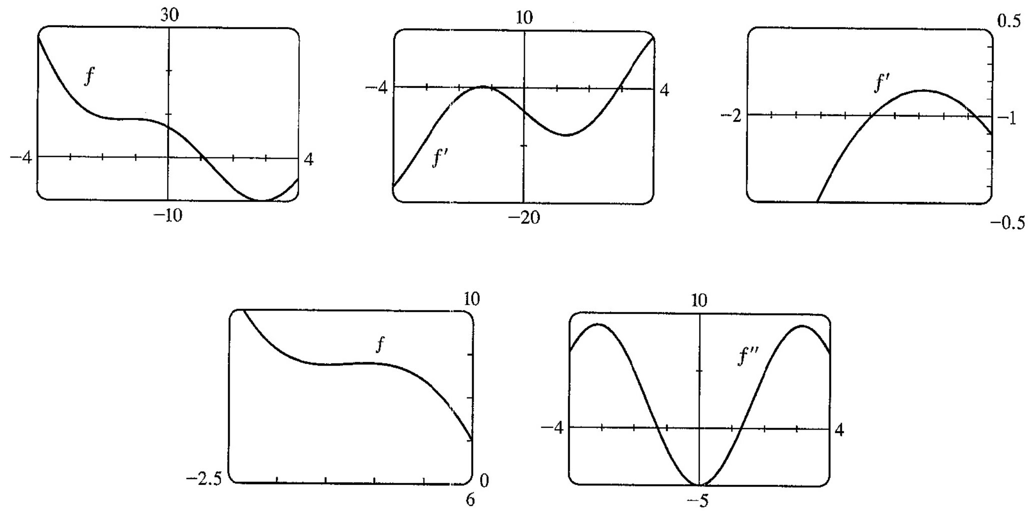
5. $f(x) = \dfrac{x}{x^3 - x^2 - 4x + 1} \Rightarrow f'(x) = \dfrac{-2x^3 + x^2 + 1}{(x^3 - x^2 - 4x + 1)^2} \Rightarrow$

$$f''(x) = \frac{2(3x^5 - 3x^4 + 5x^3 - 6x^2 + 3x + 4)}{(x^3 - x^2 - 4x + 1)^3}$$



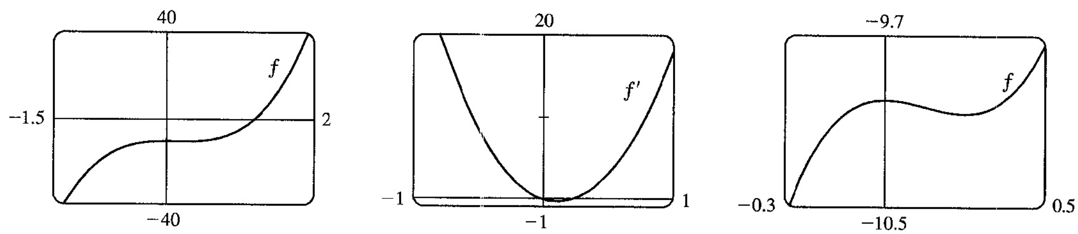
We estimate from the graph of f that $y = 0$ is a horizontal asymptote, and that there are vertical asymptotes at $x = -1.7$, $x = 0.24$, and $x = 2.46$. From the graph of f', we estimate that f is increasing on $(-\infty, -1.7)$, $(-1.7, 0.24)$, and $(0.24, 1)$, and that f is decreasing on $(1, 2.46)$ and $(2.46, \infty)$. There is a local maximum value at $f(1) = -\frac{1}{3}$. From the graph of f'', we estimate that f is CU on $(-\infty, -1.7)$, $(-0.506, 0.24)$, and $(2.46, \infty)$, and that f is CD on $(-1.7, -0.506)$ and $(0.24, 2.46)$. There is an inflection point at $(-0.506, -0.192)$.

7. $f(x) = x^2 - 4x + 7\cos x$, $-4 \le x \le 4$. $f'(x) = 2x - 4 - 7\sin x \;\Rightarrow\; f''(x) = 2 - 7\cos x$.
$f(x) = 0 \;\Leftrightarrow\; x \approx 1.10$; $f'(x) = 0 \;\Leftrightarrow\; x \approx -1.49, -1.07$, or 2.89; $f''(x) = 0 \;\Leftrightarrow$
$x = \pm\cos^{-1}\left(\frac{2}{7}\right) \approx \pm 1.28$.

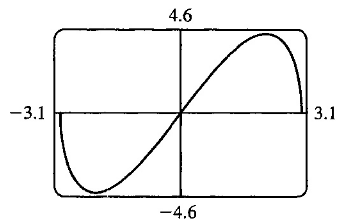


From the graphs of f', we estimate that f is decreasing ($f' < 0$) on $(-4, -1.49)$, increasing on $(-1.49, -1.07)$, decreasing on $(-1.07, 2.89)$, and increasing on $(2.89, 4)$, with local minimum values $f(-1.49) \approx 8.75$ and $f(2.89) \approx -9.99$ and local maximum value $f(-1.07) \approx 8.79$ (notice the second graph of f). From the graph of f'', we estimate that f is CU ($f'' > 0$) on $(-4, -1.28)$, CD on $(-1.28, 1.28)$, and CU on $(1.28, 4)$. There are inflection points at about $(-1.28, 8.77)$ and $(1.28, -1.48)$.

9. $f(x) = 8x^3 - 3x^2 - 10 \;\Rightarrow\; f'(x) = 24x^2 - 6x \;\Rightarrow\; f''(x) = 48x - 6$



From the graphs, it appears that $f(x) = 8x^3 - 3x^2 - 10$ increases on $(-\infty, 0)$ and $(0.25, \infty)$ and decreases on $(0, 0.25)$; that f has a local maximum value of $f(0) = -10.0$ and a local minimum value of $f(0.25) \approx -10.1$; that f is CU on $(0.1, \infty)$ and CD on $(-\infty, 0.1)$; and that f has an IP at $(0.1, -10)$. To find the exact values, note that $f'(x) = 24x^2 - 6x = 6x(4x - 1)$, which is positive ($f$ is increasing) for $(-\infty, 0)$ and $\left(\frac{1}{4}, \infty\right)$, and negative ($f$ is decreasing) on $\left(0, \frac{1}{4}\right)$. By the FDT, f has a local maximum at $x = 0$: $f(0) = -10$; and f has a local minimum at $\frac{1}{4}$: $f\left(\frac{1}{4}\right) = \frac{1}{8} - \frac{3}{16} - 10 = -\frac{161}{16}$. $f''(x) = 48x - 6 = 6(8x - 1)$, which is positive ($f$ is CU) on $\left(\frac{1}{8}, \infty\right)$ and negative (f is CD) on $\left(-\infty, \frac{1}{8}\right)$. f has an IP at $\left(\frac{1}{8}, f\left(\frac{1}{8}\right)\right) = \left(\frac{1}{8}, -\frac{321}{32}\right)$.



11. From the graph, it appears that f increases on $(-2.1, 2.1)$ and decreases on $(-3, -2.1)$ and $(2.1, 3)$; that f has a local maximum of $f(2.1) \approx 4.5$ and a local minimum of $f(-2.1) \approx -4.5$; that f is CU on $(-3.0, 0)$ and CD on $(0, 3.0)$, and that f has an IP at $(0, 0)$. $f(x) = x\sqrt{9-x^2} \Rightarrow$

$f'(x) = \dfrac{-x^2}{\sqrt{9-x^2}} + \sqrt{9-x^2} = \dfrac{9-2x^2}{\sqrt{9-x^2}}$, which is positive

(f is increasing) on $\left(\frac{-3\sqrt{2}}{2}, \frac{3\sqrt{2}}{2}\right)$ and negative (f is decreasing) on $\left(-3, \frac{-3\sqrt{2}}{2}\right)$ and $\left(\frac{3\sqrt{2}}{2}, 3\right)$. By the FDT,

f has a local maximum value of $f\left(\frac{3\sqrt{2}}{2}\right) = \frac{3\sqrt{2}}{2}\sqrt{9 - \left(\frac{3\sqrt{2}}{2}\right)^2} = \frac{9}{2}$; and f has a local minimum value of

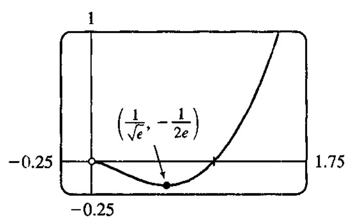
$f\left(\frac{-3\sqrt{2}}{2}\right) = -\frac{9}{2}$ (since f is an odd function). $f'(x) = \dfrac{-x^2}{\sqrt{9-x^2}} + \sqrt{9-x^2} \Rightarrow$

$$f''(x) = \frac{\sqrt{9-x^2}\,(-2x) + x^2\left(\frac{1}{2}\right)\left(9-x^2\right)^{-1/2}(-2x)}{9-x^2} - x\left(9-x^2\right)^{-1/2} = \frac{-2x - x^3\left(9-x^2\right)^{-1} - x}{\sqrt{9-x^2}}$$

$$= \frac{-3x}{\sqrt{9-x^2}} - \frac{x^3}{(9-x^2)^{3/2}} = \frac{x\left(2x^2-27\right)}{(9-x^2)^{3/2}}$$

which is positive (f is CU) on $(-3, 0)$ and negative (f is CD) on $(0, 3)$. f has an IP at $(0, 0)$.

13. (a) $f(x) = x^2 \ln x$. The domain of f is $(0, \infty)$.



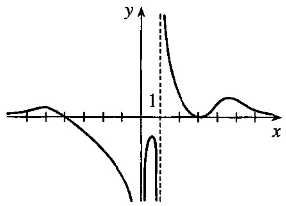
(b) $\displaystyle\lim_{x\to 0^+} x^2 \ln x = \lim_{x\to 0^+} \frac{\ln x}{1/x^2} \overset{H}{=} \lim_{x\to 0^+} \frac{1/x}{-2/x^3} = \lim_{x\to 0^+}\left(-\frac{x^2}{2}\right) = 0$. There is a hole at $(0, 0)$.

(c) It appears that there is an IP at about $(0.2, -0.06)$ and a local minimum at $(0.6, -0.18)$. $f(x) = x^2 \ln x \Rightarrow$

$f'(x) = x^2(1/x) + (\ln x)(2x) = x(2\ln x + 1) > 0 \Leftrightarrow \ln x > -\frac{1}{2} \Leftrightarrow x > e^{-1/2}$, so f is increasing on $(1/\sqrt{e}, \infty)$, decreasing on $(0, 1/\sqrt{e})$. By the FDT, $f(1/\sqrt{e}) = -1/(2e)$ is a local minimum value. This point is approximately $(0.6065, -0.1839)$, which agrees with our estimate.

$f''(x) = x(2/x) + (2\ln x + 1) = 2\ln x + 3 > 0 \Leftrightarrow \ln x > -\frac{3}{2} \Leftrightarrow x > e^{-3/2}$, so f is CU on $\left(e^{-3/2}, \infty\right)$ and CD on $\left(0, e^{-3/2}\right)$. IP is $\left(e^{-3/2}, -3/(2e^3)\right) \approx (0.2231, -0.0747)$.

15.

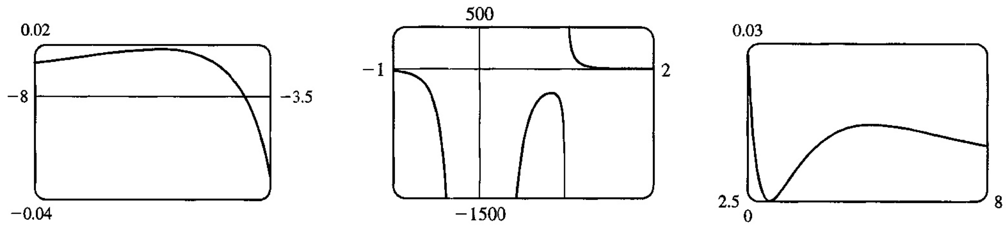


$f(x) = \dfrac{(x+4)(x-3)^2}{x^4(x-1)}$ has VA at $x = 0$ and at $x = 1$ since $\lim\limits_{x \to 0} f(x) = -\infty$, $\lim\limits_{x \to 1^-} f(x) = -\infty$ and $\lim\limits_{x \to 1^+} f(x) = \infty$.

$$f(x) = \frac{\dfrac{x+4}{x} \cdot \dfrac{(x-3)^2}{x^2}}{\dfrac{x^4}{x^3} \cdot (x-1)} \qquad \text{[dividing numerator and denominator by } x^3\text{]}$$

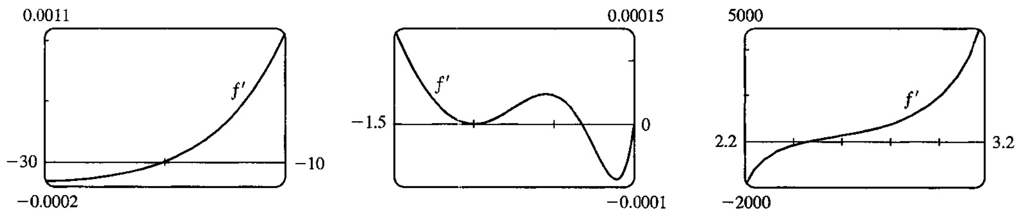
$$= \frac{(1+4/x)(1-3/x)^2}{x(x-1)} \to 0 \text{ as } x \to \pm\infty, \text{ so } f \text{ is asymptotic}$$

to the x-axis. Since f is undefined at $x = 0$, it has no y-intercept. $f(x) = 0 \Rightarrow (x+4)(x-3)^2 = 0 \Rightarrow x = -4$ or $x = 3$, so f has x-intercepts -4 and 3. Note, however, that the graph of f is only tangent to the x-axis and does not cross it at $x = 3$, since f is positive as $x \to 3^-$ and as $x \to 3^+$.



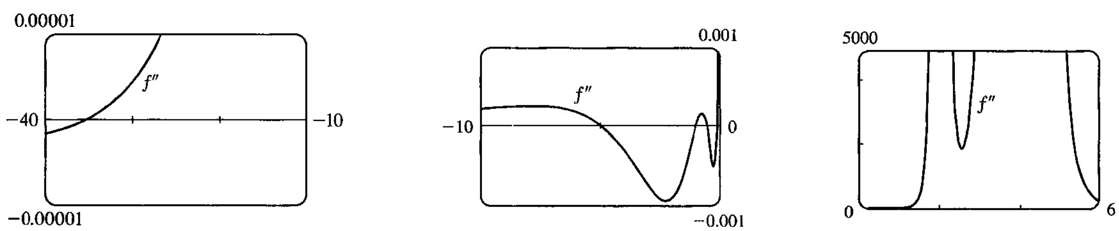
From these graphs, it appears that f has three maximum values and one minimum value. The maximum values are approximately $f(-5.6) = 0.0182$, $f(0.82) = -281.5$ and $f(5.2) = 0.0145$ and we know (since the graph is tangent to the x-axis at $x = 3$) that the minimum value is $f(3) = 0$.

17. $f(x) = \dfrac{x^2(x+1)^3}{(x-2)^2(x-4)^4} \quad \Rightarrow \quad f'(x) = -\dfrac{x(x+1)^2(x^3+18x^2-44x-16)}{(x-2)^3(x-4)^5}$ (from CAS).



From the graphs of f', it seems that the critical points which indicate extrema occur at $x \approx -20$, -0.3, and 2.5, as estimated in Example 3. (There is another critical point at $x = -1$, but the sign of f' does not change there.)

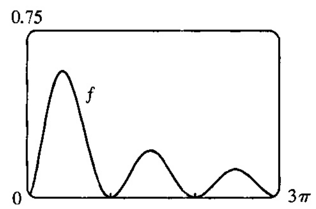
We differentiate again, obtaining $f''(x) = 2\dfrac{(x+1)(x^6+36x^5+6x^4-628x^3+684x^2+672x+64)}{(x-2)^4(x-4)^6}$.

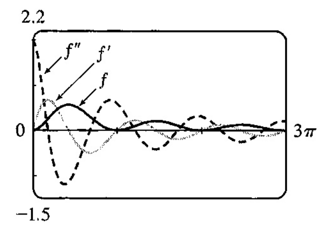


From the graphs of f'', it appears that f is CU on $(-35.3, -5.0)$, $(-1, -0.5)$, $(-0.1, 2)$, $(2, 4)$ and $(4, \infty)$ and CD on $(-\infty, -35.3)$, $(-5.0, -1)$ and $(-0.5, -0.1)$. We check back on the graphs of f to find the y-coordinates of the inflection points, and find that these points are approximately $(-35.3, -0.015)$, $(-5.0, -0.005)$, $(-1, 0)$, $(-0.5, 0.00001)$, and $(-0.1, 0.0000066)$.

19. $y = f(x) = \dfrac{\sin^2 x}{\sqrt{x^2+1}}$ with $0 \le x \le 3\pi$. From a CAS, $y' = \dfrac{\sin x\,[2(x^2+1)\cos x - x\sin x]}{(x^2+1)^{3/2}}$ and

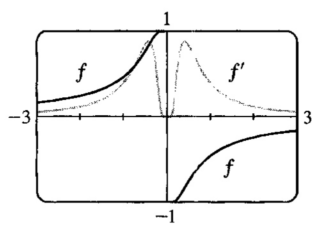
$$y'' = \frac{(4x^4 + 6x^2 + 5)\cos^2 x - 4x(x^2+1)\sin x \cos x - 2x^4 - 2x^2 - 3}{(x^2+1)^{5/2}}.$$

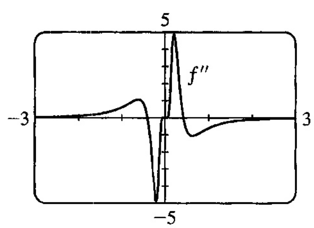




From the graph of f' and the formula for y', we determine that $y' = 0$ when $x = \pi, 2\pi, 3\pi$, or $x \approx 1.3, 4.6$, or 7.8. So f is increasing on $(0, 1.3)$, $(\pi, 4.6)$, and $(2\pi, 7.8)$. f is decreasing on $(1.3, \pi)$, $(4.6, 2\pi)$, and $(7.8, 3\pi)$. Local maximum values: $f(1.3) \approx 0.6$, $f(4.6) \approx 0.21$, and $f(7.8) \approx 0.13$. Local minimum values: $f(\pi) = f(2\pi) = 0$. From the graph of f'', we see that $y'' = 0 \quad \Leftrightarrow \quad x \approx 0.6, 2.1, 3.8, 5.4, 7.0$, or 8.6. So f is CU on $(0, 0.6)$, $(2.1, 3.8)$, $(5.4, 7.0)$, and $(8.6, 3\pi)$. f is CD on $(0.6, 2.1)$, $(3.8, 5.4)$, and $(7.0, 8.6)$. There are IP at $(0.6, 0.25)$, $(2.1, 0.31)$, $(3.8, 0.10)$, $(5.4, 0.11)$, $(7.0, 0.061)$, and $(8.6, 0.065)$.

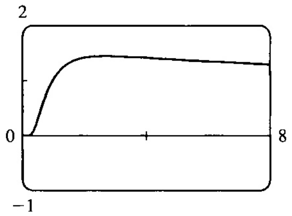
21. $y = f(x) = \dfrac{1 - e^{1/x}}{1 + e^{1/x}}$. From a CAS, $y' = \dfrac{2e^{1/x}}{x^2(1+e^{1/x})^2}$ and $y'' = \dfrac{-2e^{1/x}(1 - e^{1/x} + 2x + 2xe^{1/x})}{x^4(1+e^{1/x})^3}$.





f is an odd function defined on $(-\infty, 0) \cup (0, \infty)$. Its graph has no x- or y-intercepts. Since $\lim\limits_{x \to \pm\infty} f(x) = 0$, the x-axis is a HA. $f'(x) > 0$ for $x \ne 0$, so f is increasing on $(-\infty, 0)$ and $(0, \infty)$. It has no local extreme values. $f''(x) = 0$ for $x \approx \pm 0.417$, so f is CU on $(-\infty, -0.417)$, CD on $(-0.417, 0)$, CU on $(0, 0.417)$, and CD on $(0.417, \infty)$. f has IPs at $(-0.417, 0.834)$ and $(0.417, -0.834)$.

23. (a) $f(x) = x^{1/x}$

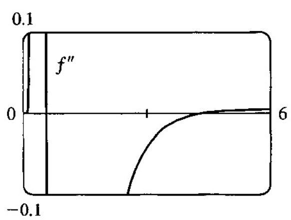


(b) Recall that $a^b = e^{b \ln a}$. $\lim\limits_{x \to 0^+} x^{1/x} = \lim\limits_{x \to 0^+} e^{(1/x) \ln x}$. As $x \to 0^+$, $\dfrac{\ln x}{x} \to -\infty$, so $x^{1/x} = e^{(1/x) \ln x} \to 0$. This indicates that there is a hole at $(0, 0)$. As $x \to \infty$, we have the indeterminate form ∞^0. $\lim\limits_{x \to \infty} x^{1/x} = \lim\limits_{x \to \infty} e^{(1/x) \ln x}$, but $\lim\limits_{x \to \infty} \dfrac{\ln x}{x} \overset{\text{H}}{=} \lim\limits_{x \to \infty} \dfrac{1/x}{1} = 0$, so $\lim\limits_{x \to \infty} x^{1/x} = e^0 = 1$. This indicates that $y = 1$ is a HA.

(c) Estimated maximum: $(2.72, 1.45)$. No estimated minimum. We use logarithmic differentiation to find any critical numbers. $y = x^{1/x} \quad \Rightarrow \quad \ln y = \dfrac{1}{x} \ln x \quad \Rightarrow \quad \dfrac{y'}{y} = \dfrac{1}{x} \cdot \dfrac{1}{x} + (\ln x)\left(-\dfrac{1}{x^2}\right) \quad \Rightarrow$

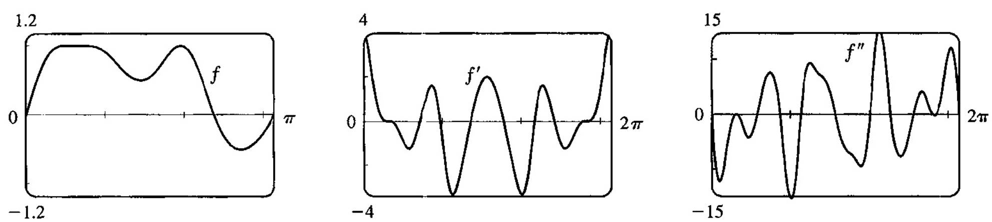
$y' = x^{1/x}\left(\dfrac{1 - \ln x}{x^2}\right) = 0 \quad \Rightarrow \quad \ln x = 1 \quad \Rightarrow \quad x = e$. For $0 < x < e$, $y' > 0$ and for $x > e$, $y' < 0$, so $f(e) = e^{1/e}$ is a local maximum value. This point is approximately $(2.7183, 1.4447)$, which agrees with our estimate.

(d)

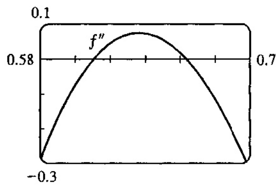


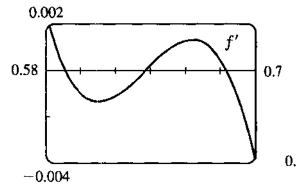
From the graph, we see that $f''(x) = 0$ at $x \approx 0.58$ and $x \approx 4.37$. Since f'' changes sign at these values, they are x-coordinates of inflection points.

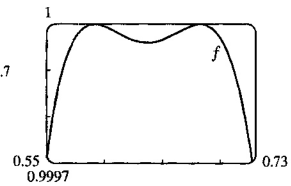
25.

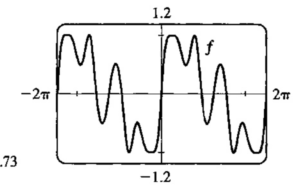


From the graph of $f(x) = \sin(x + \sin 3x)$ in the viewing rectangle $[0, \pi]$ by $[-1.2, 1.2]$, it looks like f has two maxima and two minima. If we calculate and graph $f'(x) = [\cos(x + \sin 3x)]\,(1 + 3\cos 3x)$ on $[0, 2\pi]$, we see that the graph of f' appears to be almost tangent to the x-axis at about $x = 0.7$. The graph of $f'' = -[\sin(x + \sin 3x)]\,(1 + 3\cos 3x)^2 + \cos(x + \sin 3x)(-9\sin 3x)$ is even more interesting near this x-value: it seems to just touch the x-axis.



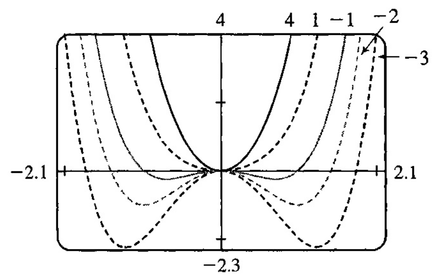




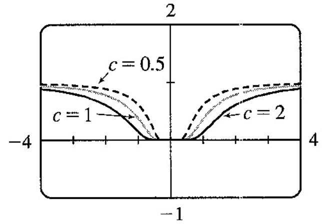


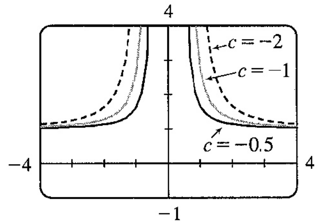
If we zoom in on this place on the graph of f'', we see that f'' actually does cross the axis twice near $x = 0.65$, indicating a change in concavity for a very short interval. If we look at the graph of f' on the same interval, we see that it changes sign three times near $x = 0.65$, indicating that what we had thought was a broad extremum at about $x = 0.7$ actually consists of three extrema (two maxima and a minimum). These maximum values are roughly $f(0.59) = 1$ and $f(0.68) = 1$, and the minimum value is roughly $f(0.64) = 0.99996$. There are also a maximum value of about $f(1.96) = 1$ and minimum values of about $f(1.46) = 0.49$ and $f(2.73) = -0.51$. The points of inflection on $(0, \pi)$ are about $(0.61, 0.99998)$, $(0.66, 0.99998)$, $(1.17, 0.72)$, $(1.75, 0.77)$, and $(2.28, 0.34)$. On $(\pi, 2\pi)$, they are about $(4.01, -0.34)$, $(4.54, -0.77)$, $(5.11, -0.72)$, $(5.62, -0.99998)$, and $(5.67, -0.99998)$. There are also IP at $(0, 0)$ and $(\pi, 0)$. Note that the function is odd and periodic with period 2π, and it is also rotationally symmetric about all points of the form $((2n + 1)\pi, 0)$, n an integer.

27. $f(x) = x^4 + cx^2 = x^2(x^2 + c)$. Note that f is an even function. For $c \geq 0$, the only x-intercept is the point $(0, 0)$. We calculate $f'(x) = 4x^3 + 2cx = 4x(x^2 + \frac{1}{2}c) \Rightarrow f''(x) = 12x^2 + 2c$. If $c \geq 0$, $x = 0$ is the only critical point and there is no inflection point. As we can see from the examples, there is no change in the basic shape of the graph for $c \geq 0$; it merely becomes steeper as c increases. For $c = 0$, the graph is the simple curve $y = x^4$. For $c < 0$, there are x-intercepts at 0 and at $\pm\sqrt{-c}$. Also, there is a maximum at $(0, 0)$, and there are minima at $\left(\pm\sqrt{-\frac{1}{2}c}, -\frac{1}{4}c^2\right)$. As $c \to -\infty$, the x-coordinates of these minima get larger in absolute value, and the minimum points move downward. There are inflection points at $\left(\pm\sqrt{-\frac{1}{6}c}, -\frac{5}{36}c^2\right)$, which also move away from the origin as $c \to -\infty$.



29.



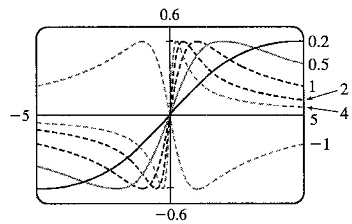


$c = 0$ is a transitional value—we get the graph of $y = 1$. For $c > 0$, we see that there is a HA at $y = 1$, and that the graph spreads out as c increases. At first glance there appears to be a minimum at $(0, 0)$, but $f(0)$ is undefined, so there is no minimum or maximum. For $c < 0$, we still have the HA at $y = 1$, but the range is $(1, \infty)$ rather than $(0, 1)$. We also have a VA at $x = 0$. $f(x) = e^{-c/x^2} \Rightarrow f'(x) = e^{-c/x^2}\left(\dfrac{2c}{x^3}\right) \Rightarrow$
$f''(x) = \dfrac{2c(2c - 3x^2)}{x^6 e^{c/x^2}}$. $f'(x) \neq 0$ and $f'(x)$ exists for all $x \neq 0$ (and 0 is not in the domain of f), so there are no maxima or minima. $f''(x) = 0 \Rightarrow x = \pm\sqrt{2c/3}$, so if $c > 0$, the inflection points spread out as c increases, and if $c < 0$, there are no IP. For $c > 0$, there are IP at $\left(\pm\sqrt{2c/3}, e^{-3/2}\right)$. Note that the y-coordinate of the IP is constant.

31. Note that $c = 0$ is a transitional value at which the graph consists of the x-axis. Also, we can see that if we substitute $-c$ for c, the function $f(x) = \dfrac{cx}{1 + c^2x^2}$ will be reflected in the x-axis, so we investigate only positive values of c (except $c = -1$, as a demonstration of this reflective property). Also, f is an odd function. $\lim\limits_{x \to \pm\infty} f(x) = 0$, so $y = 0$ is a horizontal asymptote for all c. We calculate

$f'(x) = \dfrac{(1 + c^2x^2)c - cx(2c^2x)}{(1 + c^2x^2)^2} = -\dfrac{c(c^2x^2 - 1)}{(1 + c^2x^2)^2}$. $f'(x) = 0 \Leftrightarrow c^2x^2 - 1 = 0 \Leftrightarrow x = \pm 1/c$. So there is an absolute maximum value of $f(1/c) = \frac{1}{2}$ and an absolute minimum value of $f(-1/c) = -\frac{1}{2}$. These extrema have the same value regardless of c, but the maximum points move closer to the y-axis as c increases.

$$f''(x) = \frac{(-2c^3x)(1 + c^2x^2)^2 - (-c^3x^2 + c)[2(1 + c^2x^2)(2c^2x)]}{(1 + c^2x^2)^4}$$
$$= \frac{(-2c^3x)(1 + c^2x^2) + (c^3x^2 - c)(4c^2x)}{(1 + c^2x^2)^3} = \frac{2c^3x(c^2x^2 - 3)}{(1 + c^2x^2)^3}$$



$f''(x) = 0 \Leftrightarrow x = 0$ or $\pm\sqrt{3}/c$, so there are inflection points at $(0, 0)$ and at $\left(\pm\sqrt{3}/c, \pm\sqrt{3}/4\right)$.
Again, the y-coordinate of the inflection points does not depend on c, but as c increases, both inflection points approach the y-axis.

33. $f(x) = cx + \sin x \Rightarrow f'(x) = c + \cos x \Rightarrow f''(x) = -\sin x$

$f(-x) = -f(x)$, so f is an odd function and its graph is symmetric with respect to the origin.

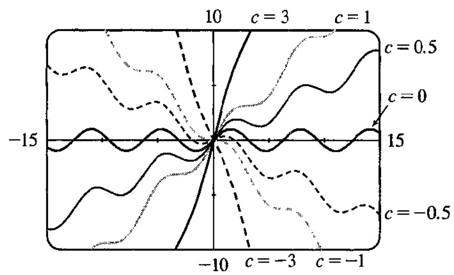
$f(x) = 0 \Leftrightarrow \sin x = -cx$, so 0 is always an x-intercept.

$f'(x) = 0 \Leftrightarrow \cos x = -c$, so there is no critical number when $|c| > 1$. If $|c| \leq 1$, then there are infinitely many critical numbers. If x_1 is the unique solution of $\cos x = -c$ in the interval $[0, \pi]$, then the critical numbers are $2n\pi \pm x_1$, where n ranges over the integers. (Special cases: When $c = 1$, $x_1 = 0$; when $c = 0$, $x = \frac{\pi}{2}$; and when $c = -1$, $x_1 = \pi$.)

$f''(x) < 0 \quad \Leftrightarrow \quad \sin x > 0$, so f is CD on intervals of the form $(2n\pi, (2n+1)\pi)$. f is CU on intervals of the form $((2n-1)\pi, 2n\pi)$. The inflection points of f are the points $(2n\pi, 2n\pi c)$, where n is an integer.

If $c \geq 1$, then $f'(x) \geq 0$ for all x, so f is increasing and has no extremum. If $c \leq -1$, then $f'(x) \leq 0$ for all x, so f is decreasing and has no extremum. If $|c| < 1$, then $f'(x) > 0 \quad \Leftrightarrow \quad \cos x > -c \quad \Leftrightarrow \quad x$ is in an interval of the form $(2n\pi - x_1, 2n\pi + x_1)$ for some integer n. These are the intervals on which f is increasing. Similarly, we find that f is decreasing on the intervals of the form $(2n\pi + x_1, 2(n+1)\pi - x_1)$. Thus, f has local maxima at the points $2n\pi + x_1$, where f has the values $c(2n\pi + x_1) + \sin x_1 = c(2n\pi + x_1) + \sqrt{1-c^2}$, and f has local minima at the points $2n\pi - x_1$, where we have $f(2n\pi - x_1) = c(2n\pi - x_1) - \sin x_1 = c(2n\pi - x_1) - \sqrt{1-c^2}$.

The transitional values of c are -1 and 1. The inflection points move vertically, but not horizontally, when c changes. When $|c| \geq 1$, there is no extremum. For $|c| < 1$, the maxima are spaced 2π apart horizontally, as are the minima. The horizontal spacing between maxima and adjacent minima is regular (and equals π) when $c = 0$, but the horizontal space between a local maximum and the nearest local minimum shrinks as $|c|$ approaches 1.

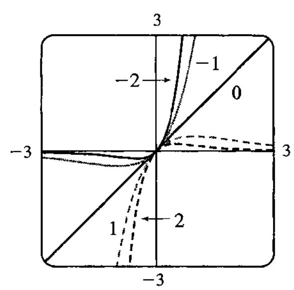


35. If $c < 0$, then $\lim\limits_{x \to -\infty} f(x) = \lim\limits_{x \to -\infty} \dfrac{x}{e^{cx}} \overset{\text{H}}{=} \lim\limits_{x \to -\infty} \dfrac{1}{ce^{cx}} = 0$, and $\lim\limits_{x \to \infty} f(x) = \infty$.

If $c > 0$, then $\lim\limits_{x \to -\infty} f(x) = -\infty$, and $\lim\limits_{x \to \infty} f(x) \overset{\text{H}}{=} \lim\limits_{x \to \infty} \dfrac{1}{ce^{cx}} = 0$.

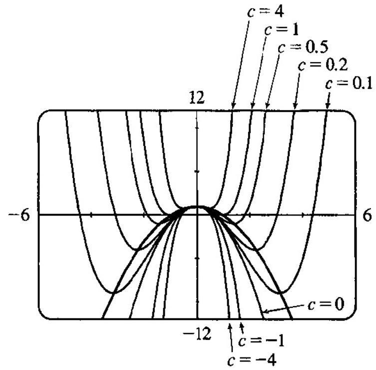
If $c = 0$, then $f(x) = x$, so $\lim\limits_{x \to \pm\infty} f(x) = \pm\infty$ respectively.

So we see that $c = 0$ is a transitional value. We now exclude the case $c = 0$, since we know how the function behaves in that case. To find the maxima and minima of f, we differentiate: $f(x) = xe^{-cx} \quad \Rightarrow$ $f'(x) = x(-ce^{-cx}) + e^{-cx} = (1 - cx)e^{-cx}$. This is 0 when $1 - cx = 0 \quad \Leftrightarrow \quad x = 1/c$. If $c < 0$ then this represents a minimum value of $f(1/c) = 1/(ce)$, since $f'(x)$ changes from negative to positive at $x = 1/c$; and if $c > 0$, it represents a maximum value. As $|c|$ increases, the maximum or minimum point gets closer to the origin. To find the inflection points, we differentiate again: $f'(x) = e^{-cx}(1 - cx) \quad \Rightarrow$ $f''(x) = e^{-cx}(-c) + (1 - cx)(-ce^{-cx}) = (cx - 2)ce^{-cx}$. This changes sign when $cx - 2 = 0 \quad \Leftrightarrow \quad x = 2/c$. So as $|c|$ increases, the points of inflection get closer to the origin.



37. (a) $f(x) = cx^4 - 2x^2 + 1$. For $c = 0$, $f(x) = -2x^2 + 1$, a parabola whose vertex, $(0, 1)$, is the absolute maximum. For $c > 0$, $f(x) = cx^4 - 2x^2 + 1$ opens upward with two minimum points. As $c \to 0$, the minimum points spread apart and move downward; they are below the x-axis for $0 < c < 1$ and above for $c > 1$. For $c < 0$, the graph opens downward, and has an absolute maximum at $x = 0$ and no local minimum.

(b) $f'(x) = 4cx^3 - 4x = 4cx(x^2 - 1/c)$ $(c \neq 0)$. If $c \leq 0$, 0 is the only critical number. $f''(x) = 12cx^2 - 4$, so $f''(0) = -4$ and there is a local maximum at $(0, f(0)) = (0, 1)$, which lies on $y = 1 - x^2$. If $c > 0$, the critical numbers are 0 and $\pm 1/\sqrt{c}$. As before, there is a local maximum at $(0, f(0)) = (0, 1)$, which lies on $y = 1 - x^2$. $f''(\pm 1/\sqrt{c}) = 12 - 4 = 8 > 0$, so there is a local minimum at $x = \pm 1/\sqrt{c}$. Here $f(\pm 1/\sqrt{c}) = c(1/c^2) - 2/c + 1 = -1/c + 1$. But $(\pm 1/\sqrt{c}, -1/c + 1)$ lies on $y = 1 - x^2$ since $1 - (\pm 1/\sqrt{c})^2 = 1 - 1/c$.



4.7 Optimization Problems

1. (a)

First Number	Second Number	Product
1	22	22
2	21	42
3	20	60
4	19	76
5	18	90
6	17	102
7	16	112
8	15	120
9	14	126
10	13	130
11	12	132

We needn't consider pairs where the first number is larger than the second, since we can just interchange the numbers in such cases. The answer appears to be 11 and 12, but we have considered only integers in the table.

(b) Call the two numbers x and y. Then $x + y = 23$, so $y = 23 - x$. Call the product P. Then $P = xy = x(23 - x) = 23x - x^2$, so we wish to maximize the function $P(x) = 23x - x^2$. Since $P'(x) = 23 - 2x$, we see that $P'(x) = 0 \iff x = \frac{23}{2} = 11.5$. Thus, the maximum value of P is $P(11.5) = (11.5)^2 = 132.25$ and it occurs when $x = y = 11.5$.

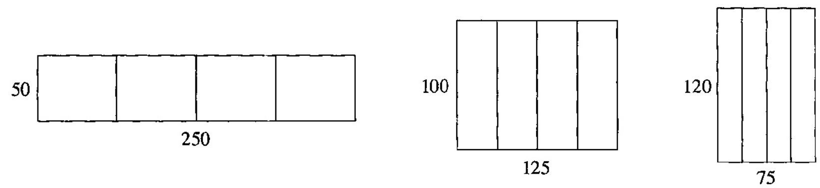
Or: Note that $P''(x) = -2 < 0$ for all x, so P is everywhere concave downward and the local maximum at $x = 11.5$ must be an absolute maximum.

3. The two numbers are x and $\frac{100}{x}$, where $x > 0$. Minimize $f(x) = x + \frac{100}{x}$. $f'(x) = 1 - \frac{100}{x^2} = \frac{x^2 - 100}{x^2}$.

The critical number is $x = 10$. Since $f'(x) < 0$ for $0 < x < 10$ and $f'(x) > 0$ for $x > 10$, there is an absolute minimum at $x = 10$. The numbers are 10 and 10.

5. If the rectangle has dimensions x and y, then its perimeter is $2x + 2y = 100$ m, so $y = 50 - x$. Thus, the area is $A = xy = x(50 - x)$. We wish to maximize the function $A(x) = x(50 - x) = 50x - x^2$, where $0 < x < 50$. Since $A'(x) = 50 - 2x = -2(x - 25)$, $A'(x) > 0$ for $0 < x < 25$ and $A'(x) < 0$ for $25 < x < 50$. Thus, A has an absolute maximum at $x = 25$, and $A(25) = 25^2 = 625\text{ m}^2$. The dimensions of the rectangle that maximize its area are $x = y = 25$ m. (The rectangle is a square.)

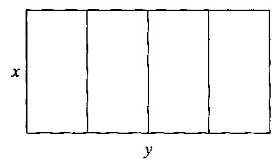
7. (a)



The areas of the three figures are 12,500, 12,500, and 9000 ft^2. There appears to be a maximum area of at least 12,500 ft^2.

(b) Let x denote the length of each of two sides and three dividers.

Let y denote the length of the other two sides.



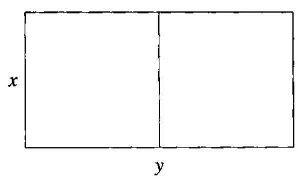
(c) Area $A =$ length $\times$ width $= y \cdot x$

(d) Length of fencing $= 750 \;\Rightarrow\; 5x + 2y = 750$

(e) $5x + 2y = 750 \;\Rightarrow\; y = 375 - \frac{5}{2}x \;\Rightarrow\; A(x) = \left(375 - \frac{5}{2}x\right)x = 375x - \frac{5}{2}x^2$

(f) $A'(x) = 375 - 5x = 0 \;\Rightarrow\; x = 75$. Since $A''(x) = -5 < 0$ there is an absolute maximum when $x = 75$. Then $y = \frac{375}{2} = 187.5$. The largest area is $75\left(\frac{375}{2}\right) = 14{,}062.5\text{ ft}^2$. These values of x and y are between the values in the first and second figures in part (a). Our original estimate was low.

9.

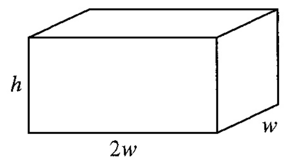


$xy = 1.5 \times 10^6$, so $y = 1.5 \times 10^6/x$. Minimize the amount of fencing, which is $3x + 2y = 3x + 2(1.5 \times 10^6/x) = 3x + 3 \times 10^6/x = F(x)$. $F'(x) = 3 - 3 \times 10^6/x^2 = 3(x^2 - 10^6)/x^2$. The critical number is $x = 10^3$ and $F'(x) < 0$ for $0 < x < 10^3$ and $F'(x) > 0$ if $x > 10^3$, so the absolute minimum occurs when $x = 10^3$ and $y = 1.5 \times 10^3$.

The field should be 1000 feet by 1500 feet with the middle fence parallel to the short side of the field.

11. Let b be the length of the base of the box and h the height. The surface area is $1200 = b^2 + 4hb \Rightarrow$ $h = (1200 - b^2)/(4b)$. The volume is $V = b^2 h = b^2(1200 - b^2)/4b = 300b - b^3/4 \Rightarrow V'(b) = 300 - \frac{3}{4}b^2$. $V'(b) = 0 \Rightarrow 300 = \frac{3}{4}b^2 \Rightarrow b^2 = 400 \Rightarrow b = \sqrt{400} = 20$. Since $V'(b) > 0$ for $0 < b < 20$ and $V'(b) < 0$ for $b > 20$, there is an absolute maximum when $b = 20$ by the First Derivative Test for Absolute Extreme Values (see page 334). If $b = 20$, then $h = (1200 - 20^2)/(4 \cdot 20) = 10$, so the largest possible volume is $b^2 h = (20)^2(10) = 4000\text{ cm}^3$.

13.



$10 = (2w)(w)h = 2w^2 h$, so $h = 5/w^2$. The cost is

$$C(w) = 10(2w^2) + 6[2(2wh) + 2hw] + 6(2w^2)$$
$$= 32w^2 + 36wh = 32w^2 + 180/w$$

$C'(w) = 64w - 180/w^2 = 4(16w^3 - 45)/w^2 \Rightarrow w = \sqrt[3]{\frac{45}{16}}$ is the critical number. $C'(w) < 0$ for $0 < w < \sqrt[3]{\frac{45}{16}}$ and $C'(w) > 0$ for $w > \sqrt[3]{\frac{45}{16}}$. The minimum cost is $C\left(\sqrt[3]{\frac{45}{16}}\right) = 32(2.8125)^{2/3} + 180/\sqrt{2.8125} \approx \$191.28$.

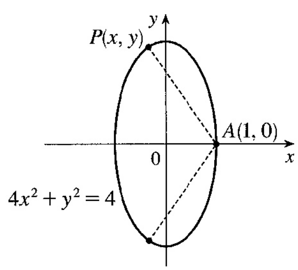
15. The distance from a point (x, y) on the line $y = 4x + 7$ to the origin is $\sqrt{(x-0)^2 + (y-0)^2} = \sqrt{x^2 + y^2}$. However, it is easier to work with the *square* of the distance; that is,

$D(x) = \left(\sqrt{x^2 + y^2}\right)^2 = x^2 + y^2 = x^2 + (4x + 7)^2$. Because the distance is positive, its minimum value will occur at the same point as the minimum value of D.

$D'(x) = 2x + 2(4x + 7)(4) = 34x + 56$, so $D'(x) = 0 \Leftrightarrow x = -\frac{28}{17}$.

$D''(x) = 34 > 0$, so D is concave upward for all x. Thus, D has an absolute minimum at $x = -\frac{28}{17}$. The point closest to the origin is $(x, y) = \left(-\frac{28}{17}, 4\left(-\frac{28}{17}\right) + 7\right) = \left(-\frac{28}{17}, \frac{7}{17}\right)$.

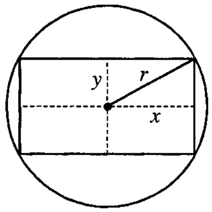
17.



From the figure, we see that there are two points that are farthest away from $A(1, 0)$. The distance d from A to an arbitrary point $P(x, y)$ on the ellipse is $d = \sqrt{(x-1)^2 + (y-0)^2}$ and the square of the distance is $S = d^2 = x^2 - 2x + 1 + y^2 = x^2 - 2x + 1 + (4 - 4x^2) = -3x^2 - 2x + 5$. $S' = -6x - 2$ and $S' = 0 \Rightarrow x = -\frac{1}{3}$. Now $S'' = -6 < 0$, so we know that S has a maximum at $x = -\frac{1}{3}$. Since $-1 \le x \le 1$, $S(-1) = 4$, $S\left(-\frac{1}{3}\right) = \frac{16}{3}$, and $S(1) = 0$, we see that the maximum distance is $\sqrt{\frac{16}{3}}$. The corresponding y-values are

$y = \pm\sqrt{4 - 4\left(-\frac{1}{3}\right)^2} = \pm\sqrt{\frac{32}{9}} = \pm\frac{4}{3}\sqrt{2} \approx \pm 1.89$. The points are $\left(-\frac{1}{3}, \pm\frac{4}{3}\sqrt{2}\right)$.

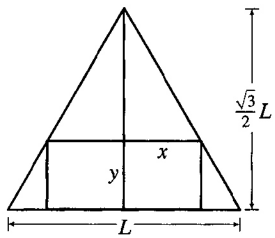
19.



The area of the rectangle is $(2x)(2y) = 4xy$. Also $r^2 = x^2 + y^2$ so $y = \sqrt{r^2 - x^2}$, so the area is $A(x) = 4x\sqrt{r^2 - x^2}$. Now $A'(x) = 4\left(\sqrt{r^2 - x^2} - \dfrac{x^2}{\sqrt{r^2 - x^2}}\right) = 4\dfrac{r^2 - 2x^2}{\sqrt{r^2 - x^2}}$. The critical number is $x = \frac{1}{\sqrt{2}}r$. Clearly this gives a maximum.

$y = \sqrt{r^2 - \left(\frac{1}{\sqrt{2}}r\right)^2} = \sqrt{\frac{1}{2}r^2} = \frac{1}{\sqrt{2}}r = x$, which tells us that the rectangle is a square. The dimensions are $2x = \sqrt{2}\,r$ and $2y = \sqrt{2}\,r$.

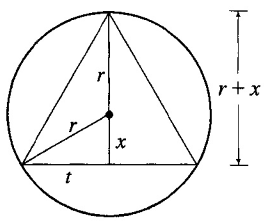
21.



The height h of the equilateral triangle with sides of length L is $\frac{\sqrt{3}}{2}L$, since $h^2 + (L/2)^2 = L^2 \;\Rightarrow\; h^2 = L^2 - \frac{1}{4}L^2 = \frac{3}{4}L^2 \;\Rightarrow$ $h = \frac{\sqrt{3}}{2}L$. Using similar triangles, $\dfrac{\frac{\sqrt{3}}{2}L - y}{x} = \dfrac{\frac{\sqrt{3}}{2}L}{L/2} = \sqrt{3} \;\Rightarrow$ $\sqrt{3}\,x = \dfrac{\sqrt{3}}{2}L - y \;\Rightarrow\; y = \dfrac{\sqrt{3}}{2}L - \sqrt{3}\,x \;\Rightarrow\; y = \dfrac{\sqrt{3}}{2}(L - 2x)$.

The area of the inscribed rectangle is $A(x) = (2x)y = \sqrt{3}\,x(L - 2x) = \sqrt{3}\,Lx - 2\sqrt{3}\,x^2$, where $0 \le x \le L/2$. Now $0 = A'(x) = \sqrt{3}\,L - 4\sqrt{3}\,x \;\Rightarrow\; x = \sqrt{3}\,L/(4\sqrt{3}) = L/4$. Since $A(0) = A(L/2) = 0$, the maximum occurs when $x = L/4$, and $y = \frac{\sqrt{3}}{2}L - \frac{\sqrt{3}}{4}L = \frac{\sqrt{3}}{4}L$, so the dimensions are $L/2$ and $\frac{\sqrt{3}}{4}L$.

23.



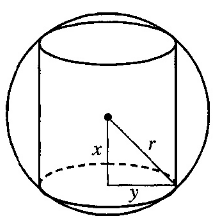
The area of the triangle is $A(x) = \frac{1}{2}(2t)(r + x) = t(r + x) = \sqrt{r^2 - x^2}(r + x)$. Then

$$0 = A'(x) = r\frac{-2x}{2\sqrt{r^2 - x^2}} + \sqrt{r^2 - x^2} + x\frac{-2x}{2\sqrt{r^2 - x^2}} = -\frac{x^2 + rx}{\sqrt{r^2 - x^2}} + \sqrt{r^2 - x^2} \;\Rightarrow$$

$$\frac{x^2 + rx}{\sqrt{r^2 - x^2}} = \sqrt{r^2 - x^2} \;\Rightarrow\; x^2 + rx = r^2 - x^2 \;\Rightarrow\; 0 = 2x^2 + rx - r^2 = (2x - r)(x + r) \;\Rightarrow$$

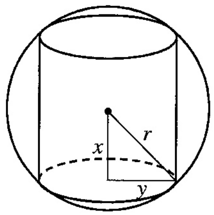
$x = \frac{1}{2}r$ or $x = -r$. Now $A(r) = 0 = A(-r) \;\Rightarrow$ the maximum occurs where $x = \frac{1}{2}r$, so the triangle has height $r + \frac{1}{2}r = \frac{3}{2}r$ and base $2\sqrt{r^2 - \left(\frac{1}{2}r\right)^2} = 2\sqrt{\frac{3}{4}r^2} = \sqrt{3}\,r$.

25.



The cylinder has volume $V = \pi y^2(2x)$. Also $x^2 + y^2 = r^2 \;\Rightarrow$ $y^2 = r^2 - x^2$, so $V(x) = \pi(r^2 - x^2)(2x) = 2\pi(r^2x - x^3)$, where $0 \le x \le r$. $V'(x) = 2\pi(r^2 - 3x^2) = 0 \;\Rightarrow\; x = r/\sqrt{3}$. Now $V(0) = V(r) = 0$, so there is a maximum when $x = r/\sqrt{3}$ and $V(r/\sqrt{3}) = \pi(r^2 - r^2/3)(2r/\sqrt{3}) = 4\pi r^3/(3\sqrt{3})$.

27.



The cylinder has surface area

2(area of the base) + (lateral surface area)

$= 2\pi(\text{radius})^2 + 2\pi(\text{radius})(\text{height}) = 2\pi y^2 + 2\pi y(2x)$.

Now $x^2 + y^2 = r^2 \;\Rightarrow\; y^2 = r^2 - x^2 \;\Rightarrow\; y = \sqrt{r^2 - x^2}$, so the surface area is

$$S(x) = 2\pi(r^2 - x^2) + 4\pi x\sqrt{r^2 - x^2}, \quad 0 \le x \le r$$
$$= 2\pi r^2 - 2\pi x^2 + 4\pi\left(x\sqrt{r^2 - x^2}\right)$$

Thus, $S'(x) = 0 - 4\pi x + 4\pi\left[x \cdot \frac{1}{2}(r^2 - x^2)^{-1/2}(-2x) + (r^2 - x^2)^{1/2} \cdot 1\right]$

$$= 4\pi\left[-x - \frac{x^2}{\sqrt{r^2 - x^2}} + \sqrt{r^2 - x^2}\right] = 4\pi \cdot \frac{-x\sqrt{r^2 - x^2} - x^2 + r^2 - x^2}{\sqrt{r^2 - x^2}}$$

$S'(x) = 0 \;\Rightarrow\; x\sqrt{r^2 - x^2} = r^2 - 2x^2 \quad (\star) \;\Rightarrow\; \left(x\sqrt{r^2 - x^2}\right)^2 = (r^2 - 2x^2)^2 \;\Rightarrow$

$x^2(r^2 - x^2) = r^4 - 4r^2x^2 + 4x^4 \;\Rightarrow\; r^2x^2 - x^4 = r^4 - 4r^2x^2 + 4x^4 \;\Rightarrow\; 5x^4 - 5r^2x^2 + r^4 = 0.$

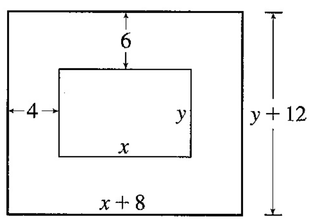
This is a quadratic equation in x^2. By the quadratic formula, $x^2 = \frac{5 \pm \sqrt{5}}{10}r^2$, but we reject the root with the + sign since it doesn't satisfy $(\star)$. [The right side is negative and the left side is positive.] So $x = \sqrt{\frac{5 - \sqrt{5}}{10}}\,r$. Since $S(0) = S(r) = 0$, the maximum surface area occurs at the critical number and $x^2 = \frac{5 - \sqrt{5}}{10}r^2 \;\Rightarrow$

$y^2 = r^2 - \frac{5 - \sqrt{5}}{10}r^2 = \frac{5 + \sqrt{5}}{10}r^2 \;\Rightarrow$ the surface area is

$2\pi\left(\frac{5+\sqrt{5}}{10}\right)r^2 + 4\pi\sqrt{\frac{5-\sqrt{5}}{10}}\sqrt{\frac{5+\sqrt{5}}{10}}\,r^2 = \pi r^2\left[2 \cdot \frac{5+\sqrt{5}}{10} + 4\frac{\sqrt{(5-\sqrt{5})(5+\sqrt{5})}}{10}\right] = \pi r^2\left[\frac{5+\sqrt{5}}{5} + \frac{2\sqrt{20}}{5}\right] =$

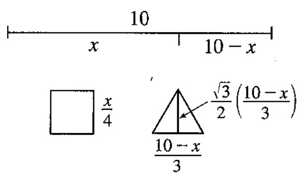
$\pi r^2\left[\frac{5+\sqrt{5}+2\cdot 2\sqrt{5}}{5}\right] = \pi r^2\left[\frac{5+5\sqrt{5}}{5}\right] = \pi r^2(1+\sqrt{5}).$

29.



$xy = 384 \;\Rightarrow\; y = 384/x$. Total area is $A(x) = (8 + x)(12 + 384/x) = 12(40 + x + 256/x)$, so $A'(x) = 12(1 - 256/x^2) = 0 \;\Rightarrow\; x = 16$. There is an absolute minimum when $x = 16$ since $A'(x) < 0$ for $0 < x < 16$ and $A'(x) > 0$ for $x > 16$. When $x = 16$, $y = 384/16 = 24$, so the dimensions are 24 cm and 36 cm.

31.



Let x be the length of the wire used for the square. The total area is

$$A(x) = \left(\frac{x}{4}\right)^2 + \frac{1}{2}\left(\frac{10 - x}{3}\right)\frac{\sqrt{3}}{2}\left(\frac{10 - x}{3}\right)$$
$$= \frac{1}{16}x^2 + \frac{\sqrt{3}}{36}(10 - x)^2, \quad 0 \le x \le 10$$

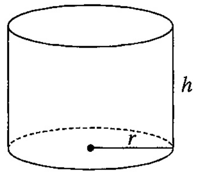
$A'(x) = \frac{1}{8}x - \frac{\sqrt{3}}{18}(10 - x) = 0 \;\Leftrightarrow\; \frac{9}{72}x + \frac{4\sqrt{3}}{72}x - \frac{40\sqrt{3}}{72} = 0 \;\Leftrightarrow\; x = \frac{40\sqrt{3}}{9 + 4\sqrt{3}}$. Now

$A(0) = \left(\frac{\sqrt{3}}{36}\right)100 \approx 4.81$, $A(10) = \frac{100}{16} = 6.25$ and $A\left(\frac{40\sqrt{3}}{9+4\sqrt{3}}\right) \approx 2.72$, so

(a) The maximum area occurs when $x = 10$ m, and all the wire is used for the square.

(b) The minimum area occurs when $x = \frac{40\sqrt{3}}{9 + 4\sqrt{3}} \approx 4.35$ m.

33.



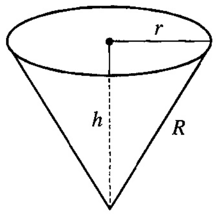
The volume is $V = \pi r^2 h$ and the surface area is

$$S(r) = \pi r^2 + 2\pi r h = \pi r^2 + 2\pi r\left(\frac{V}{\pi r^2}\right) = \pi r^2 + \frac{2V}{r}.$$

$$S'(r) = 2\pi r - \frac{2V}{r^2} = 0 \quad\Rightarrow\quad 2\pi r^3 = 2V \quad\Rightarrow\quad r = \sqrt[3]{\frac{V}{\pi}}\ \text{cm}.$$

This gives an absolute minimum since $S'(r) < 0$ for $0 < r < \sqrt[3]{\frac{V}{\pi}}$ and $S'(r) > 0$ for $r > \sqrt[3]{\frac{V}{\pi}}$. When $r = \sqrt[3]{\frac{V}{\pi}}$, $h = \frac{V}{\pi r^2} = \frac{V}{\pi (V/\pi)^{2/3}} = \sqrt[3]{\frac{V}{\pi}}$ cm.

35.

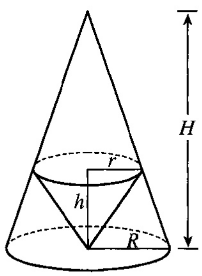


$h^2 + r^2 = R^2 \quad\Rightarrow\quad V = \frac{\pi}{3} r^2 h = \frac{\pi}{3}(R^2 - h^2)h = \frac{\pi}{3}(R^2 h - h^3).$

$V'(h) = \frac{\pi}{3}(R^2 - 3h^2) = 0$ when $h = \frac{1}{\sqrt{3}}R$. This gives an absolute maximum, since $V'(h) > 0$ for $0 < h < \frac{1}{\sqrt{3}}R$ and $V'(h) < 0$ for $h > \frac{1}{\sqrt{3}}R$. The maximum volume is

$$V\left(\frac{1}{\sqrt{3}}R\right) = \frac{\pi}{3}\left(\frac{1}{\sqrt{3}}R^3 - \frac{1}{3\sqrt{3}}R^3\right) = \frac{2}{9\sqrt{3}}\pi R^3.$$

37.



By similar triangles, $\frac{H}{R} = \frac{H-h}{r}$ **(1)**. The volume of the inner cone is $V = \frac{1}{3}\pi r^2 h$, so we'll solve **(1)** for h. $\frac{Hr}{R} = H - h \quad\Rightarrow$

$$h = H - \frac{Hr}{R} = \frac{HR - Hr}{R} = \frac{H}{R}(R - r) \quad \textbf{(2)}.$$

Thus, $V(r) = \frac{\pi}{3} r^2 \cdot \frac{H}{R}(R - r) = \frac{\pi H}{3R}(Rr^2 - r^3) \quad\Rightarrow$

$$V'(r) = \frac{\pi H}{3R}(2Rr - 3r^2) = \frac{\pi H}{3R} r(2R - 3r).$$

$V'(r) = 0 \quad\Rightarrow\quad r = 0$ or $2R = 3r \quad\Rightarrow\quad r = \frac{2}{3}R$ and from **(2)**, $h = \frac{H}{R}\left(R - \frac{2}{3}R\right) = \frac{H}{R}\left(\frac{1}{3}R\right) = \frac{1}{3}H$.

$V'(r)$ changes from positive to negative at $r = \frac{2}{3}R$, so the inner cone has a maximum volume of

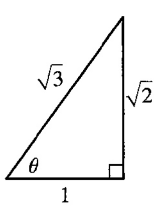
$V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi\left(\frac{2}{3}R\right)^2\left(\frac{1}{3}H\right) = \frac{4}{27}\cdot\frac{1}{3}\pi R^2 H$, which is approximately 15% of the volume of the larger cone.

39. $S = 6sh - \frac{3}{2}s^2 \cot\theta + 3s^2 \frac{\sqrt{3}}{2} \csc\theta$

(a) $\frac{dS}{d\theta} = \frac{3}{2}s^2 \csc^2\theta - 3s^2 \frac{\sqrt{3}}{2} \csc\theta \cot\theta$ or $\frac{3}{2}s^2 \csc\theta \left(\csc\theta - \sqrt{3}\cot\theta\right)$.

(b) $\frac{dS}{d\theta} = 0$ when $\csc\theta - \sqrt{3}\cot\theta = 0 \quad\Rightarrow\quad \frac{1}{\sin\theta} - \sqrt{3}\frac{\cos\theta}{\sin\theta} = 0 \quad\Rightarrow\quad \cos\theta = \frac{1}{\sqrt{3}}$. The First Derivative Test shows that the minimum surface area occurs when $\theta = \cos^{-1}\left(\frac{1}{\sqrt{3}}\right) \approx 55°$.

(c)



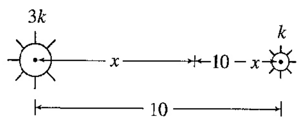
If $\cos\theta = \frac{1}{\sqrt{3}}$, then $\cot\theta = \frac{1}{\sqrt{2}}$ and $\csc\theta = \frac{\sqrt{3}}{\sqrt{2}}$, so the surface area is

$$S = 6sh - \frac{3}{2}s^2\frac{1}{\sqrt{2}} + 3s^2\frac{\sqrt{3}}{2}\frac{\sqrt{3}}{\sqrt{2}} = 6sh - \frac{3}{2\sqrt{2}}s^2 + \frac{9}{2\sqrt{2}}s^2$$

$$= 6sh + \frac{6}{2\sqrt{2}}s^2 = 6s\left(h + \frac{1}{2\sqrt{2}}s\right)$$

41. Here $T(x) = \dfrac{\sqrt{x^2+25}}{6} + \dfrac{5-x}{8}$, $0 \le x \le 5 \;\Rightarrow\; T'(x) = \dfrac{x}{6\sqrt{x^2+25}} - \dfrac{1}{8} = 0 \;\Leftrightarrow\; 8x = 6\sqrt{x^2+25}$ $\Leftrightarrow\; 16x^2 = 9(x^2+25) \;\Leftrightarrow\; x = \frac{15}{\sqrt{7}}$. But $\frac{15}{\sqrt{7}} > 5$, so T has no critical number. Since $T(0) \approx 1.46$ and $T(5) \approx 1.18$, he should row directly to B.

43.



The total illumination is $I(x) = \dfrac{3k}{x^2} + \dfrac{k}{(10-x)^2}$, $0 < x < 10$. Then

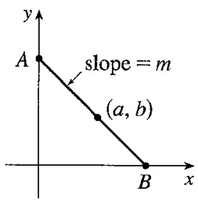
$I'(x) = \dfrac{-6k}{x^3} + \dfrac{2k}{(10-x)^3} = 0 \;\Rightarrow\; 6k(10-x)^3 = 2kx^3 \;\Rightarrow$

$3(10-x)^3 = x^3 \;\Rightarrow\; \sqrt[3]{3}\,(10-x) = x \;\Rightarrow\; 10\sqrt[3]{3} - \sqrt[3]{3}\,x = x$

$\Rightarrow\; 10\sqrt[3]{3} = x + \sqrt[3]{3}\,x \;\Rightarrow\; 10\sqrt[3]{3} = (1+\sqrt[3]{3})x \;\Rightarrow$

$x = \dfrac{10\sqrt[3]{3}}{1+\sqrt[3]{3}} \approx 5.9$ ft. This gives a minimum since $I''(x) > 0$ for $0 < x < 10$.

45.



Every line segment in the first quadrant passing through (a, b) with endpoints on the x- and y-axes satisfies an equation of the form $y - b = m(x - a)$, where $m < 0$. By setting $x = 0$ and then $y = 0$, we find its endpoints, $A(0, b - am)$ and $B\left(a - \frac{b}{m}, 0\right)$. The distance d from A to B is given by $d = \sqrt{[(a - \frac{b}{m}) - 0]^2 + [0 - (b - am)]^2}$.

It follows that the square of the length of the line segment, as a function of m, is given by

$$S(m) = \left(a - \frac{b}{m}\right)^2 + (am - b)^2 = a^2 - \frac{2ab}{m} + \frac{b^2}{m^2} + a^2m^2 - 2abm + b^2.\text{ Thus,}$$

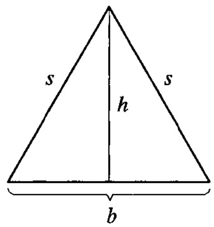
$$\begin{aligned} S'(m) &= \frac{2ab}{m^2} - \frac{2b^2}{m^3} + 2a^2m - 2ab = \frac{2}{m^3}(abm - b^2 + a^2m^4 - abm^3) \\ &= \frac{2}{m^3}[b(am - b) + am^3(am - b)] = \frac{2}{m^3}(am - b)(b + am^3) \end{aligned}$$

Thus, $S'(m) = 0 \;\Leftrightarrow\; m = b/a$ or $m = -\sqrt[3]{\frac{b}{a}}$. Since $b/a > 0$ and $m < 0$, m must equal $-\sqrt[3]{\frac{b}{a}}$. Since $\dfrac{2}{m^3} < 0$, we see that $S'(m) < 0$ for $m < -\sqrt[3]{\frac{b}{a}}$ and $S'(m) > 0$ for $m > -\sqrt[3]{\frac{b}{a}}$. Thus, S has its absolute minimum value when $m = -\sqrt[3]{\frac{b}{a}}$. That value is

$$\begin{aligned} S\left(-\sqrt[3]{\tfrac{b}{a}}\right) &= \left(a + b\sqrt[3]{\tfrac{a}{b}}\right)^2 + \left(-a\sqrt[3]{\tfrac{b}{a}} - b\right)^2 = \left(a + \sqrt[3]{ab^2}\right)^2 + \left(\sqrt[3]{a^2b} + b\right)^2 \\ &= a^2 + 2a^{4/3}b^{2/3} + a^{2/3}b^{4/3} + a^{4/3}b^{2/3} + 2a^{2/3}b^{4/3} + b^2 = a^2 + 3a^{4/3}b^{2/3} + 3a^{2/3}b^{4/3} + b^2 \end{aligned}$$

The last expression is of the form $x^3 + 3x^2y + 3xy^2 + y^3$ $\;[= (x+y)^3]\;$ with $x = a^{2/3}$ and $y = b^{2/3}$, so we can write it as $(a^{2/3} + b^{2/3})^3$ and the shortest such line segment has length $\sqrt{S} = (a^{2/3} + b^{2/3})^{3/2}$.

47.



Here $s^2 = h^2 + b^2/4$, so $h^2 = s^2 - b^2/4$. The area is $A = \frac{1}{2}b\sqrt{s^2 - b^2/4}$.

Let the perimeter be p, so $2s + b = p$ or $s = (p-b)/2 \;\Rightarrow$

$A(b) = \frac{1}{2}b\sqrt{(p-b)^2/4 - b^2/4} = b\sqrt{p^2 - 2pb}/4$. Now

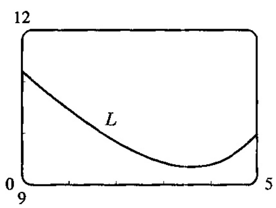
$A'(b) = \dfrac{\sqrt{p^2-2pb}}{4} - \dfrac{bp/4}{\sqrt{p^2-2pb}} = \dfrac{-3pb+p^2}{4\sqrt{p^2-2pb}}$. Therefore, $A'(b) = 0 \;\Rightarrow$

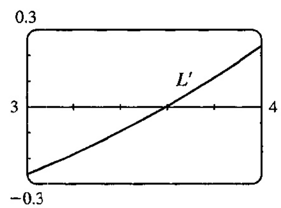
$-3pb + p^2 = 0 \;\Rightarrow\; b = p/3$. Since $A'(b) > 0$ for $b < p/3$ and $A'(b) < 0$ for $b > p/3$, there is an absolute maximum when $b = p/3$. But then $2s + p/3 = p$, so $s = p/3 \;\Rightarrow\; s = b \;\Rightarrow\;$ the triangle is equilateral.

49. Note that $|AD| = |AP| + |PD| \;\Rightarrow\; 5 = x + |PD| \;\Rightarrow\; |PD| = 5 - x$. Using the Pythagorean Theorem for ΔPDB and ΔPDC gives us

$$L(x) = |AP| + |BP| + |CP| = x + \sqrt{(5-x)^2 + 2^2} + \sqrt{(5-x)^2 + 3^2}$$
$$= x + \sqrt{x^2 - 10x + 29} + \sqrt{x^2 - 10x + 34}$$

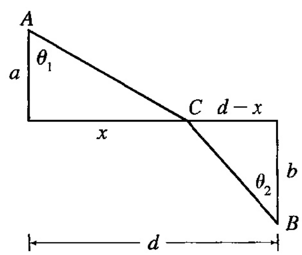
$$\Rightarrow\quad L'(x) = 1 + \frac{x-5}{\sqrt{x^2-10x+29}} + \frac{x-5}{\sqrt{x^2-10x+34}}.$$





From the graphs of L and L', it seems that the minimum value of L is about $L(3.59) = 9.35$ m.

51.



The total time is

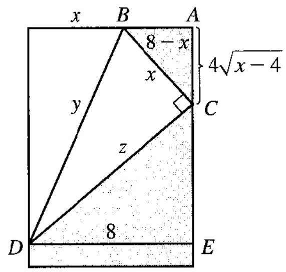
$$T(x) = (\text{time from } A \text{ to } C) + (\text{time from } C \text{ to } B)$$
$$= \frac{\sqrt{a^2+x^2}}{v_1} + \frac{\sqrt{b^2+(d-x)^2}}{v_2}, \quad 0 < x < d$$

$$T'(x) = \frac{x}{v_1\sqrt{a^2+x^2}} - \frac{d-x}{v_2\sqrt{b^2+(d-x)^2}} = \frac{\sin\theta_1}{v_1} - \frac{\sin\theta_2}{v_2}$$

The minimum occurs when $T'(x) = 0 \;\Rightarrow\; \dfrac{\sin\theta_1}{v_1} = \dfrac{\sin\theta_2}{v_2}$.

[*Note:* $T''(x) > 0$]

53.

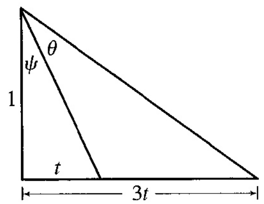


$y^2 = x^2 + z^2$, but triangles CDE and BCA are similar, so $z/8 = x/\left(4\sqrt{x-4}\right) \;\Rightarrow\; z = 2x/\sqrt{x-4}$. Thus, we minimize $f(x) = y^2 = x^2 + 4x^2/(x-4) = x^3/(x-4)$, $\;4 < x \le 8$.

$$f'(x) = \frac{(x-4)(3x^2) - x^3}{(x-4)^2} = \frac{x^2[3(x-4)-x]}{(x-4)^2} = \frac{2x^2(x-6)}{(x-4)^2} = 0$$

when $x = 6$. $f'(x) < 0$ when $x < 6$, $f'(x) > 0$ when $x > 6$, so the minimum occurs when $x = 6$ in.

55.



It suffices to maximize $\tan\theta$. Now

$$\frac{3t}{1} = \tan(\psi + \theta) = \frac{\tan\psi + \tan\theta}{1 - \tan\psi\,\tan\theta} = \frac{t + \tan\theta}{1 - t\tan\theta}.$$

So $3t(1 - t\tan\theta) = t + \tan\theta \;\Rightarrow\; 2t = \left(1 + 3t^2\right)\tan\theta \;\Rightarrow$

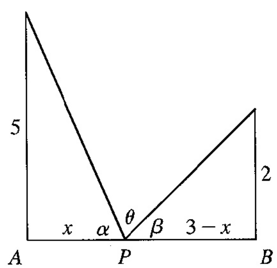
$\tan\theta = \dfrac{2t}{1+3t^2}$. Let $f(t) = \tan\theta = \dfrac{2t}{1+3t^2} \;\Rightarrow$

$$f'(t) = \frac{2(1+3t^2) - 2t(6t)}{(1+3t^2)^2} = \frac{2(1-3t^2)}{(1+3t^2)^2} = 0 \;\Leftrightarrow$$

$1 - 3t^2 = 0 \;\Leftrightarrow\; t = \frac{1}{\sqrt{3}}$ since $t \ge 0$.

Now $f'(t) > 0$ for $0 \le t < \frac{1}{\sqrt{3}}$ and $f'(t) < 0$ for $t > \frac{1}{\sqrt{3}}$, so f has an absolute maximum when $t = \frac{1}{\sqrt{3}}$ and $\tan\theta = \dfrac{2\left(1/\sqrt{3}\right)}{1 + 3\left(1/\sqrt{3}\right)^2} = \frac{1}{\sqrt{3}} \;\Rightarrow\; \theta = \frac{\pi}{6}$. Substituting for t and θ in $3t = \tan(\psi + \theta)$ gives us $\sqrt{3} = \tan\left(\psi + \frac{\pi}{6}\right) \;\Rightarrow\; \psi = \frac{\pi}{6}$.

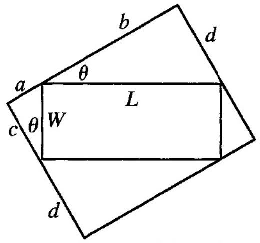
57.



From the figure, $\tan\alpha = \dfrac{5}{x}$ and $\tan\beta = \dfrac{2}{3-x}$. Since $\alpha + \beta + \theta = 180° = \pi$, $\theta = \pi - \tan^{-1}\left(\dfrac{5}{x}\right) - \tan^{-1}\left(\dfrac{2}{3-x}\right) \;\Rightarrow$

$$\frac{d\theta}{dx} = -\frac{1}{1 + \left(\frac{5}{x}\right)^2}\left(-\frac{5}{x^2}\right) - \frac{1}{1 + \left(\frac{2}{3-x}\right)^2}\left[\frac{2}{(3-x)^2}\right] = \frac{x^2}{x^2+25}\cdot\frac{5}{x^2} - \frac{(3-x)^2}{(3-x)^2+4}\cdot\frac{2}{(3-x)^2}.$$

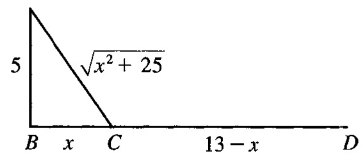
Now $\dfrac{d\theta}{dx} = 0 \;\Rightarrow\; \dfrac{5}{x^2+25} = \dfrac{2}{x^2 - 6x + 13} \;\Rightarrow\; 2x^2 + 50 = 5x^2 - 30x + 65 \;\Rightarrow$ $3x^2 - 30x + 15 = 0 \;\Rightarrow\; x^2 - 10x + 5 = 0 \;\Rightarrow\; x = 5 \pm 2\sqrt{5}$. We reject the root with the $+$ sign, since it is larger than 3. $d\theta/dx > 0$ for $x < 5 - 2\sqrt{5}$ and $d\theta/dx < 0$ for $x > 5 - 2\sqrt{5}$, so θ is maximized when $|AP| = x = 5 - 2\sqrt{5} \approx 0.53$.

59. 

In the small triangle with sides a and c and hypotenuse W, $\sin\theta = \dfrac{a}{W}$ and $\cos\theta = \dfrac{c}{W}$. In the triangle with sides b and d and hypotenuse L, $\sin\theta = \dfrac{d}{L}$ and $\cos\theta = \dfrac{b}{L}$. Thus, $a = W\sin\theta$, $c = W\cos\theta$, $d = L\sin\theta$, and $b = L\cos\theta$, so the area of the circumscribed rectangle is

$$\begin{aligned} A(\theta) &= (a+b)(c+d) = (W\sin\theta + L\cos\theta)(W\cos\theta + L\sin\theta) \\ &= W^2\sin\theta\,\cos\theta + WL\sin^2\theta + LW\cos^2\theta + L^2\sin\theta\,\cos\theta \\ &= LW\sin^2\theta + LW\cos^2\theta + (L^2+W^2)\sin\theta\,\cos\theta \\ &= LW(\sin^2\theta + \cos^2\theta) + (L^2+W^2)\cdot\tfrac{1}{2}\cdot 2\sin\theta\,\cos\theta \\ &= LW + \tfrac{1}{2}(L^2+W^2)\sin 2\theta,\quad 0 \le \theta \le \tfrac{\pi}{2} \end{aligned}$$

This expression shows, without calculus, that the maximum value of $A(\theta)$ occurs when $\sin 2\theta = 1 \;\Leftrightarrow\; 2\theta = \frac{\pi}{2}$ $\Rightarrow\; \theta = \frac{\pi}{4}$. So the maximum area is $A\left(\frac{\pi}{4}\right) = LW + \frac{1}{2}(L^2+W^2) = \frac{1}{2}(L^2 + 2LW + W^2) = \frac{1}{2}(L+W)^2$.

61. (a) 

If k = energy/km over land, then energy/km over water $= 1.4k$. So the total energy is $E = 1.4k\sqrt{25+x^2} + k(13-x)$, $0 \le x \le 13$, and so $\dfrac{dE}{dx} = \dfrac{1.4kx}{(25+x^2)^{1/2}} - k$.

Set $\dfrac{dE}{dx} = 0$: $1.4kx = k(25+x^2)^{1/2} \;\Rightarrow\; 1.96x^2 = x^2 + 25 \;\Rightarrow\; 0.96x^2 = 25 \;\Rightarrow\; x = \frac{5}{\sqrt{0.96}} \approx 5.1$.

Testing against the value of E at the endpoints: $E(0) = 1.4k(5) + 13k = 20k$, $E(5.1) \approx 17.9k$, $E(13) \approx 19.5k$. Thus, to minimize energy, the bird should fly to a point about 5.1 km from B.

(b) If W/L is large, the bird would fly to a point C that is closer to B than to D to minimize the energy used flying over water. If W/L is small, the bird would fly to a point C that is closer to D than to B to minimize the distance of the flight. $E = W\sqrt{25+x^2} + L(13-x) \;\Rightarrow\; \dfrac{dE}{dx} = \dfrac{Wx}{\sqrt{25+x^2}} - L = 0$ when $\dfrac{W}{L} = \dfrac{\sqrt{25+x^2}}{x}$. By the same sort of argument as in part (a), this ratio will give the minimal expenditure of energy if the bird heads for the point x km from B.

(c) For flight direct to D, $x = 13$, so from part (b), $W/L = \frac{\sqrt{25+13^2}}{13} \approx 1.07$. There is no value of W/L for which the bird should fly directly to B. But note that $\lim\limits_{x\to 0^+}(W/L) = \infty$, so if the point at which E is a minimum is close to B, then W/L is large.

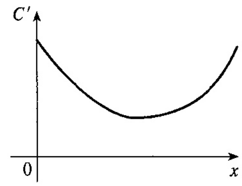
(d) Assuming that the birds instinctively choose the path that minimizes the energy expenditure, we can use the equation for $dE/dx = 0$ from part (a) with $1.4k = c$, $x = 4$, and $k = 1$: $(c)(4) = 1\cdot(25+4^2)^{1/2} \;\Rightarrow\; c = \sqrt{41}/4 \approx 1.6$.

4.8 Applications to Business and Economics

1. (a) $C(0)$ represents the fixed costs of production, such as rent, utilities, machinery etc., which are incurred even when nothing is produced.

(b) The inflection point is the point at which $C''(x)$ changes from negative to positive; that is, the marginal cost $C'(x)$ changes from decreasing to increasing. Thus, the marginal cost is minimized at the inflection point.

(c) The marginal cost function is $C'(x)$. We graph it as in Example 1 in Section 2.9.



3. $c(x) = 21.4 - 0.002x$ and $c(x) = C(x)/x \quad \Rightarrow \quad C(x) = 21.4x - 0.002x^2$. $C'(x) = 21.4 - 0.004x$ and $C'(1000) = 17.4$. This means that the cost of producing the 1001st unit is about \$17.40.

5. (a) The cost function is $C(x) = 40{,}000 + 300x - x^2$, so the cost at a production level of 1000 is $C(1000) = \$1{,}340{,}000$. The average cost function is $c(x) = \dfrac{C(x)}{x} = \dfrac{40{,}000}{x} + 300 + x$ and $c(1000) = \$1340/\text{unit}$. The marginal cost function is $C'(x) = 300 + 2x$ and $C'(1000) = \$2300/\text{unit}$.

(b) See the box preceding Example 1. We must have $C'(x) = c(x) \quad \Leftrightarrow \quad 300 + 2x = \dfrac{40{,}000}{x} + 300 + x \quad \Leftrightarrow$ $x = \dfrac{40{,}000}{x} \quad \Rightarrow \quad x^2 = 40{,}000 \quad \Rightarrow \quad x = \sqrt{40{,}000} = 200$. This gives a minimum value of the average cost function $c(x)$ since $c''(x) = \dfrac{80{,}000}{x^3} > 0$.

(c) The minimum average cost is $c(200) = \$700/\text{unit}$.

7. (a) $C(x) = 16{,}000 + 200x + 4x^{3/2}$, $C(1000) = 16{,}000 + 200{,}000 + 40{,}000\sqrt{10} \approx 216{,}000 + 126{,}491$, so $C(1000) \approx \$342{,}491$. $c(x) = C(x)/x = \dfrac{16{,}000}{x} + 200 + 4x^{1/2}$, $c(1000) \approx \$342.49/\text{unit}$. $C'(x) = 200 + 6x^{1/2}$, $C'(1000) = 200 + 60\sqrt{10} \approx \$389.74/\text{unit}$.

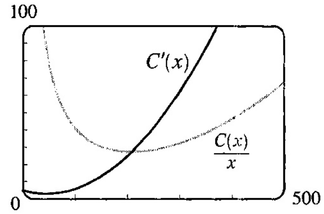
(b) We must have $C'(x) = c(x) \quad \Leftrightarrow \quad 200 + 6x^{1/2} = \dfrac{16{,}000}{x} + 200 + 4x^{1/2} \quad \Leftrightarrow \quad 2x^{3/2} = 16{,}000 \quad \Leftrightarrow$ $x = (8{,}000)^{2/3} = 400$ units. To check that this is a minimum, we calculate $c'(x) = \dfrac{-16{,}000}{x^2} + \dfrac{2}{\sqrt{x}} = \dfrac{2}{x^2}(x^{3/2} - 8000)$. This is negative for $x < (8000)^{2/3} = 400$, zero at $x = 400$, and positive for $x > 400$, so c is decreasing on $(0, 400)$ and increasing on $(400, \infty)$. Thus, c has an absolute minimum at $x = 400$. [*Note:* $c''(x)$ is *not* positive for all $x > 0$.]

(c) The minimum average cost is $c(400) = 40 + 200 + 80 = \$320/\text{unit}$.

9. (a) $C(x) = 3700 + 5x - 0.04x^2 + 0.0003x^3 \quad \Rightarrow \quad C'(x) = 5 - 0.08x + 0.0009x^2$ (marginal cost).

$c(x) = \dfrac{C(x)}{x} = \dfrac{3700}{x} + 5 - 0.04x + 0.0003x^2$ (average cost).

(b)



The graphs intersect at $(208.51, 27.45)$, so the production level that minimizes average cost is about 209 units.

(c) $c'(x) = -\dfrac{3700}{x^2} - 0.04 + 0.0006x = 0 \quad \Rightarrow \quad 3700 + 0.04x^2 - 0.0006x^3 = 0 \quad \Rightarrow \quad x_1 \approx 208.51.$

$c(x_1) \approx \$27.45/\text{unit}.$

(d) The marginal cost is given by $C'(x)$, so to find its minimum value we'll find the derivative of C'; that is, C''.

$C''(x) = -0.08 + 0.0018x = 0 \quad \Rightarrow \quad x_1 = \frac{800}{18} = 44.4\overline{4}.\quad C'(x_1) = \$3.22/\text{unit}.$

$C'''(x) = 0.0018 > 0$ for all x, so this is the minimum marginal cost. C''' is the second derivative of C'.

11. $C(x) = 680 + 4x + 0.01x^2$, $p(x) = 12 \quad \Rightarrow \quad R(x) = xp(x) = 12x$. If the profit is maximum, then $R'(x) = C'(x) \quad \Rightarrow \quad 12 = 4 + 0.02x \quad \Rightarrow \quad 0.02x = 8 \quad \Rightarrow \quad x = 400$. The profit is maximized if $P''(x) < 0$, but since $P''(x) = R''(x) - C''(x)$, we can just check the condition $R''(x) < C''(x)$. Now $R''(x) = 0 < 0.02 = C''(x)$, so $x = 400$ gives a maximum.

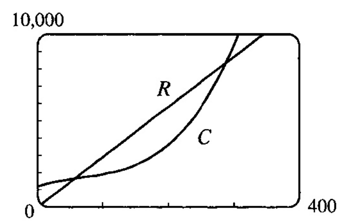
13. $C(x) = 1450 + 36x - x^2 + 0.001x^3$, $p(x) = 60 - 0.01x$. Then $R(x) = xp(x) = 60x - 0.01x^2$. If the profit is maximum, then $R'(x) = C'(x) \quad \Leftrightarrow \quad 60 - 0.02x = 36 - 2x + 0.003x^2 \quad \Rightarrow \quad 0.003x^2 - 1.98x - 24 = 0$. By the quadratic formula, $x = \dfrac{1.98 \pm \sqrt{(-1.98)^2 + 4(0.003)(24)}}{2(0.003)} = \dfrac{1.98 \pm \sqrt{4.2084}}{0.006}$. Since $x > 0$, $x \approx (1.98 + 2.05)/0.006 \approx 672$. Now $R''(x) = -0.02$ and $C''(x) = -2 + 0.006x \quad \Rightarrow \quad C''(672) = 2.032 \quad \Rightarrow \quad R''(672) < C''(672) \quad \Rightarrow \quad$ there is a maximum at $x = 672$.

15. $C(x) = 0.001x^3 - 0.3x^2 + 6x + 900$. The marginal cost is $C'(x) = 0.003x^2 - 0.6x + 6$. $C'(x)$ is increasing when $C''(x) > 0 \quad \Leftrightarrow \quad 0.006x - 0.6 > 0 \quad \Leftrightarrow \quad x > 0.6/0.006 = 100$. So $C'(x)$ starts to increase when $x = 100$.

17. (a) $C(x) = 1200 + 12x - 0.1x^2 + 0.0005x^3$.

$R(x) = xp(x) = 29x - 0.00021x^2$.

Since the profit is maximized when $R'(x) = C'(x)$, we examine the curves R and C in the figure, looking for x-values at which the slopes of the tangent lines are equal. It appears that $x = 200$ is a good estimate.



(b) $R'(x) = C'(x) \quad \Rightarrow \quad 29 - 0.00042x = 12 - 0.2x + 0.0015x^2 \quad \Rightarrow \quad 0.0015x^2 - 0.19958x - 17 = 0 \quad \Rightarrow$ $x \approx 192.06$ (for $x > 0$). As in Exercise 11, $R''(x) < C''(x) \quad \Rightarrow \quad -0.00042 < -0.2 + 0.003x \quad \Leftrightarrow$ $0.003x > 0.19958 \quad \Leftrightarrow \quad x > 66.5$. Our value of 192 is in this range, so we have a maximum profit when we produce 192 yards of fabric.

19. (a) We are given that the demand function p is linear and $p(27{,}000) = 10$, $p(33{,}000) = 8$, so the slope is
$\frac{10-8}{27{,}000-33{,}000} = -\frac{1}{3000}$ and an equation of the line is $y - 10 = \left(-\frac{1}{3000}\right)(x - 27{,}000) \quad \Rightarrow$
$y = p(x) = -\frac{1}{3000}x + 19 = 19 - (x/3000)$.

(b) The revenue is $R(x) = xp(x) = 19x - (x^2/3000) \quad \Rightarrow \quad R'(x) = 19 - (x/1500) = 0$ when $x = 28{,}500$.
Since $R''(x) = -1/1500 < 0$, the maximum revenue occurs when $x = 28{,}500 \quad \Rightarrow \quad$ the price is
$p(28{,}500) = \$9.50$.

21. (a) As in Example 3, we see that the demand function p is linear. We are given that $p(1000) = 450$ and deduce that
$p(1100) = 440$, since a \$10 reduction in price increases sales by 100 per week. The slope for p is
$\frac{440-450}{1100-1000} = -\frac{1}{10}$, so an equation is $p - 450 = -\frac{1}{10}(x - 1000)$ or $p(x) = -\frac{1}{10}x + 550$.

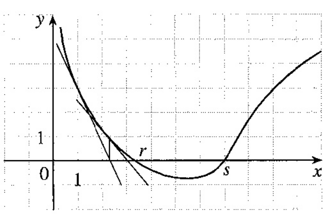
(b) $R(x) = xp(x) = -\frac{1}{10}x^2 + 550x$. $R'(x) = -\frac{1}{5}x + 550 = 0$ when $x = 5(550) = 2750$.
$p(2750) = 275$, so the rebate should be $450 - 275 = \$175$.

(c) $C(x) = 68{,}000 + 150x \quad \Rightarrow$
$P(x) = R(x) - C(x) = -\frac{1}{10}x^2 + 550x - 68{,}000 - 150x = -\frac{1}{10}x^2 + 400x - 68{,}000$,
$P'(x) = -\frac{1}{5}x + 400 = 0$ when $x = 2000$. $p(2000) = 350$. Therefore, the rebate to maximize profits should be $450 - 350 = \$100$.

23. If the reorder quantity is x, then the manager places $\dfrac{800}{x}$ orders per year. Storage costs for the year are
$\frac{1}{2}x \cdot 4 = 2x$ dollars. Handling costs are \$100 per delivery, for a total of $\dfrac{800}{x} \cdot 100 = \dfrac{80{,}000}{x}$ dollars. The total
costs for the year are $C(x) = 2x + \dfrac{80{,}000}{x}$. To minimize $C(x)$, we calculate
$C'(x) = 2 - \dfrac{80{,}000}{x^2} = \dfrac{2}{x^2}(x^2 - 40{,}000)$. This is negative when $x < 200$, zero when $x = 200$, and positive when
$x > 200$, so C is decreasing on $(0, 200)$ and increasing on $(200, \infty)$, reaching its absolute minimum when
$x = 200$. Thus, the optimal reorder quantity is 200 cases. The manager will place 4 orders per year for a total cost
of $C(200) = \$800$.

4.9 Newton's Method

1. (a)

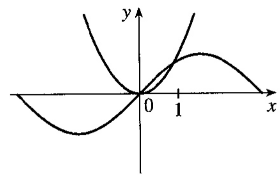


The tangent line at $x = 1$ intersects the x-axis at $x \approx 2.3$, so $x_2 \approx 2.3$. The tangent line at $x = 2.3$ intersects the x-axis at $x \approx 3$, so $x_3 \approx 3.0$.

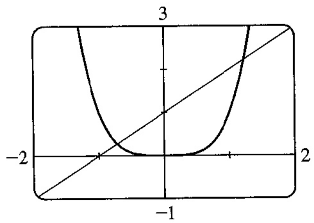
(b) $x_1 = 5$ would *not* be a better first approximation than $x_1 = 1$ since the tangent line is nearly horizontal. In fact, the second approximation for $x_1 = 5$ appears to be to the left of $x = 1$.

13. $f(x) = 2x^3 - 6x^2 + 3x + 1 \quad \Rightarrow \quad f'(x) = 6x^2 - 12x + 3 \quad \Rightarrow \quad x_{n+1} = x_n - \dfrac{2x_n^3 - 6x_n^2 + 3x_n + 1}{6x_n^2 - 12x_n + 3}$. We need to find approximations until they agree to six decimal places. $x_1 = 2.5 \quad \Rightarrow \quad x_2 \approx 2.285714$, $x_3 \approx 2.228824$, $x_4 \approx 2.224765$, $x_5 \approx 2.224745 \approx x_6$. So the root is 2.224745, to six decimal places.

15. $\sin x = x^2$, so $f(x) = \sin x - x^2 \quad \Rightarrow \quad f'(x) = \cos x - 2x \quad \Rightarrow$ $x_{n+1} = x_n - \dfrac{\sin x_n - x_n^2}{\cos x_n - 2x_n}$. From the figure, the positive root of $\sin x = x^2$ is near 1. $x_1 = 1 \quad \Rightarrow \quad x_2 \approx 0.891396$, $x_3 \approx 0.876985$, $x_4 \approx 0.876726 \approx x_5$. So the positive root is 0.876726, to six decimal places.



17.

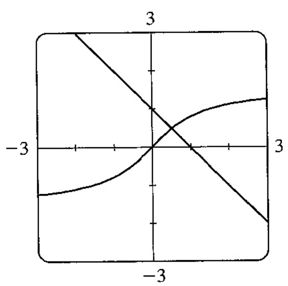


From the graph, we see that there appear to be points of intersection near $x = -0.7$ and $x = 1.2$. Solving $x^4 = 1 + x$ is the same as solving $f(x) = x^4 - x - 1 = 0$. $f(x) = x^4 - x - 1 \quad \Rightarrow \quad f'(x) = 4x^3 - 1$, so $x_{n+1} = x_n - \dfrac{x_n^4 - x_n - 1}{4x_n^3 - 1}$.

$$\begin{array}{ll} x_1 = -0.7 & x_1 = 1.2 \\ x_2 \approx -0.725253 & x_2 \approx 1.221380 \\ x_3 \approx -0.724493 & x_3 \approx 1.220745 \\ x_4 \approx -0.724492 \approx x_5 & x_4 \approx 1.220744 \approx x_5 \end{array}$$

To six decimal places, the roots of the equation are -0.724492 and 1.220744.

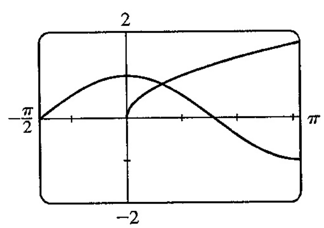
19.



From the graph, there appears to be a point of intersection near $x = 0.5$. Solving $\tan^{-1} x = 1 - x$ is the same as solving $f(x) = \tan^{-1} x + x - 1 = 0$. $f(x) = \tan^{-1} x + x - 1 \quad \Rightarrow$ $f'(x) = \dfrac{1}{1 + x^2} + 1$, so $x_{n+1} = x_n - \dfrac{\tan^{-1} x_n + x_n - 1}{1/\left(1 + x_n^2\right) + 1}$.

$x_1 = 0.5 \quad \Rightarrow \quad x_2 \approx 0.520196$, $x_3 \approx 0.520269 \approx x_4$. So to six decimal places, the root of the equation is 0.520269.

21. From the graph, there appears to be a point of intersection near $x = 0.6$. Solving $\cos x = \sqrt{x}$ is the same as solving $f(x) = \cos x - \sqrt{x} = 0$. $f(x) = \cos x - \sqrt{x} \quad \Rightarrow \quad f'(x) = -\sin x - 1/(2\sqrt{x})$, so $x_{n+1} = x_n - \dfrac{\cos x_n - \sqrt{x_n}}{-\sin x_n - 1/(2\sqrt{x})}$. Now $x_1 = 0.6 \quad \Rightarrow$ $x_2 \approx 0.641928$, $x_3 \approx 0.641714 \approx x_4$. To six decimal places, the root of the equation is 0.641714.



3. Since $x_1 = 3$ and $y = 5x - 4$ is tangent to $y = f(x)$ at $x = 3$, we simply need to find where the tangent line intersects the x-axis. $y = 0 \quad\Rightarrow\quad 5x_2 - 4 = 0 \quad\Rightarrow\quad x_2 = \frac{4}{5}$.

5. $f(x) = x^3 + 2x - 4 \quad\Rightarrow\quad f'(x) = 3x^2 + 2$, so $x_{n+1} = x_n - \dfrac{x_n^3 + 2x_n - 4}{3x_n^2 + 2}$. Now $x_1 = 1 \quad\Rightarrow$

$$x_2 = 1 - \frac{1 + 2 - 4}{3 \cdot 1^2 + 2} = 1 - \frac{-1}{5} = 1.2 \quad\Rightarrow\quad x_3 = 1.2 - \frac{(1.2)^3 + 2(1.2) - 4}{3(1.2)^2 + 2} \approx 1.1797.$$

7. $f(x) = x^4 - 20 \quad\Rightarrow\quad f'(x) = 4x^3$, so $x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)} = x_n - \dfrac{x_n^4 - 20}{4x_n^3}$.

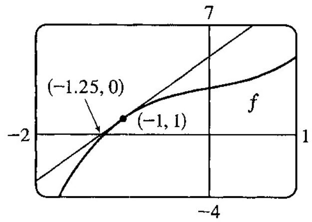
Now $x_1 = 2 \quad\Rightarrow\quad x_2 = 2 - \dfrac{2^4 - 20}{4(2)^3} = 2.125 \quad\Rightarrow\quad x_3 = 2.125 - \dfrac{(2.125)^4 - 20}{4(2.125)^3} \approx 2.1148.$

9. $f(x) = x^3 + x + 3 \quad\Rightarrow\quad f'(x) = 3x^2 + 1$, so

$x_{n+1} = x_n - \dfrac{x_n^3 + x_n + 3}{3x_n^2 + 1}$. Now $x_1 = -1 \quad\Rightarrow$

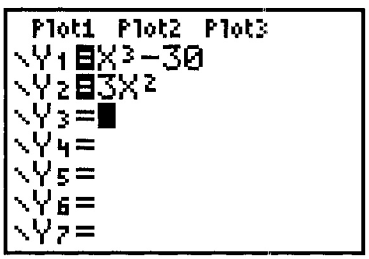
$$x_2 = -1 - \frac{(-1)^3 + (-1) + 3}{3(-1)^2 + 1} = -1 - \frac{-1 - 1 + 3}{3 + 1} = -1 - \frac{1}{4} = -1.25.$$

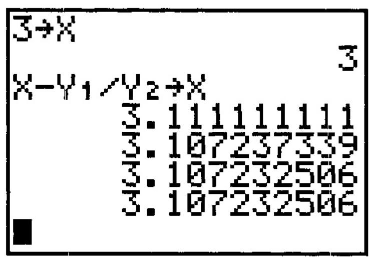
Newton's method follows the tangent line at $(-1, 1)$ down to its intersection with the x-axis at $(-1.25, 0)$, giving the second approximation $x_2 = -1.25$.



11. To approximate $x = \sqrt[3]{30}$ (so that $x^3 = 30$), we can take $f(x) = x^3 - 30$. So $f'(x) = 3x^2$, and thus, $x_{n+1} = x_n - \dfrac{x_n^3 - 30}{3x_n^2}$. Since $\sqrt[3]{27} = 3$ and 27 is close to 30, we'll use $x_1 = 3$. We need to find approximations until they agree to eight decimal places. $x_1 = 3 \quad\Rightarrow\quad x_2 \approx 3.11111111$, $x_3 \approx 3.10723734$, $x_4 \approx 3.10723251 \approx x_5$. So $\sqrt[3]{30} \approx 3.10723251$, to eight decimal places.

Here is a quick and easy method for finding the iterations for Newton's method on a programmable calculator. (The screens shown are from the TI-83 Plus, but the method is similar on other calculators.) Assign $f(x) = x^3 - 30$ to Y_1, and $f'(x) = 3x^2$ to Y_2. Now store $x_1 = 3$ in X and then enter $X - Y_1/Y_2 \to X$ to get $x_2 = 3.\overline{1}$. By successively pressing the ENTER key, you get the approximations $x_3, x_4, \ldots$.





In Derive, load the utility file SOLVE. Enter `NEWTON(x^3-30,x,3)` and then `APPROXIMATE` to get [3, 3.11111111, 3.10723733, 3.10723250, 3.10723250]. You can request a specific iteration by adding a fourth argument. For example, `NEWTON(x^3-30,x,3,2)` gives [3, 3.11111111, 3.10723733].

In Maple, make the assignments $f := x \to x\hat{\ }3 - 30;$, $g := x \to x - f(x)/D(f)(x);$, and $x := 3.;$. Repeatedly execute the command $x := g(x);$ to generate successive approximations.

In Mathematica, make the assignments $f[x\_] := x\hat{\ }3 - 30$, $g[x\_] := x - f[x]/f'[x]$, and $x = 3$. Repeatedly execute the command $x = g[x]$ to generate successive approximations.

29. (a) $f(x) = x^2 - a \quad \Rightarrow \quad f'(x) = 2x$, so Newton's method gives

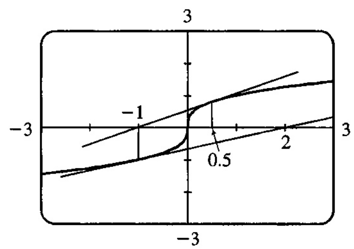
$$x_{n+1} = x_n - \frac{x_n^2 - a}{2x_n} = x_n - \frac{1}{2}x_n + \frac{a}{2x_n} = \frac{1}{2}x_n + \frac{a}{2x_n} = \frac{1}{2}\left(x_n + \frac{a}{x_n}\right).$$

(b) Using (a) with $a = 1000$ and $x_1 = \sqrt{900} = 30$, we get $x_2 \approx 31.666667$, $x_3 \approx 31.622807$, and $x_4 \approx 31.622777 \approx x_5$. So $\sqrt{1000} \approx 31.622777$.

31. $f(x) = x^3 - 3x + 6 \quad \Rightarrow \quad f'(x) = 3x^2 - 3$. If $x_1 = 1$, then $f'(x_1) = 0$ and the tangent line used for approximating x_2 is horizontal. Attempting to find x_2 results in trying to divide by zero.

33. For $f(x) = x^{1/3}$, $f'(x) = \frac{1}{3}x^{-2/3}$ and

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n^{1/3}}{\frac{1}{3}x_n^{-2/3}} = x_n - 3x_n = -2x_n.$$



Therefore, each successive approximation becomes twice as large as the previous one in absolute value, so the sequence of approximations fails to converge to the root, which is 0. In the figure, we have $x_1 = 0.5$, $x_2 = -2(0.5) = -1$, and $x_3 = -2(-1) = 2$.

35. (a) $f(x) = 3x^4 - 28x^3 + 6x^2 + 24x \quad \Rightarrow \quad f'(x) = 12x^3 - 84x^2 + 12x + 24 \quad \Rightarrow$

$f''(x) = 36x^2 - 168x + 12$. Now to solve $f'(x) = 0$, try $x_1 = \frac{1}{2} \quad \Rightarrow \quad x_2 = x_1 - \dfrac{f'(x_1)}{f''(x_1)} = \dfrac{2}{3} \quad \Rightarrow$

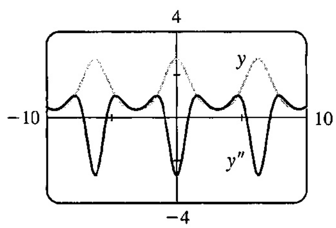
$x_3 \approx 0.6455 \quad \Rightarrow \quad x_4 \approx 0.6452 \quad \Rightarrow \quad x_5 \approx 0.6452$. Now try $x_1 = 6 \quad \Rightarrow \quad x_2 = 7.12 \quad \Rightarrow$

$x_3 \approx 6.8353 \quad \Rightarrow \quad x_4 \approx 6.8102 \quad \Rightarrow \quad x_5 \approx 6.8100$. Finally try $x_1 = -0.5 \quad \Rightarrow \quad x_2 \approx -0.4571 \quad \Rightarrow$

$x_3 \approx -0.4552 \quad \Rightarrow \quad x_4 \approx -0.4552$. Therefore, $x = -0.455$, 6.810 and 0.645 are all critical numbers correct to three decimal places.

(b) $f(-1) = 13$, $f(7) = -1939$, $f(6.810) \approx -1949.07$, $f(-0.455) \approx -6.912$, $f(0.645) \approx 10.982$. Therefore, $f(6.810) \approx -1949.07$ is the absolute minimum correct to two decimal places.

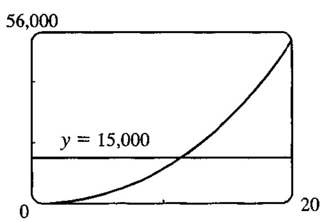
37.



From the figure, we see that $y = f(x) = e^{\cos x}$ is periodic with period 2π. To find the x-coordinates of the IP, we only need to approximate the zeros of y'' on $[0, \pi]$. $f'(x) = -e^{\cos x}\sin x \quad \Rightarrow$

$f''(x) = e^{\cos x}(\sin^2 x - \cos x)$. Since $e^{\cos x} \neq 0$, we will use Newton's method with $g(x) = \sin^2 x - \cos x$, $g'(x) = 2\sin x\,\cos x + \sin x$, and $x_1 = 1$. $x_2 \approx 0.904173$, $x_3 \approx 0.904557 \approx x_4$. Thus, $(0.904557, 1.855277)$ is the IP.

39.



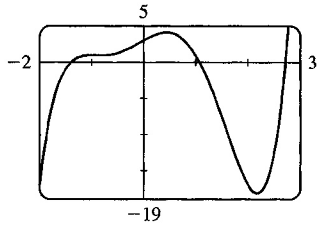
The volume of the silo, in terms of its radius, is

$V(r) = \pi r^2(30) + \frac{1}{2}\left(\frac{4}{3}\pi r^3\right) = 30\pi r^2 + \frac{2}{3}\pi r^3$.

From a graph of V, we see that $V(r) = 15{,}000$ at $r \approx 11$ ft. Now we use Newton's method to solve the equation $V(r) - 15{,}000 = 0$.

$\dfrac{dV}{dr} = 60\pi r + 2\pi r^2$, so $r_{n+1} = r_n - \dfrac{30\pi r_n^2 + \frac{2}{3}\pi r_n^3 - 15{,}000}{60\pi r_n + 2\pi r_n^2}$. Taking $r_1 = 11$, we get $r_2 \approx 11.2853$, $r_3 \approx 11.2807 \approx r_4$. So in order for the silo to hold $15{,}000$ ft^3 of grain, its radius must be about 11.2807 ft.

23.



$f(x) = x^5 - x^4 - 5x^3 - x^2 + 4x + 3 \;\Rightarrow$

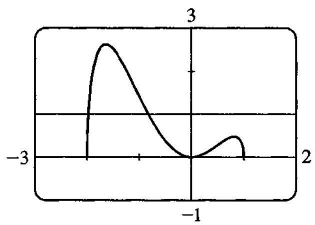
$f'(x) = 5x^4 - 4x^3 - 15x^2 - 2x + 4 \;\Rightarrow$

$x_{n+1} = x_n - \dfrac{x_n^5 - x_n^4 - 5x_n^3 - x_n^2 + 4x_n + 3}{5x_n^4 - 4x_n^3 - 15x_n^2 - 2x_n + 4}$. From the graph of f, there appear to be roots near -1.4, 1.1, and 2.7.

$x_1 = -1.4$	$x_1 = 1.1$	$x_1 = 2.7$
$x_2 \approx -1.39210970$	$x_2 \approx 1.07780402$	$x_2 \approx 2.72046250$
$x_3 \approx -1.39194698$	$x_3 \approx 1.07739442$	$x_3 \approx 2.71987870$
$x_4 \approx -1.39194691 \approx x_5$	$x_4 \approx 1.07739428 \approx x_5$	$x_4 \approx 2.71987822 \approx x_5$

To eight decimal places, the roots of the equation are -1.39194691, 1.07739428, and 2.71987822.

25.



From the graph, $y = x^2\sqrt{2 - x - x^2}$ and $y = 1$ intersect twice, at $x \approx -2$ and at $x \approx -1$. $f(x) = x^2\sqrt{2 - x - x^2} - 1 \;\Rightarrow$

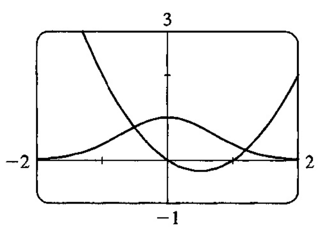
$$\begin{aligned} f'(x) &= x^2 \cdot \tfrac{1}{2}(2 - x - x^2)^{-1/2}(-1 - 2x) + (2 - x - x^2)^{1/2} \cdot 2x \\ &= \tfrac{1}{2}x(2 - x - x^2)^{-1/2}\left[x(-1 - 2x) + 4(2 - x - x^2)\right] \\ &= \frac{x(8 - 5x - 6x^2)}{2\sqrt{(2 + x)(1 - x)}}, \end{aligned}$$

so $x_{n+1} = x_n - \dfrac{x_n^2\sqrt{2 - x_n - x_n^2} - 1}{\dfrac{x_n(8 - 5x_n - 6x_n^2)}{2\sqrt{(2 + x_n)(1 - x_n)}}}$. Trying $x_1 = -2$ won't work because $f'(-2)$ is undefined, so we'll try $x_1 = -1.95$.

$x_1 = -1.95$	$x_1 = -0.8$
$x_2 \approx -1.98580357$	$x_2 \approx -0.82674444$
$x_3 \approx -1.97899778$	$x_3 \approx -0.82646236$
$x_4 \approx -1.97807848$	$x_4 \approx -0.82646233 \approx x_5$
$x_5 \approx -1.97806682$	
$x_6 \approx -1.97806681 \approx x_7$	

To eight decimal places, the roots of the equation are -1.97806681 and -0.82646233.

27.



From the graph, we see that $y = e^{-x^2}$ and $y = x^2 - x$ intersect twice. Good first approximations are $x = -0.5$ and $x = 1.1$.

$f(x) = e^{-x^2} - x^2 + x \;\Rightarrow\; f'(x) = -2xe^{-x^2} - 2x + 1$, so

$$x_{n+1} = x_n - \frac{e^{-x_n^2} - x_n^2 + x_n}{-2x_n e^{-x_n^2} - 2x_n + 1}.$$

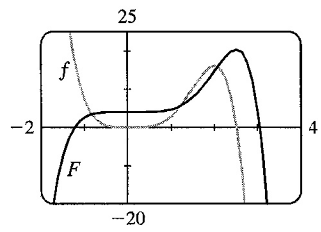
$x_1 = -0.5$	$x_1 = 1.1$
$x_2 \approx -0.51036446$	$x_2 \approx 1.20139754$
$x_3 \approx -0.51031156 \approx x_4$	$x_3 \approx 1.19844118$
	$x_4 \approx 1.19843871 \approx x_5$

To eight decimal places, the roots of the equation are -0.51031156 and 1.19843871.

17. $f(x) = 5x^4 - 2x^5 \quad \Rightarrow \quad F(x) = 5 \cdot \dfrac{x^5}{5} - 2 \cdot \dfrac{x^6}{6} + C = x^5 - \frac{1}{3}x^6 + C.$

$F(0) = 4 \quad \Rightarrow \quad 0^5 - \frac{1}{3} \cdot 0^6 + C = 4 \quad \Rightarrow \quad C = 4$, so

$F(x) = x^5 - \frac{1}{3}x^6 + 4$. The graph confirms our answer since $f(x) = 0$ when F has a local maximum, f is positive when F is increasing, and f is negative when F is decreasing.



19. $f''(x) = 6x + 12x^2 \quad \Rightarrow \quad f'(x) = 6 \cdot \dfrac{x^2}{2} + 12 \cdot \dfrac{x^3}{3} + C = 3x^2 + 4x^3 + C \quad \Rightarrow$

$f(x) = 3 \cdot \dfrac{x^3}{3} + 4 \cdot \dfrac{x^4}{4} + Cx + D = x^3 + x^4 + Cx + D$ [C and D are just arbitrary constants]

21. $f''(x) = 1 + x^{4/5} \quad \Rightarrow \quad f'(x) = x + \frac{5}{9}x^{9/5} + C \quad \Rightarrow$

$f(x) = \frac{1}{2}x^2 + \frac{5}{9} \cdot \frac{5}{14}x^{14/5} + Cx + D = \frac{1}{2}x^2 + \frac{25}{126}x^{14/5} + Cx + D$

23. $f'''(t) = e^t \quad \Rightarrow \quad f''(t) = e^t + C \quad \Rightarrow \quad f'(t) = e^t + Ct + D \quad \Rightarrow \quad f(t) = e^t + \frac{1}{2}Ct^2 + Dt + E$

25. $f'(x) = 1 - 6x \quad \Rightarrow \quad f(x) = x - 3x^2 + C$. $f(0) = C$ and $f(0) = 8 \quad \Rightarrow \quad C = 8$, so $f(x) = x - 3x^2 + 8$.

27. $f'(x) = \sqrt{x}(6 + 5x) = 6x^{1/2} + 5x^{3/2} \quad \Rightarrow \quad f(x) = 4x^{3/2} + 2x^{5/2} + C.$

$f(1) = 6 + C$ and $f(1) = 10 \quad \Rightarrow \quad C = 4$, so $f(x) = 4x^{3/2} + 2x^{5/2} + 4$.

29. $f'(t) = 2\cos t + \sec^2 t \quad \Rightarrow \quad f(t) = 2\sin t + \tan t + C$ because $-\pi/2 < t < \pi/2$.

$f\left(\frac{\pi}{3}\right) = 2\left(\sqrt{3}/2\right) + \sqrt{3} + C = 2\sqrt{3} + C$ and $f\left(\frac{\pi}{3}\right) = 4 \quad \Rightarrow \quad C = 4 - 2\sqrt{3}$, so

$f(t) = 2\sin t + \tan t + 4 - 2\sqrt{3}$.

31. $f'(x) = 2/x \quad \Rightarrow \quad f(x) = 2\ln|x| + C = 2\ln(-x) + C$ (since $x < 0$). Now

$f(-1) = 2\ln 1 + C = 2(0) + C = 7 \quad \Rightarrow \quad C = 7$. Therefore, $f(x) = 2\ln(-x) + 7$, $x < 0$.

33. $f''(x) = 24x^2 + 2x + 10 \quad \Rightarrow \quad f'(x) = 8x^3 + x^2 + 10x + C$. $f'(1) = 8 + 1 + 10 + C$ and $f'(1) = -3 \quad \Rightarrow$

$19 + C = -3 \quad \Rightarrow \quad C = -22$, so $f'(x) = 8x^3 + x^2 + 10x - 22$ and hence,

$f(x) = 2x^4 + \frac{1}{3}x^3 + 5x^2 - 22x + D$. $f(1) = 2 + \frac{1}{3} + 5 - 22 + D$ and $f(1) = 5 \quad \Rightarrow \quad D = 22 - \frac{7}{3} = \frac{59}{3}$,

so $f(x) = 2x^4 + \frac{1}{3}x^3 + 5x^2 - 22x + \frac{59}{3}$.

35. $f''(\theta) = \sin\theta + \cos\theta \quad \Rightarrow \quad f'(\theta) = -\cos\theta + \sin\theta + C$. $f'(0) = -1 + C$ and $f'(0) = 4 \quad \Rightarrow \quad C = 5$, so

$f'(\theta) = -\cos\theta + \sin\theta + 5$ and hence, $f(\theta) = -\sin\theta - \cos\theta + 5\theta + D$. $f(0) = -1 + D$ and $f(0) = 3 \quad \Rightarrow$

$D = 4$, so $f(\theta) = -\sin\theta - \cos\theta + 5\theta + 4$.

37. $f''(x) = 2 - 12x \quad \Rightarrow \quad f'(x) = 2x - 6x^2 + C \quad \Rightarrow \quad f(x) = x^2 - 2x^3 + Cx + D.$

$f(0) = D$ and $f(0) = 9 \quad \Rightarrow \quad D = 9$. $f(2) = 4 - 16 + 2C + 9 = 2C - 3$ and $f(2) = 15 \quad \Rightarrow \quad 2C = 18 \quad \Rightarrow$

$C = 9$, so $f(x) = x^2 - 2x^3 + 9x + 9$.

41. In this case, $A = 18{,}000$, $R = 375$, and $n = 5(12) = 60$. So the formula $A = \frac{R}{i}\left[1-(1+i)^{-n}\right]$ becomes

$18{,}000 = \frac{375}{x}\left[1-(1+x)^{-60}\right] \quad \Leftrightarrow \quad 48x = 1-(1+x)^{-60}$ [multiply each term by $(1+x)^{60}$] $\Leftrightarrow$

$48x(1+x)^{60} - (1+x)^{60} + 1 = 0$. Let the LHS be called $f(x)$, so that

$$\begin{aligned} f'(x) &= 48x(60)(1+x)^{59} + 48(1+x)^{60} - 60(1+x)^{59} \\ &= 12(1+x)^{59}\left[4x(60) + 4(1+x) - 5\right] = 12(1+x)^{59}(244x-1) \end{aligned}$$

$x_{n+1} = x_n - \dfrac{48x_n(1+x_n)^{60} - (1+x_n)^{60} + 1}{12(1+x_n)^{59}(244x_n - 1)}$. An interest rate of 1% per month seems like a reasonable estimate for $x = i$. So let $x_1 = 1\% = 0.01$, and we get $x_2 \approx 0.0082202$, $x_3 \approx 0.0076802$, $x_4 \approx 0.0076291$, $x_5 \approx 0.0076286 \approx x_6$. Thus, the dealer is charging a monthly interest rate of 0.76286% (or 9.55% per year, compounded monthly).

4.10 Antiderivatives

1. $f(x) = 6x^2 - 8x + 3 \quad \Rightarrow \quad F(x) = 6\dfrac{x^{2+1}}{2+1} - 8\dfrac{x^{1+1}}{1+1} + 3x + C = 2x^3 - 4x^2 + 3x + C$

Check: $F'(x) = 2\cdot 3x^2 - 4\cdot 2x + 3 + 0 = 6x^2 - 8x + 3 = f(x)$

3. $f(x) = 1 - x^3 + 5x^5 - 3x^7 \quad \Rightarrow \quad F(x) = x - \dfrac{x^{3+1}}{3+1} + 5\dfrac{x^{5+1}}{5+1} - 3\dfrac{x^{7+1}}{7+1} + C = x - \frac{1}{4}x^4 + \frac{5}{6}x^6 - \frac{3}{8}x^8 + C$

5. $f(x) = 5x^{1/4} - 7x^{3/4} \quad \Rightarrow \quad F(x) = 5\dfrac{x^{1/4+1}}{\frac{1}{4}+1} - 7\dfrac{x^{3/4+1}}{\frac{3}{4}+1} + C = 5\dfrac{x^{5/4}}{5/4} - 7\dfrac{x^{7/4}}{7/4} + C = 4x^{5/4} - 4x^{7/4} + C$

7. $f(x) = 6\sqrt{x} - \sqrt[6]{x} = 6x^{1/2} - x^{1/6} \quad \Rightarrow$

$$F(x) = 6\,\frac{x^{1/2+1}}{\frac{1}{2}+1} - \frac{x^{1/6+1}}{\frac{1}{6}+1} + C = 6\,\frac{x^{3/2}}{3/2} - \frac{x^{7/6}}{7/6} + C = 4x^{3/2} - \tfrac{6}{7}x^{7/6} + C$$

9. $f(x) = \dfrac{10}{x^9} = 10x^{-9}$ has domain $(-\infty, 0) \cup (0, \infty)$, so $F(x) = \begin{cases} \dfrac{10x^{-8}}{-8} + C_1 = -\dfrac{5}{4x^8} + C_1 & \text{if } x < 0 \\ -\dfrac{5}{4x^8} + C_2 & \text{if } x > 0 \end{cases}$

See Example 1 for a similar problem.

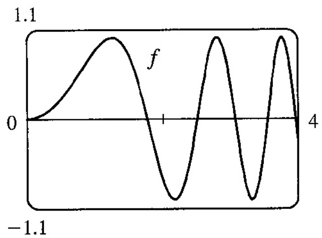
11. $f(u) = \dfrac{u^4 + 3\sqrt{u}}{u^2} = \dfrac{u^4}{u^2} + \dfrac{3u^{1/2}}{u^2} = u^2 + 3u^{-3/2} \quad \Rightarrow$

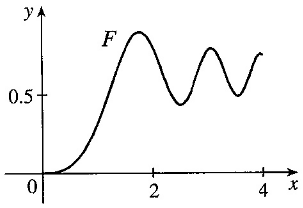
$$F(u) = \frac{u^3}{3} + 3\,\frac{u^{-3/2+1}}{-3/2+1} + C = \frac{1}{3}u^3 + 3\frac{u^{-1/2}}{-1/2} + C = \frac{1}{3}u^3 - \frac{6}{\sqrt{u}} + C$$

13. $g(\theta) = \cos\theta - 5\sin\theta \quad \Rightarrow \quad G(\theta) = \sin\theta - 5(-\cos\theta) + C = \sin\theta + 5\cos\theta + C$

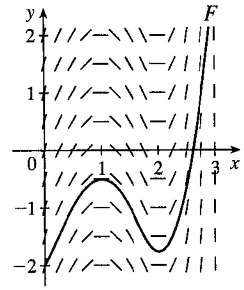
15. $f(x) = 2x + 5(1-x^2)^{-1/2} = 2x + \dfrac{5}{\sqrt{1-x^2}} \quad \Rightarrow \quad F(x) = x^2 + 5\sin^{-1}x + C$

51. $f(x) = \sin(x^2)$, $0 \le x \le 4$





53.

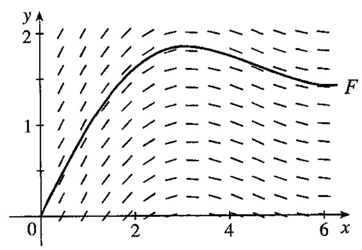


55.

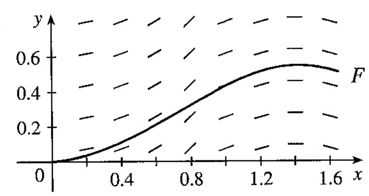
x	$f(x)$
0	1
0.5	0.959
1.0	0.841
1.5	0.665
2.0	0.455
2.5	0.239
3.0	0.047

x	$f(x)$
3.5	−0.100
4.0	−0.189
4.5	−0.217
5.0	−0.192
5.5	−0.128
6.0	−0.047

We compute slopes [values of $f(x) = (\sin x)/x$ for $0 < x < 2\pi$] as in the table [$\lim_{x\to 0^+} f(x) = 1$] and draw a direction field as in Example 6. Then we use the direction field to graph F starting at $(0, 0)$.



57.



Remember that the given table values of f are the slopes of F at any x. For example, at $x = 1.4$, the slope of F is $f(1.4) = 0$.

59. $v(t) = s'(t) = \sin t - \cos t \Rightarrow s(t) = -\cos t - \sin t + C$. $s(0) = -1 + C$ and $s(0) = 0 \Rightarrow C = 1$, so $s(t) = -\cos t - \sin t + 1$.

61. $a(t) = v'(t) = t - 2 \Rightarrow v(t) = \frac{1}{2}t^2 - 2t + C$. $v(0) = C$ and $v(0) = 3 \Rightarrow C = 3$, so $v(t) = \frac{1}{2}t^2 - 2t + 3$ and $s(t) = \frac{1}{6}t^3 - t^2 + 3t + D$. $s(0) = D$ and $s(0) = 1 \Rightarrow D = 1$, and $s(t) = \frac{1}{6}t^3 - t^2 + 3t + 1$.

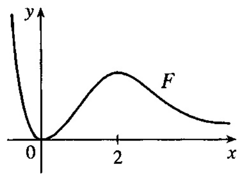
39. $f''(x) = 2 + \cos x \;\Rightarrow\; f'(x) = 2x + \sin x + C \;\Rightarrow\; f(x) = x^2 - \cos x + Cx + D$. $f(0) = -1 + D$ and $f(0) = -1 \;\Rightarrow\; D = 0$. $f\left(\frac{\pi}{2}\right) = \pi^2/4 + \left(\frac{\pi}{2}\right)C$ and $f\left(\frac{\pi}{2}\right) = 0 \;\Rightarrow\; \left(\frac{\pi}{2}\right)C = -\pi^2/4 \;\Rightarrow\; C = -\frac{\pi}{2}$, so $f(x) = x^2 - \cos x - \left(\frac{\pi}{2}\right)x$.

41. $f''(x) = x^{-2}, x > 0 \;\Rightarrow\; f'(x) = -1/x + C \;\Rightarrow\; f(x) = -\ln|x| + Cx + D = -\ln x + Cx + D$ (since $x > 0$). $f(1) = 0 \;\Rightarrow\; C + D = 0$ and $f(2) = 0 \;\Rightarrow\; -\ln 2 + 2C + D = 0 \;\Rightarrow$ $-\ln 2 + 2C - C = 0$ [since $D = -C$] $\Rightarrow\; -\ln 2 + C = 0 \;\Rightarrow\; C = \ln 2$ and $D = -\ln 2$.
So $f(x) = -\ln x + (\ln 2)x - \ln 2$.

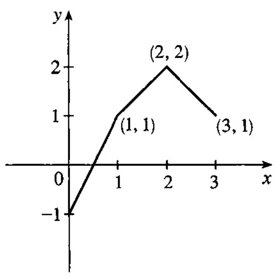
43. Given $f'(x) = 2x + 1$, we have $f(x) = x^2 + x + C$. Since f passes through $(1, 6)$, $f(1) = 6 \;\Rightarrow\; 1^2 + 1 + C = 6 \;\Rightarrow\; C = 4$. Therefore, $f(x) = x^2 + x + 4$ and $f(2) = 2^2 + 2 + 4 = 10$.

45. b is the antiderivative of f. For small x, f is negative, so the graph of its antiderivative must be decreasing. But both a and c are increasing for small x, so only b can be f's antiderivative. Also, f is positive where b is increasing, which supports our conclusion.

47. The graph of F will have a minimum at 0 and a maximum at 2, since $f = F'$ goes from negative to positive at $x = 0$, and from positive to negative at $x = 2$.



49.



$$f'(x) = \begin{cases} 2 & \text{if } 0 \le x < 1 \\ 1 & \text{if } 1 < x < 2 \\ -1 & \text{if } 2 < x \le 3 \end{cases} \;\Rightarrow\; f(x) = \begin{cases} 2x + C & \text{if } 0 \le x < 1 \\ x + D & \text{if } 1 < x < 2 \\ -x + E & \text{if } 2 < x \le 3 \end{cases}$$

$f(0) = -1 \;\Rightarrow\; 2(0) + C = -1 \;\Rightarrow\; C = -1$. Starting at the point $(0, -1)$ and moving to the right on a line with slope 2 gets us to the point $(1, 1)$. The slope for $1 < x < 2$ is 1, so we get to the point $(2, 2)$. Here we have used the fact that f is continuous. We can include the point $x = 1$ on either the first or the second part of f. The line connecting $(1, 1)$ to $(2, 2)$ is $y = x$, so $D = 0$. The slope for $2 < x \le 3$ is -1, so we get to $(3, 1)$. $f(3) = 1 \;\Rightarrow\; -3 + E = 1 \;\Rightarrow\; E = 4$. Thus,

$$f(x) = \begin{cases} 2x - 1 & \text{if } 0 \le x \le 1 \\ x & \text{if } 1 < x < 2 \\ -x + 4 & \text{if } 2 \le x \le 3 \end{cases}$$

Note that $f'(x)$ does not exist at $x = 1$ or at $x = 2$.

77. Let the acceleration be $a(t) = k$ km/h^2. We have $v(0) = 100$ km/h and we can take the initial position $s(0)$ to be 0. We want the time t_f for which $v(t) = 0$ to satisfy $s(t) < 0.08$ km. In general, $v'(t) = a(t) = k$, so $v(t) = kt + C$, where $C = v(0) = 100$. Now $s'(t) = v(t) = kt + 100$, so $s(t) = \frac{1}{2}kt^2 + 100t + D$, where $D = s(0) = 0$. Thus, $s(t) = \frac{1}{2}kt^2 + 100t$. Since $v(t_f) = 0$, we have $kt_f + 100 = 0$ or $t_f = -100/k$, so $s(t_f) = \frac{1}{2}k\left(-\frac{100}{k}\right)^2 + 100\left(-\frac{100}{k}\right) = 10{,}000\left(\frac{1}{2k} - \frac{1}{k}\right) = -\frac{5{,}000}{k}$. The condition $s(t_f)$ must satisfy is $-\frac{5{,}000}{k} < 0.08 \Rightarrow -\frac{5{,}000}{0.08} > k$ [k is negative] $\Rightarrow k < -62{,}500$ km/h^2, or equivalently, $k < -\frac{3125}{648} \approx -4.82$ m/s^2.

79. (a) First note that 90 mi/h $= 90 \times \frac{5280}{3600}$ ft/s $= 132$ ft/s. Then $a(t) = 4$ ft/s^2 $\Rightarrow$ $v(t) = 4t + C$, but $v(0) = 0$ $\Rightarrow$ $C = 0$. Now $4t = 132$ when $t = \frac{132}{4} = 33$ s, so it takes 33 s to reach 132 ft/s. Therefore, taking $s(0) = 0$, we have $s(t) = 2t^2$, $0 \le t \le 33$. So $s(33) = 2178$ ft. 15 minutes $= 15(60) = 900$ s, so for $33 < t \le 933$ we have $v(t) = 132$ ft/s $\Rightarrow$ $s(933) = 132(900) + 2178 = 120{,}978$ ft $= 22.9125$ mi.

(b) As in part (a), the train accelerates for 33 s and travels 2178 ft while doing so. Similarly, it decelerates for 33 s and travels 2178 ft at the end of its trip. During the remaining $900 - 66 = 834$ s it travels at 132 ft/s, so the distance traveled is $132 \cdot 834 = 110{,}088$ ft. Thus, the total distance is $2178 + 110{,}088 + 2178 = 114{,}444$ ft $= 21.675$ mi.

(c) 45 mi $= 45(5280) = 237{,}600$ ft. Subtract $2(2178)$ to take care of the speeding up and slowing down, and we have 233,244 ft at 132 ft/s for a trip of $233{,}244/132 = 1767$ s at 90 mi/h. The total time is $1767 + 2(33) = 1833$ s $= 30$ min 33 s $= 30.55$ min.

(d) $37.5(60) = 2250$ s. $2250 - 2(33) = 2184$ s at maximum speed. $2184(132) + 2(2178) = 292{,}644$ total feet or $292{,}644/5280 = 55.425$ mi.

4 Review

CONCEPT CHECK

1. A function f has an **absolute maximum** at $x = c$ if $f(c)$ is the largest function value on the entire domain of f, whereas f has a **local maximum** at c if $f(c)$ is the largest function value when x is near c. See Figure 4 in Section 4.1.

2. (a) See Theorem 4.1.3.

(b) See the Closed Interval Method before Example 8 in Section 4.1.

3. (a) See Theorem 4.1.4.

(b) See Definition 4.1.6.

63. $a(t) = v'(t) = 10\sin t + 3\cos t \quad \Rightarrow \quad v(t) = -10\cos t + 3\sin t + C \quad \Rightarrow$
$s(t) = -10\sin t - 3\cos t + Ct + D$. $s(0) = -3 + D = 0$ and $s(2\pi) = -3 + 2\pi C + D = 12 \quad \Rightarrow \quad D = 3$ and
$C = \frac{6}{\pi}$. Thus, $s(t) = -10\sin t - 3\cos t + \frac{6}{\pi}t + 3$.

65. (a) We first observe that since the stone is dropped 450 m above the ground, $v(0) = 0$ and $s(0) = 450$.
$v'(t) = a(t) = -9.8 \quad \Rightarrow \quad v(t) = -9.8t + C$. Now $v(0) = 0 \quad \Rightarrow \quad C = 0$, so $v(t) = -9.8t \quad \Rightarrow$
$s(t) = -4.9t^2 + D$. Last, $s(0) = 450 \quad \Rightarrow \quad D = 450 \quad \Rightarrow \quad s(t) = 450 - 4.9t^2$.

(b) The stone reaches the ground when $s(t) = 0$. $450 - 4.9t^2 = 0 \quad \Rightarrow \quad t^2 = 450/4.9 \quad \Rightarrow$
$t_1 = \sqrt{450/4.9} \approx 9.58$ s.

(c) The velocity with which the stone strikes the ground is $v(t_1) = -9.8\sqrt{450/4.9} \approx -93.9$ m/s.

(d) This is just reworking parts (a) and (b) with $v(0) = -5$. Using $v(t) = -9.8t + C$, $v(0) = -5 \quad \Rightarrow$
$0 + C = -5 \quad \Rightarrow \quad v(t) = -9.8t - 5$. So $s(t) = -4.9t^2 - 5t + D$ and $s(0) = 450 \quad \Rightarrow \quad D = 450 \quad \Rightarrow$
$s(t) = -4.9t^2 - 5t + 450$. Solving $s(t) = 0$ by using the quadratic formula gives us
$t = \left(5 \pm \sqrt{8845}\right)/(-9.8) \quad \Rightarrow \quad t_1 \approx 9.09$ s.

67. By Exercise 66 with $a = -9.8$, $s(t) = -4.9t^2 + v_0t + s_0$ and $v(t) = s'(t) = -9.8t + v_0$. So
$[v(t)]^2 = (-9.8t + v_0)^2 = (9.8)^2 t^2 - 19.6v_0t + v_0^2 = v_0^2 + 96.04t^2 - 19.6v_0t = v_0^2 - 19.6\left(-4.9t^2 + v_0t\right)$.
But $-4.9t^2 + v_0t$ is just $s(t)$ without the s_0 term; that is, $s(t) - s_0$. Thus, $[v(t)]^2 = v_0^2 - 19.6\,[s(t) - s_0]$.

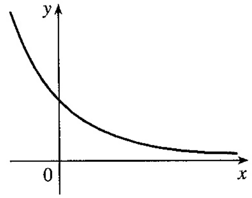
69. Using Exercise 66 with $a = -32$, $v_0 = 0$, and $s_0 = h$ (the height of the cliff), we know that the height at time t is
$s(t) = -16t^2 + h$. $v(t) = s'(t) = -32t$ and $v(t) = -120 \quad \Rightarrow \quad -32t = -120 \quad \Rightarrow \quad t = 3.75$, so
$0 = s(3.75) = -16(3.75)^2 + h \quad \Rightarrow \quad h = 16(3.75)^2 = 225$ ft.

71. Marginal cost $= 1.92 - 0.002x = C'(x) \quad \Rightarrow \quad C(x) = 1.92x - 0.001x^2 + K$. But
$C(1) = 1.92 - 0.001 + K = 562 \quad \Rightarrow \quad K = 560.081$. Therefore, $C(x) = 1.92x - 0.001x^2 + 560.081 \quad \Rightarrow$
$C(100) = 742.081$, so the cost of producing 100 items is \$742.08.

73. Taking the upward direction to be positive we have that for $0 \le t \le 10$ (using the subscript 1 to refer to
$0 \le t \le 10$), $a_1(t) = -(9 - 0.9t) = v_1'(t) \quad \Rightarrow \quad v_1(t) = -9t + 0.45t^2 + v_0$, but $v_1(0) = v_0 = -10 \quad \Rightarrow$
$v_1(t) = -9t + 0.45t^2 - 10 = s_1'(t) \quad \Rightarrow \quad s_1(t) = -\frac{9}{2}t^2 + 0.15t^3 - 10t + s_0$. But $s_1(0) = 500 = s_0 \quad \Rightarrow$
$s_1(t) = -\frac{9}{2}t^2 + 0.15t^3 - 10t + 500$. $s_1(10) = -450 + 150 - 100 + 500 = 100$, so it takes more
than 10 seconds for the raindrop to fall. Now for $t > 10$, $a(t) = 0 = v'(t) \quad \Rightarrow$
$v(t) = \text{constant} = v_1(10) = -9(10) + 0.45(10)^2 - 10 = -55 \quad \Rightarrow \quad v(t) = -55$. At 55 ft/s, it will take
$100/55 \approx 1.8$ s to fall the last 100 ft. Hence, the total time is $10 + \frac{100}{55} = \frac{130}{11} \approx 11.8$ s.

75. $a(t) = k$, the initial velocity is 30 mi/h $= 30 \cdot \frac{5280}{3600} = 44$ ft/s, and the final velocity (after 5 seconds) is
50 mi/h $= 50 \cdot \frac{5280}{3600} = \frac{220}{3}$ ft/s. So $v(t) = kt + C$ and $v(0) = 44 \quad \Rightarrow \quad C = 44$. Thus, $v(t) = kt + 44 \quad \Rightarrow$
$v(5) = 5k + 44$. But $v(5) = \frac{220}{3}$, so $5k + 44 = \frac{220}{3} \quad \Rightarrow \quad 5k = \frac{88}{3} \quad \Rightarrow \quad k = \frac{88}{15} \approx 5.87$ ft/s^2.

9. True. The graph of one such function is sketched.



11. True. Let $x_1 < x_2$ where $x_1, x_2 \in I$. Then $f(x_1) < f(x_2)$ and $g(x_1) < g(x_2)$ (since f and g are increasing on I), so $(f+g)(x_1) = f(x_1) + g(x_1) < f(x_2) + g(x_2) = (f+g)(x_2)$.

13. False. Take $f(x) = x$ and $g(x) = x - 1$. Then both f and g are increasing on $(0,1)$. But $f(x)g(x) = x(x-1)$ is not increasing on $(0,1)$.

15. True. Let $x_1, x_2 \in I$ and $x_1 < x_2$. Then $f(x_1) < f(x_2)$ (f is increasing) $\Rightarrow$ $\dfrac{1}{f(x_1)} > \dfrac{1}{f(x_2)}$ (f is positive) $\Rightarrow$ $g(x_1) > g(x_2)$ $\Rightarrow$ $g(x) = 1/f(x)$ is decreasing on I.

17. True. By the Mean Value Theorem, there exists a number c in $(0,1)$ such that $f(1) - f(0) = f'(c)(1-0) = f'(c)$. Since $f'(c)$ is nonzero, $f(1) - f(0) \neq 0$, so $f(1) \neq f(0)$.

EXERCISES

1. $f(x) = 10 + 27x - x^3$, $0 \le x \le 4$. $f'(x) = 27 - 3x^2 = -3(x^2 - 9) = -3(x+3)(x-3) = 0$ only when $x = 3$ (since -3 is not in the domain). $f'(x) > 0$ for $x < 3$ and $f'(x) < 0$ for $x > 3$, so $f(3) = 64$ is a local maximum value. Checking the endpoints, we find $f(0) = 10$ and $f(4) = 54$. Thus, $f(0) = 10$ is the absolute minimum value and $f(3) = 64$ is the absolute maximum value.

3. $f(x) = \dfrac{x}{x^2+x+1}$, $-2 \le x \le 0$. $f'(x) = \dfrac{(x^2+x+1)(1) - x(2x+1)}{(x^2+x+1)^2} = \dfrac{1-x^2}{(x^2+x+1)^2} = 0 \iff$ $x = -1$ (since 1 is not in the domain). $f'(x) < 0$ for $-2 < x < -1$ and $f'(x) > 0$ for $-1 < x < 0$, so $f(-1) = -1$ is a local and absolute minimum value. $f(-2) = -\frac{2}{3}$ and $f(0) = 0$, so $f(0) = 0$ is an absolute maximum value.

5. $f(x) = x + \sin 2x$, $[0, \pi]$. $f'(x) = 1 + 2\cos 2x = 0 \iff \cos 2x = -\frac{1}{2} \iff 2x = \frac{2\pi}{3}$ or $\frac{4\pi}{3} \iff x = \frac{\pi}{3}$ or $\frac{2\pi}{3}$. $f''(x) = -4\sin 2x$, so $f''\left(\frac{\pi}{3}\right) = -4\sin\frac{2\pi}{3} = -2\sqrt{3} < 0$ and $f''\left(\frac{2\pi}{3}\right) = -4\sin\frac{4\pi}{3} = 2\sqrt{3} > 0$, so $f\left(\frac{\pi}{3}\right) = \frac{\pi}{3} + \frac{\sqrt{3}}{2} \approx 1.91$ is a local maximum value and $f\left(\frac{2\pi}{3}\right) = \frac{2\pi}{3} - \frac{\sqrt{3}}{2} \approx 1.23$ is a local minimum value. Also $f(0) = 0$ and $f(\pi) = \pi$, so $f(0) = 0$ is the absolute minimum value and $f(\pi) = \pi$ is the absolute maximum value.

7. $\displaystyle \lim_{x\to 0} \frac{\tan \pi x}{\ln(1+x)} \overset{\text{H}}{=} \lim_{x\to 0} \frac{\pi \sec^2 \pi x}{1/(1+x)} = \frac{\pi \cdot 1^2}{1/1} = \pi$

9. $\displaystyle \lim_{x\to 0} \frac{e^{4x} - 1 - 4x}{x^2} \overset{\text{H}}{=} \lim_{x\to 0} \frac{4e^{4x} - 4}{2x} \overset{\text{H}}{=} \lim_{x\to 0} \frac{16e^{4x}}{2} = \lim_{x\to 0} 8e^{4x} = 8 \cdot 1 = 8$

11. $\displaystyle \lim_{x\to\infty} x^3 e^{-x} = \lim_{x\to\infty} \frac{x^3}{e^x} \overset{\text{H}}{=} \lim_{x\to\infty} \frac{3x^2}{e^x} \overset{\text{H}}{=} \lim_{x\to\infty} \frac{6x}{e^x} \overset{\text{H}}{=} \lim_{x\to\infty} \frac{6}{e^x} = 0$

4. (a) See Rolle's Theorem at the beginning of Section 4.2.

(b) See the Mean Value Theorem in Section 4.2. Geometric interpretation—there is some point P on the graph of a function f [on the interval (a, b)] where the tangent line is parallel to the secant line that connects $(a, f(a))$ and $(b, f(b))$.

5. (a) See the I/D Test before Example 1 in Section 4.3.

(b) See the Concavity Test before Example 4 in Section 4.3.

6. (a) See the First Derivative Test after Example 1 in Section 4.3.

(b) See the Second Derivative Test before Example 6 in Section 4.3.

(c) See the note before Example 7 in Section 4.3.

7. (a) See l'Hospital's Rule and the three notes that follow it in Section 4.4.

(b) Write fg as $\dfrac{f}{1/g}$ or $\dfrac{g}{1/f}$.

(c) Convert the difference into a quotient using a common denominator, rationalizing, factoring, or some other method.

(d) Convert the power to a product by taking the natural logarithm of both sides of $y = f^g$ or by writing f^g as $e^{g \ln f}$.

8. Without calculus you could get misleading graphs that fail to show the most interesting features of a function. See the discussion following Figure 3 in Section 4.5 and the first paragraph in Section 4.6.

9. (a) See Figure 3 in Section 4.9.

(b) $x_2 = x_1 - \dfrac{f(x_1)}{f'(x_1)}$

(c) $x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)}$

(d) Newton's method is likely to fail or to work very slowly when $f'(x_1)$ is close to 0.

10. (a) See the definition at the beginning of Section 4.10.

(b) If F_1 and F_2 are both antiderivatives of f on an interval I, then they differ by a constant.

TRUE-FALSE QUIZ

1. False. For example, take $f(x) = x^3$, then $f'(x) = 3x^2$ and $f'(0) = 0$, but $f(0) = 0$ is not a maximum or minimum; $(0, 0)$ is an inflection point.

3. False. For example, $f(x) = x$ is continuous on $(0, 1)$ but attains neither a maximum nor a minimum value on $(0, 1)$. Don't confuse this with f being continuous on the *closed* interval $[a, b]$, which would make the statement true.

5. True. This is an example of part (b) of the I/D Test.

7. False. $f'(x) = g'(x) \;\Rightarrow\; f(x) = g(x) + C$. For example, if $f(x) = x + 2$ and $g(x) = x + 1$, then $f'(x) = g'(x) = 1$, but $f(x) \neq g(x)$.

23. $y = f(x) = \dfrac{1}{x(x-3)^2}$ **A.** $D = \{x \mid x \neq 0, 3\} = (-\infty, 0) \cup (0, 3) \cup (3, \infty)$ **B.** No intercepts.

C. No symmetry. **D.** $\displaystyle\lim_{x\to\pm\infty} \frac{1}{x(x-3)^2} = 0$, so $y = 0$ is a HA. $\displaystyle\lim_{x\to 0^+} \frac{1}{x(x-3)^2} = \infty$,

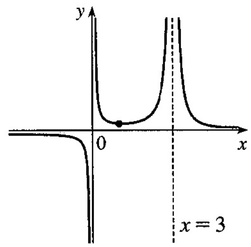
$\displaystyle\lim_{x\to 0^-} \frac{1}{x(x-3)^2} = -\infty$, $\displaystyle\lim_{x\to 3} \frac{1}{x(x-3)^2} = \infty$, so $x = 0$ and $x = 3$ are VA.

E. $f'(x) = -\dfrac{(x-3)^2 + 2x(x-3)}{x^2(x-3)^4} = \dfrac{3(1-x)}{x^2(x-3)^3} \Rightarrow$

$f'(x) > 0 \Leftrightarrow 1 < x < 3$, so f is increasing on $(1, 3)$ and decreasing on $(-\infty, 0)$, $(0, 1)$, and $(3, \infty)$. **F.** Local minimum value $f(1) = \frac{1}{4}$

G. $f''(x) = \dfrac{6(2x^2 - 4x + 3)}{x^3(x-3)^4}$. Note that $2x^2 - 4x + 3 > 0$ for all x since it has negative discriminant. So $f''(x) > 0 \Leftrightarrow x > 0 \Rightarrow f$ is CU on $(0, 3)$ and $(3, \infty)$ and CD on $(-\infty, 0)$. No IP

H.



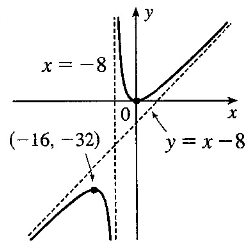
25. $y = f(x) = \dfrac{x^2}{x+8} = x - 8 + \dfrac{64}{x+8}$ **A.** $D = \{x \mid x \neq -8\}$ **B.** Intercepts are 0 **C.** No symmetry

D. $\displaystyle\lim_{x\to\infty} \frac{x^2}{x+8} = \infty$, but $f(x) - (x - 8) = \dfrac{64}{x+8} \to 0$ as $x \to \infty$, so $y = x - 8$ is a slant asymptote.

$\displaystyle\lim_{x\to -8^+} \frac{x^2}{x+8} = \infty$ and $\displaystyle\lim_{x\to -8^-} \frac{x^2}{x+8} = -\infty$, so $x = -8$ is a VA.

E. $f'(x) = 1 - \dfrac{64}{(x+8)^2} = \dfrac{x(x+16)}{(x+8)^2} > 0 \Leftrightarrow x > 0$ or $x < -16$, so f is increasing on $(-\infty, -16)$ and $(0, \infty)$ and decreasing on $(-16, -8)$ and $(-8, 0)$. **F.** Local maximum value $f(-16) = -32$, local minimum value $f(0) = 0$ **G.** $f''(x) = 128/(x+8)^3 > 0 \Leftrightarrow x > -8$, so f is CU on $(-8, \infty)$ and CD on $(-\infty, -8)$. No IP

H.

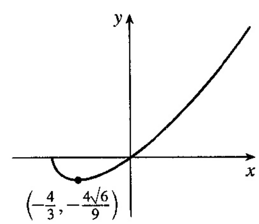


27. $y = f(x) = x\sqrt{2+x}$ **A.** $D = [-2, \infty)$ **B.** y-intercept: $f(0) = 0$; x-intercepts: -2 and 0 **C.** No symmetry **D.** No asymptote **E.** $f'(x) = \dfrac{x}{2\sqrt{2+x}} + \sqrt{2+x} = \dfrac{1}{2\sqrt{2+x}}[x + 2(2+x)] = \dfrac{3x+4}{2\sqrt{2+x}} = 0$ when $x = -\frac{4}{3}$, so f is decreasing on $\left(-2, -\frac{4}{3}\right)$ and increasing on $\left(-\frac{4}{3}, \infty\right)$. **F.** Local minimum value $f\left(-\frac{4}{3}\right) = -\frac{4}{3}\sqrt{\frac{2}{3}} = -\frac{4\sqrt{6}}{9} \approx -1.09$, no local maximum

G. $$f''(x) = \frac{2\sqrt{2+x} \cdot 3 - (3x+4)\dfrac{1}{\sqrt{2+x}}}{4(2+x)} = \frac{6(2+x) - (3x+4)}{4(2+x)^{3/2}} = \frac{3x+8}{4(2+x)^{3/2}}$$

$f''(x) > 0$ for $x > -2$, so f is CU on $(-2, \infty)$. No IP

H.



13. $\displaystyle\lim_{x\to1^+}\left(\frac{x}{x-1}-\frac{1}{\ln x}\right)=\lim_{x\to1^+}\left(\frac{x\ln x-x+1}{(x-1)\ln x}\right)\overset{\text{H}}{=}\lim_{x\to1^+}\frac{x\cdot(1/x)+\ln x-1}{(x-1)\cdot(1/x)+\ln x}$

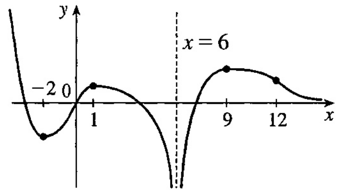
$$=\lim_{x\to1^+}\frac{\ln x}{1-1/x+\ln x}\overset{\text{H}}{=}\lim_{x\to1^+}\frac{1/x}{1/x^2+1/x}=\frac{1}{1+1}=\frac{1}{2}$$

15. $f(0)=0$, $f'(-2)=f'(1)=f'(9)=0$, $\lim\limits_{x\to\infty}f(x)=0$,

$\lim\limits_{x\to6}f(x)=-\infty$, $f'(x)<0$ on $(-\infty,-2)$, $(1,6)$, and $(9,\infty)$,

$f'(x)>0$ on $(-2,1)$ and $(6,9)$, $f''(x)>0$ on $(-\infty,0)$

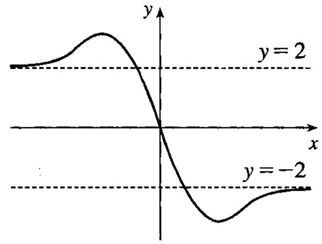
and $(12,\infty)$, $f''(x)<0$ on $(0,6)$ and $(6,12)$



17. f is odd, $f'(x)<0$ for $0<x<2$, $f'(x)>0$ for $x>2$,

$f''(x)>0$ for $0<x<3$, $f''(x)<0$ for $x>3$,

$\lim_{x\to\infty}f(x)=-2$



19. $y=f(x)=2-2x-x^3$ **A.** $D=\mathbb{R}$ **B.** y-intercept: $f(0)=2$.

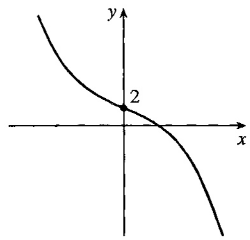
The x-intercept (approximately 0.770917) can be found using Newton's Method. **C.** No symmetry **D.** No asymptote

E. $f'(x)=-2-3x^2=-(3x^2+2)<0$, so f is decreasing on $\mathbb{R}$.

F. No extreme value **G.** $f''(x)=-6x<0$ on $(0,\infty)$ and $f''(x)>0$ on $(-\infty,0)$, so f is CD on $(0,\infty)$ and CU on $(-\infty,0)$.

There is an IP at $(0,2)$.

H.



21. $y=f(x)=x^4-3x^3+3x^2-x=x(x-1)^3$ **A.** $D=\mathbb{R}$ **B.** y-intercept: $f(0)=0$; x-intercepts: $f(x)=0$ $\Leftrightarrow$ $x=0$ or $x=1$ **C.** No symmetry **D.** f is a polynomial function and hence, it has no asymptote.

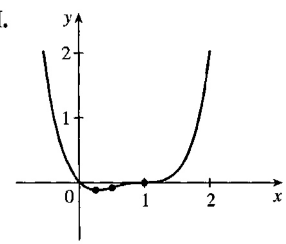
E. $f'(x)=4x^3-9x^2+6x-1$. Since the sum of the coefficients is 0, 1 is a root of f', so

$f'(x)=(x-1)\left(4x^2-5x+1\right)=(x-1)^2(4x-1)$. $f'(x)<0$ $\Rightarrow$ $x<\frac{1}{4}$, so f is decreasing on $\left(-\infty,\frac{1}{4}\right)$ and f is increasing on $\left(\frac{1}{4},\infty\right)$. **F.** $f'(x)$ does not change sign at $x=1$, so there is not a local extremum there. $f\left(\frac{1}{4}\right)=-\frac{27}{256}$ is a local minimum value. **G.** $f''(x)=12x^2-18x+6=6(2x-1)(x-1)$.

$f''(x)=0$ $\Leftrightarrow$ $x=\frac{1}{2}$ or 1. $f''(x)<0$ $\Leftrightarrow$ $\frac{1}{2}<x<1$ $\Rightarrow$

f is CD on $\left(\frac{1}{2},1\right)$ and CU on $\left(-\infty,\frac{1}{2}\right)$ and $(1,\infty)$. There are inflection points at $\left(\frac{1}{2},-\frac{1}{16}\right)$ and $(1,0)$.

H.

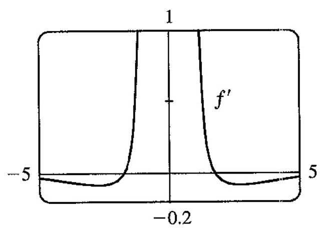


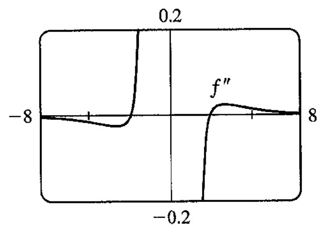
35. $f(x) = \dfrac{x^2-1}{x^3} \quad \Rightarrow \quad f'(x) = \dfrac{x^3(2x) - (x^2-1)3x^2}{x^6} = \dfrac{3-x^2}{x^4} \quad \Rightarrow$

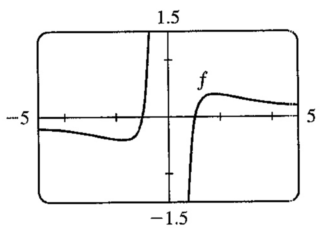
$f''(x) = \dfrac{x^4(-2x) - (3-x^2)4x^3}{x^8} = \dfrac{2x^2-12}{x^5}$

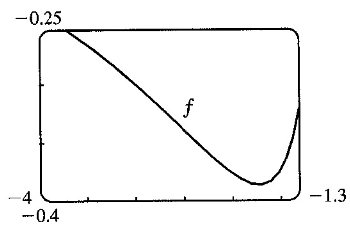
Estimates: From the graphs of f' and f'', it appears that f is increasing on $(-1.73, 0)$ and $(0, 1.73)$ and decreasing on $(-\infty, -1.73)$ and $(1.73, \infty)$; f has a local maximum of about $f(1.73) = 0.38$ and a local minimum of about $f(-1.7) = -0.38$; f is CU on $(-2.45, 0)$ and $(2.45, \infty)$, and CD on $(-\infty, -2.45)$ and $(0, 2.45)$; and f has inflection points at about $(-2.45, -0.34)$ and $(2.45, 0.34)$.

Exact: Now $f'(x) = \dfrac{3-x^2}{x^4}$ is positive for $0 < x^2 < 3$, that is, f is increasing on $\left(-\sqrt{3}, 0\right)$ and $\left(0, \sqrt{3}\right)$; and $f'(x)$ is negative (and so f is decreasing) on $\left(-\infty, -\sqrt{3}\right)$ and $\left(\sqrt{3}, \infty\right)$. $f'(x) = 0$ when $x = \pm\sqrt{3}$. f' goes from positive to negative at $x = \sqrt{3}$, so f has a local maximum of $f\left(\sqrt{3}\right) = \frac{\left(\sqrt{3}\right)^2 - 1}{\left(\sqrt{3}\right)^3} = \frac{2\sqrt{3}}{9}$; and since f is odd, we know that maxima on the interval $(0, \infty)$ correspond to minima on $(-\infty, 0)$, so f has a local minimum of $f\left(-\sqrt{3}\right) = -\frac{2\sqrt{3}}{9}$. Also, $f''(x) = \dfrac{2x^2-12}{x^5}$ is positive (so f is CU) on $\left(-\sqrt{6}, 0\right)$ and $\left(\sqrt{6}, \infty\right)$, and negative (so f is CD) on $\left(-\infty, -\sqrt{6}\right)$ and $\left(0, \sqrt{6}\right)$. There are IP at $\left(\sqrt{6}, \frac{5\sqrt{6}}{36}\right)$ and $\left(-\sqrt{6}, -\frac{5\sqrt{6}}{36}\right)$.



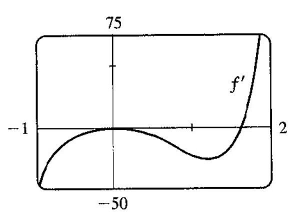


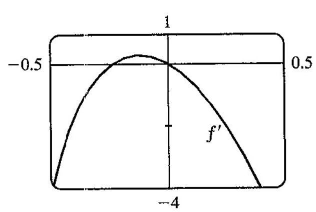


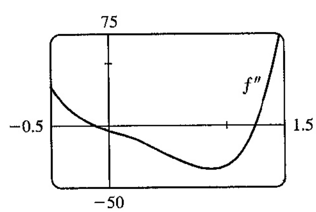


37. $f(x) = 3x^6 - 5x^5 + x^4 - 5x^3 - 2x^2 + 2 \quad \Rightarrow \quad f'(x) = 18x^5 - 25x^4 + 4x^3 - 15x^2 - 4x \quad \Rightarrow$

$f''(x) = 90x^4 - 100x^3 + 12x^2 - 30x - 4$







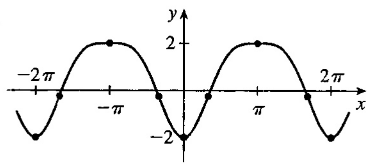
29. $y = f(x) = \sin^2 x - 2\cos x$ **A.** $D = \mathbb{R}$ **B.** y-intercept: $f(0) = -2$ **C.** $f(-x) = f(x)$, so f is symmetric with respect to the y-axis. f has period 2π. **D.** No asymptote

E. $y' = 2\sin x\cos x + 2\sin x = 2\sin x\,(\cos x + 1)$. $y' = 0 \Leftrightarrow \sin x = 0$ or $\cos x = -1 \Leftrightarrow x = n\pi$ or $x = (2n+1)\pi$. $y' > 0$ when $\sin x > 0$, since $\cos x + 1 \geq 0$ for all x. Therefore, $y' > 0$ (and so f is increasing) on $(2n\pi, (2n+1)\pi)$; $y' < 0$ (and so f is decreasing) on $((2n-1)\pi, 2n\pi)$. **F.** Local maximum values are $f((2n+1)\pi) = 2$; local minimum values are $f(2n\pi) = -2$. **G.** $y' = \sin 2x + 2\sin x \Rightarrow$

$$y'' = 2\cos 2x + 2\cos x = 2(2\cos^2 x - 1) + 2\cos x = 4\cos^2 x + 2\cos x - 2$$
$$= 2(2\cos^2 x + \cos x - 1) = 2(2\cos x - 1)(\cos x + 1)$$

$y'' = 0 \Leftrightarrow \cos x = \frac{1}{2}$ or $-1 \Leftrightarrow x = 2n\pi \pm \frac{\pi}{3}$ or $x = (2n+1)\pi$. $y'' > 0$ (and so f is CU) on $(2n\pi - \frac{\pi}{3}, 2n\pi + \frac{\pi}{3})$; $y'' \leq 0$ (and so f is CD) on $(2n\pi + \frac{\pi}{3}, 2n\pi + \frac{5\pi}{3})$. There are inflection points at $(2n\pi \pm \frac{\pi}{3}, -\frac{1}{4})$.

H.



31. $y = f(x) = \sin^{-1}(1/x)$ **A.** $D = \{x \mid -1 \leq 1/x \leq 1\} = (-\infty, -1] \cup [1, \infty)$. **B.** No intercept

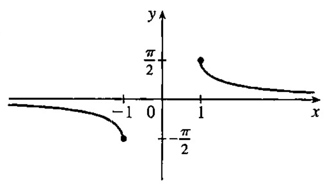
C. $f(-x) = -f(x)$, symmetric about the origin **D.** $\lim_{x\to\pm\infty} \sin^{-1}(1/x) = \sin^{-1}(0) = 0$, so $y = 0$ is a HA.

E. $f'(x) = \dfrac{1}{\sqrt{1-(1/x)^2}}\left(-\dfrac{1}{x^2}\right) = \dfrac{-1}{\sqrt{x^4 - x^2}} < 0$, so f is decreasing on $(-\infty, -1)$ and $(1, \infty)$.

F. No local extreme value, but $f(1) = \frac{\pi}{2}$ is the absolute maximum value and $f(-1) = -\frac{\pi}{2}$ is the absolute minimum value.

G. $f''(x) = \dfrac{4x^3 - 2x}{2(x^4 - x^2)^{3/2}} = \dfrac{x(2x^2 - 1)}{(x^4 - x^2)^{3/2}} > 0$ for $x > 1$ and $f''(x) < 0$ for $x < -1$, so f is CU on $(1, \infty)$ and CD on $(-\infty, -1)$. No IP

H.



33. $y = f(x) = e^x + e^{-3x}$ **A.** $D = \mathbb{R}$ **B.** y-intercept 2; no x-intercept **C.** No symmetry

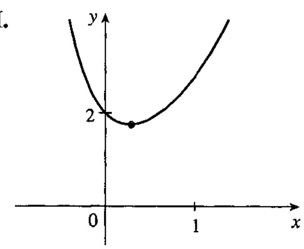
D. $\lim_{x\to\pm\infty} (e^x + e^{-3x}) = \infty$, no asymptote **E.** $y = f(x) = e^x + e^{-3x} \Rightarrow$

$f'(x) = e^x - 3e^{-3x} = e^{-3x}(e^{4x} - 3) > 0 \Leftrightarrow e^{4x} > 3 \Leftrightarrow$ $4x > \ln 3 \Leftrightarrow x > \frac{1}{4}\ln 3 \approx 0.27$, so f is increasing on $(\frac{1}{4}\ln 3, \infty)$ and decreasing on $(-\infty, \frac{1}{4}\ln 3)$.

F. Absolute minimum value $f(\frac{1}{4}\ln 3) = 3^{1/4} + 3^{-3/4} \approx 1.75$.

G. $f''(x) = e^x + 9e^{-3x} > 0$, so f is CU on $(-\infty, \infty)$. No IP

H.



45. $f(x) = x^{101} + x^{51} + x - 1 = 0$. Since f is continuous and $f(0) = -1$ and $f(1) = 2$, the equation has at least one root in $(0, 1)$, by the Intermediate Value Theorem. Suppose the equation has two roots, a and b, with $a < b$. Then $f(a) = 0 = f(b)$, so by the Mean Value Theorem, there is a number x in (a, b) such that

$f'(x) = \dfrac{f(b) - f(a)}{b - a} = \dfrac{0}{b - a} = 0$, so f' has a root in (a, b). But this is impossible since

$f'(x) = 101x^{100} + 51x^{50} + 1 \geq 1$ for all x.

47. Since f is continuous on $[32, 33]$ and differentiable on $(32, 33)$, then by the Mean Value Theorem there exists a number c in $(32, 33)$ such that $f'(c) = \frac{1}{5}c^{-4/5} = \dfrac{\sqrt[5]{33} - \sqrt[5]{32}}{33 - 32} = \sqrt[5]{33} - 2$, but $\frac{1}{5}c^{-4/5} > 0 \quad \Rightarrow \quad \sqrt[5]{33} - 2 > 0$

$\Rightarrow \quad \sqrt[5]{33} > 2$. Also f' is decreasing, so that $f'(c) < f'(32) = \frac{1}{5}(32)^{-4/5} = 0.0125 \quad \Rightarrow$

$0.0125 > f'(c) = \sqrt[5]{33} - 2 \quad \Rightarrow \quad \sqrt[5]{33} < 2.0125$. Therefore, $2 < \sqrt[5]{33} < 2.0125$.

49. (a) $g(x) = f(x^2) \quad \Rightarrow \quad g'(x) = 2xf'(x^2)$ by the Chain Rule. Since $f'(x) > 0$ for all $x \neq 0$, we must have $f'(x^2) > 0$ for $x \neq 0$, so $g'(x) = 0 \quad \Leftrightarrow \quad x = 0$. Now $g'(x)$ changes sign (from negative to positive) at $x = 0$, since one of its factors, $f'(x^2)$, is positive for all x, and its other factor, $2x$, changes from negative to positive at this point, so by the First Derivative Test, f has a local and absolute minimum at $x = 0$.

(b) $g'(x) = 2xf'(x^2) \quad \Rightarrow \quad g''(x) = 2\left[xf''(x^2)(2x) + f'(x^2)\right] = 4x^2f''(x^2) + 2f'(x^2)$ by the Product Rule and the Chain Rule. But $x^2 > 0$ for all $x \neq 0$, $f''(x^2) > 0$ (since f is CU for $x > 0$), and $f'(x^2) > 0$ for all $x \neq 0$, so since all of its factors are positive, $g''(x) > 0$ for $x \neq 0$. Whether $g''(0)$ is positive or 0 doesn't matter (since the sign of g'' does not change there); g is concave upward on $\mathbb{R}$.

51. If $B = 0$, the line is vertical and the distance from $x = -\dfrac{C}{A}$ to (x_1, y_1) is $\left|x_1 + \dfrac{C}{A}\right| = \dfrac{|Ax_1 + By_1 + C|}{\sqrt{A^2 + B^2}}$, so assume $B \neq 0$. The square of the distance from (x_1, y_1) to the line is $f(x) = (x - x_1)^2 + (y - y_1)^2$ where

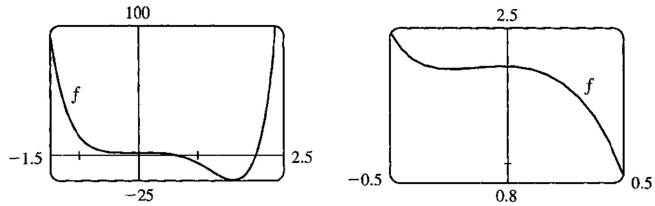
$Ax + By + C = 0$, so we minimize $f(x) = (x - x_1)^2 + \left(-\dfrac{A}{B}x - \dfrac{C}{B} - y_1\right)^2 \quad \Rightarrow$

$f'(x) = 2(x - x_1) + 2\left(-\dfrac{A}{B}x - \dfrac{C}{B} - y_1\right)\left(-\dfrac{A}{B}\right)$. $f'(x) = 0 \quad \Rightarrow \quad x = \dfrac{B^2x_1 - ABy_1 - AC}{A^2 + B^2}$ and this gives

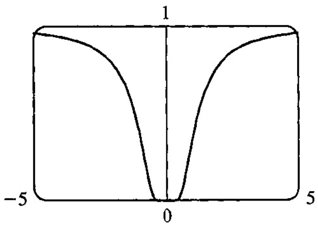
a minimum since $f''(x) = 2\left(1 + \dfrac{A^2}{B^2}\right) > 0$. Substituting this value of x into $f(x)$ and simplifying gives

$f(x) = \dfrac{(Ax_1 + By_1 + C)^2}{A^2 + B^2}$, so the minimum distance is $\sqrt{f(x)} = \dfrac{|Ax_1 + By_1 + C|}{\sqrt{A^2 + B^2}}$.

From the graphs of f' and f'', it appears that f is increasing on $(-0.23, 0)$ and $(1.62, \infty)$ and decreasing on $(-\infty, -0.23)$ and $(0, 1.62)$; f has a local maximum of about $f(0) = 2$ and local minima of about $f(-0.23) = 1.96$ and $f(1.62) = -19.2$; f is CU on $(-\infty, -0.12)$ and $(1.24, \infty)$ and CD on $(-0.12, 1.24)$; and f has inflection points at about $(-0.12, 1.98)$ and $(1.24, -12.1)$.



39.

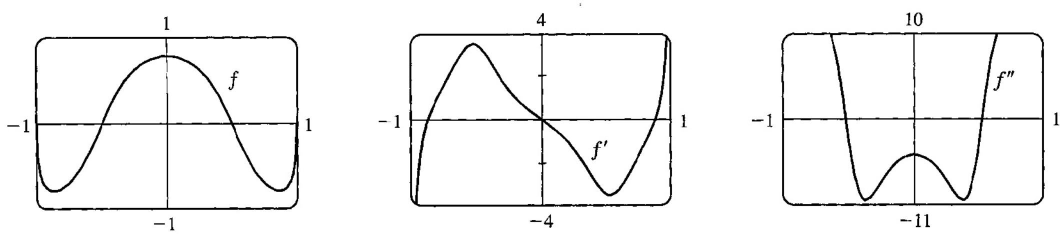


From the graph, we estimate the points of inflection to be about $(\pm 0.82, 0.22)$. $f(x) = e^{-1/x^2} \;\Rightarrow\; f'(x) = 2x^{-3}e^{-1/x^2} \;\Rightarrow$

$$f''(x) = 2\left[x^{-3}(2x^{-3})e^{-1/x^2} + e^{-1/x^2}(-3x^{-4})\right] = 2x^{-6}e^{-1/x^2}\left(2 - 3x^2\right).$$

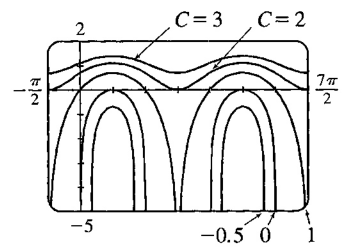
This is 0 when $2 - 3x^2 = 0 \;\Leftrightarrow\; x = \pm\sqrt{\frac{2}{3}}$, so the inflection points are $\left(\pm\sqrt{\frac{2}{3}}, e^{-3/2}\right)$.

41. $f(x) = \arctan(\cos(3\arcsin x))$. We use a CAS to compute f' and f'', and to graph f, f', and f'':



From the graph of f', it appears that the only maximum occurs at $x = 0$ and there are minima at $x = \pm 0.87$. From the graph of f'', it appears that there are inflection points at $x = \pm 0.52$.

43. The family of functions $f(x) = \ln(\sin x + C)$ all have the same period and all have maximum values at $x = \frac{\pi}{2} + 2\pi n$. Since the domain of ln is $(0, \infty)$, f has a graph only if $\sin x + C > 0$ somewhere. Since $-1 \le \sin x \le 1$, this happens if $C > -1$, that is, f has no graph if $C \le -1$. Similarly, if $C > 1$, then $\sin x + C > 0$ and f is continuous on $(-\infty, \infty)$. As C increases, the graph of f is shifted vertically upward and flattens out.



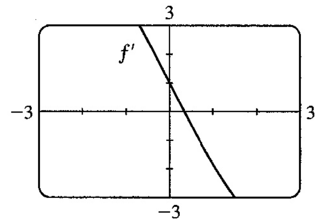
If $-1 < C \le 1$, f is defined where $\sin x + C > 0 \;\Leftrightarrow\; \sin x > -C \;\Leftrightarrow\; \sin^{-1}(-C) < x < \pi - \sin^{-1}(-C)$. Since the period is 2π, the domain of f is $\left(2n\pi + \sin^{-1}(-C), (2n+1)\pi - \sin^{-1}(-C)\right)$, n an integer.

61. $f(x) = x^5 - x^4 + 3x^2 - 3x - 2 \Rightarrow f'(x) = 5x^4 - 4x^3 + 6x - 3$, so

$x_{n+1} = x_n - \dfrac{x_n^5 - x_n^4 + 3x_n^2 - 3x_n - 2}{5x_n^4 - 4x_n^3 + 6x_n - 3}$. Now $x_1 = 1 \Rightarrow x_2 = 1.5 \Rightarrow x_3 \approx 1.343860 \Rightarrow$

$x_4 \approx 1.300320 \Rightarrow x_5 \approx 1.297396 \Rightarrow x_6 \approx 1.297383 \approx x_7$, so the root in $[1, 2]$ is 1.297383, to six decimal places.

63. $f(t) = \cos t + t - t^2 \Rightarrow f'(t) = -\sin t + 1 - 2t$. $f'(t)$ exists for all t, so to find the maximum of f, we can examine the zeros of f'. From the graph of f', we see that a good choice for t_1 is $t_1 = 0.3$. Use $g(t) = -\sin t + 1 - 2t$ and $g'(t) = -\cos t - 2$ to obtain $t_2 \approx 0.33535293$, $t_3 \approx 0.33541803 \approx t_4$. Since $f''(t) = -\cos t - 2 < 0$ for all t, $f(0.33541803) \approx 1.16718557$ is the absolute maximum.



65. $f'(x) = \sqrt{x^5} - 4/\sqrt[5]{x} = x^{5/2} - 4x^{-1/5} \Rightarrow f(x) = \frac{2}{7}x^{7/2} - 4\left(\frac{5}{4}x^{4/5}\right) + C = \frac{2}{7}x^{7/2} - 5x^{4/5} + C$

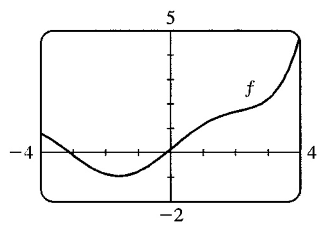
67. $f'(x) = e^x - (2/\sqrt{x}) = e^x - 2x^{-1/2} \Rightarrow$

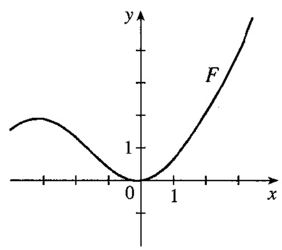
$$f(x) = e^x - 2\frac{x^{-1/2+1}}{-1/2+1} + C = e^x - 2\frac{x^{1/2}}{1/2} + C = e^x - 4\sqrt{x} + C$$

69. $f'(t) = 2t - 3\sin t \Rightarrow f(t) = t^2 + 3\cos t + C$.
$f(0) = 3 + C$ and $f(0) = 5 \Rightarrow C = 2$, so $f(t) = t^2 + 3\cos t + 2$.

71. $f''(x) = 1 - 6x + 48x^2 \Rightarrow f'(x) = x - 3x^2 + 16x^3 + C$. $f'(0) = C$ and $f'(0) = 2 \Rightarrow C = 2$, so $f'(x) = x - 3x^2 + 16x^3 + 2$ and hence, $f(x) = \frac{1}{2}x^2 - x^3 + 4x^4 + 2x + D$. $f(0) = D$ and $f(0) = 1 \Rightarrow$ $D = 1$, so $f(x) = \frac{1}{2}x^2 - x^3 + 4x^4 + 2x + 1$.

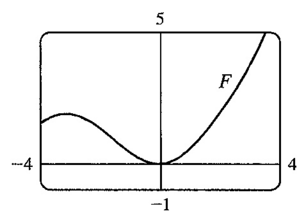
73. (a) Since f is 0 just to the left of the y-axis, we must have a minimum of F at the same place since we are increasing through $(0, 0)$ on F. There must be a local maximum to the left of $x = -3$, since f changes from positive to negative there.



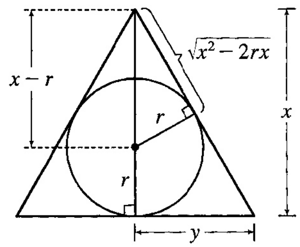


(b) $f(x) = 0.1e^x + \sin x \Rightarrow F(x) = 0.1e^x - \cos x + C$.
$F(0) = 0 \Rightarrow 0.1 - 1 + C = 0 \Rightarrow C = 0.9$, so
$F(x) = 0.1e^x - \cos x + 0.9$.

(c)



53.



By similar triangles, $\dfrac{y}{x} = \dfrac{r}{\sqrt{x^2-2rx}}$, so the area of the triangle is

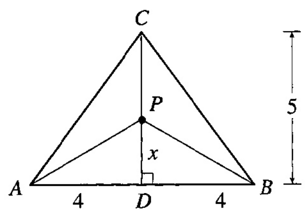
$$A(x) = \tfrac{1}{2}(2y)x = xy = \frac{rx^2}{\sqrt{x^2-2rx}} \quad\Rightarrow$$

$$A'(x) = \frac{2rx\sqrt{x^2-2rx} - rx^2(x-r)/\sqrt{x^2-2rx}}{x^2-2rx}$$

$$= \frac{rx^2(x-3r)}{(x^2-2rx)^{3/2}} = 0 \text{ when } x = 3r.$$

$A'(x) < 0$ when $2r < x < 3r$, $A'(x) > 0$ when $x > 3r$. So $x = 3r$ gives a minimum and $A(3r) = r(9r^2)/(\sqrt{3}\,r) = 3\sqrt{3}\,r^2$.

55.



We minimize

$L(x) = |PA| + |PB| + |PC| = 2\sqrt{x^2+16} + (5-x)$,

$0 \le x \le 5$. $L'(x) = 2x/\sqrt{x^2+16} - 1 = 0 \quad\Leftrightarrow$

$2x = \sqrt{x^2+16} \quad\Leftrightarrow\quad 4x^2 = x^2+16 \quad\Leftrightarrow\quad x = \frac{4}{\sqrt{3}}$.

$L(0) = 13$, $L\left(\frac{4}{\sqrt{3}}\right) \approx 11.9$, $L(5) \approx 12.8$, so the minimum

occurs when $x = \frac{4}{\sqrt{3}} \approx 2.3$.

57. $v = K\sqrt{\dfrac{L}{C} + \dfrac{C}{L}} \quad\Rightarrow\quad \dfrac{dv}{dL} = \dfrac{K}{2\sqrt{(L/C)+(C/L)}}\left(\dfrac{1}{C} - \dfrac{C}{L^2}\right) = 0 \quad\Leftrightarrow\quad \dfrac{1}{C} = \dfrac{C}{L^2} \quad\Leftrightarrow\quad L^2 = C^2 \quad\Leftrightarrow$

$L = C$. This gives the minimum velocity since $v' < 0$ for $0 < L < C$ and $v' > 0$ for $L > C$.

59. Let x denote the number of \$1 decreases in ticket price. Then the ticket price is $\$12 - \$1(x)$, and the average attendance is $11{,}000 + 1000(x)$. Now the revenue per game is

$$R(x) = (\text{price per person}) \times (\text{number of people per game})$$

$$= (12-x)(11{,}000 + 1000x) = -1000x^2 + 1000x + 132{,}000$$

for $0 \le x \le 4$ (since the seating capacity is 15,000) $\Rightarrow$ $R'(x) = -2000x + 1000 = 0 \quad\Leftrightarrow\quad x = 0.5$.

This is a maximum since $R''(x) = -2000 < 0$ for all x. Now we must check the value of

$R(x) = (12-x)(11{,}000+1000x)$ at $x = 0.5$ and at the endpoints of the domain to see which value of x gives the maximum value of R. $R(0) = (12)(11{,}000) = 132{,}000$, $R(0.5) = (11.5)(11{,}500) = 132{,}250$, and $R(4) = (8)(15{,}000) = 120{,}000$. Thus, the maximum revenue of \$132,250 per game occurs when the average attendance is 11,500 and the ticket price is \$11.50.

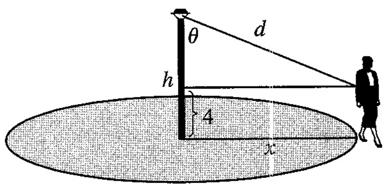
79. (a) $I = \dfrac{k\cos\theta}{d^2} = \dfrac{k(h/d)}{d^2} = k\dfrac{h}{d^3} = k\dfrac{h}{\left(\sqrt{40^2+h^2}\right)^3} = k\dfrac{h}{(1600+h^2)^{3/2}} \quad\Rightarrow$

$$\frac{dI}{dh} = k\frac{(1600+h^2)^{3/2} - h\frac{3}{2}(1600+h^2)^{1/2}\cdot 2h}{\left[(1600+h^2)^{3/2}\right]^2} = \frac{k(1600+h^2)^{1/2}(1600+h^2-3h^2)}{(1600+h^2)^3}$$

$$= \frac{k(1600-2h^2)}{(1600+h^2)^{5/2}} \quad [k \text{ is the constant of proportionality}]$$

Set $dI/dh = 0$: $1600 - 2h^2 = 0 \quad\Rightarrow\quad h^2 = 800 \quad\Rightarrow\quad h = \sqrt{800} = 20\sqrt{2}$. By the First Derivative Test, I has a local maximum at $h = 20\sqrt{2} \approx 28$ ft.

(b)



$$\frac{dx}{dt} = 4 \text{ ft/s}$$

$$I = \frac{k\cos\theta}{d^2} = \frac{k[(h-4)/d]}{d^2} = \frac{k(h-4)}{d^3} = \frac{k(h-4)}{[(h-4)^2+x^2]^{3/2}} = k(h-4)\left[(h-4)^2+x^2\right]^{-3/2}$$

$$\frac{dI}{dt} = \frac{dI}{dx}\cdot\frac{dx}{dt} = k(h-4)\left(-\tfrac{3}{2}\right)\left[(h-4)^2+x^2\right]^{-5/2}\cdot 2x\cdot\frac{dx}{dt}$$

$$= k(h-4)(-3x)\left[(h-4)^2+x^2\right]^{-5/2}\cdot 4 = \frac{-12xk(h-4)}{[(h-4)^2+x^2]^{5/2}}$$

$$\left.\frac{dI}{dt}\right|_{x=40} = -\frac{480k(h-4)}{[(h-4)^2+1600]^{5/2}}$$

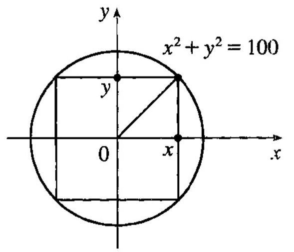
81. We first show that $\dfrac{x}{1+x^2} < \tan^{-1}x$ for $x > 0$. Let $f(x) = \tan^{-1}x - \dfrac{x}{1+x^2}$. Then

$f'(x) = \dfrac{1}{1+x^2} - \dfrac{1(1+x^2) - x(2x)}{(1+x^2)^2} = \dfrac{(1+x^2)-(1-x^2)}{(1+x^2)^2} = \dfrac{2x^2}{(1+x^2)^2} > 0$ for $x > 0$. So $f(x)$ is increasing on $(0,\infty)$. Hence, $0 < x \quad\Rightarrow\quad 0 = f(0) < f(x) = \tan^{-1}x - \dfrac{x}{1+x^2}$. So $\dfrac{x}{1+x^2} < \tan^{-1}x$ for $0 < x$. We next show that $\tan^{-1}x < x$ for $x > 0$. Let $h(x) = x - \tan^{-1}x$. Then

$h'(x) = 1 - \dfrac{1}{1+x^2} = \dfrac{x^2}{1+x^2} > 0$. Hence, $h(x)$ is increasing on $(0,\infty)$. So for $0 < x$, $0 = h(0) < h(x) = x - \tan^{-1}x$. Hence, $\tan^{-1}x < x$ for $x > 0$, and we conclude that $\dfrac{x}{1+x^2} < \tan^{-1}x < x$ for $x > 0$.

75. Choosing the positive direction to be upward, we have $a(t) = -9.8 \quad \Rightarrow \quad v(t) = -9.8t + v_0$, but $v(0) = 0 = v_0$ $\Rightarrow \quad v(t) = -9.8t = s'(t) \quad \Rightarrow \quad s(t) = -4.9t^2 + s_0$, but $s(0) = s_0 = 500 \quad \Rightarrow \quad s(t) = -4.9t^2 + 500$. When $s = 0$, $-4.9t^2 + 500 = 0 \quad \Rightarrow \quad t_1 = \sqrt{\frac{500}{4.9}} \approx 10.1 \quad \Rightarrow \quad v(t_1) = -9.8\sqrt{\frac{500}{4.9}} \approx -98.995$ m/s. Since the canister has been designed to withstand an impact velocity of 100 m/s, the canister will *not burst.*

77. (a)

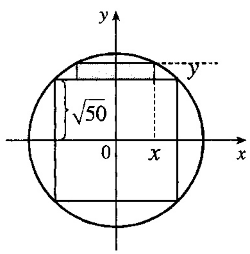


The cross-sectional area of the rectangular beam is $A = 2x \cdot 2y = 4xy = 4x\sqrt{100 - x^2}$, $0 \le x \le 10$, so

$$\frac{dA}{dx} = 4x\left(\tfrac{1}{2}\right)\left(100 - x^2\right)^{-1/2}(-2x) + \left(100 - x^2\right)^{1/2} \cdot 4 = \frac{-4x^2}{(100 - x^2)^{1/2}} + 4\left(100 - x^2\right)^{1/2} = \frac{4\left[-x^2 + \left(100 - x^2\right)\right]}{(100 - x^2)^{1/2}}.$$

$\frac{dA}{dx} = 0$ when $-x^2 + \left(100 - x^2\right) = 0 \quad \Rightarrow \quad x^2 = 50 \quad \Rightarrow \quad x = \sqrt{50} \approx 7.07 \quad \Rightarrow$ $y = \sqrt{100 - \left(\sqrt{50}\right)^2} = \sqrt{50}$. Since $A(0) = A(10) = 0$, the rectangle of maximum area is a square.

(b)



The cross-sectional area of each rectangular plank (shaded in the figure) is $A = 2x\left(y - \sqrt{50}\right) = 2x\left[\sqrt{100 - x^2} - \sqrt{50}\right]$, $0 \le x \le \sqrt{50}$, so

$$\frac{dA}{dx} = 2\left(\sqrt{100 - x^2} - \sqrt{50}\right) + 2x\left(\tfrac{1}{2}\right)\left(100 - x^2\right)^{-1/2}(-2x) = 2\left(100 - x^2\right)^{1/2} - 2\sqrt{50} - \frac{2x^2}{(100 - x^2)^{1/2}}$$

Set $\frac{dA}{dx} = 0$: $\left(100 - x^2\right) - \sqrt{50}\left(100 - x^2\right)^{1/2} - x^2 = 0 \quad \Rightarrow \quad 100 - 2x^2 = \sqrt{50}\left(100 - x^2\right)^{1/2} \quad \Rightarrow$ $10{,}000 - 400x^2 + 4x^4 = 50\left(100 - x^2\right) \quad \Rightarrow \quad 4x^4 - 350x^2 + 5000 = 0 \quad \Rightarrow$ $2x^4 - 175x^2 + 2500 = 0 \quad \Rightarrow \quad x^2 = \frac{175 \pm \sqrt{10{,}625}}{4} \approx 69.52$ or $17.98 \quad \Rightarrow \quad x \approx 8.34$ or 4.24. But $8.34 > \sqrt{50}$, so $x_1 \approx 4.24 \quad \Rightarrow \quad y - \sqrt{50} = \sqrt{100 - x_1^2} - \sqrt{50} \approx 1.99$. Each plank should have dimensions about $8\frac{1}{2}$ inches by 2 inches.

(c) From the figure in part (a), the width is $2x$ and the depth is $2y$, so the strength is $S = k(2x)(2y)^2 = 8kxy^2 = 8kx\left(100 - x^2\right) = 800kx - 8kx^3$, $0 \le x \le 10$. $dS/dx = 800k - 24kx^2 = 0$ when $24kx^2 = 800k \quad \Rightarrow \quad x^2 = \frac{100}{3} \quad \Rightarrow \quad x = \frac{10}{\sqrt{3}} \quad \Rightarrow \quad y = \sqrt{\frac{200}{3}} = \frac{10\sqrt{2}}{\sqrt{3}} = \sqrt{2}\,x$. Since $S(0) = S(10) = 0$, the maximum strength occurs when $x = \frac{10}{\sqrt{3}}$. The dimensions should be $\frac{20}{\sqrt{3}} \approx 11.55$ inches by $\frac{20\sqrt{2}}{\sqrt{3}} \approx 16.33$ inches.

9. $A = (x_1, x_1^2)$ and $B = (x_2, x_2^2)$, where x_1 and x_2 are the solutions of the quadratic equation $x^2 = mx + b$. Let $P = (x, x^2)$ and set $A_1 = (x_1, 0)$, $B_1 = (x_2, 0)$, and $P_1 = (x, 0)$. Let $f(x)$ denote the area of triangle PAB. Then $f(x)$ can be expressed in terms of the areas of three trapezoids as follows:

$$\begin{aligned} f(x) &= \text{area}\,(A_1ABB_1) - \text{area}\,(A_1APP_1) - \text{area}\,(B_1BPP_1) \\ &= \tfrac{1}{2}(x_1^2 + x_2^2)(x_2 - x_1) - \tfrac{1}{2}(x_1^2 + x^2)(x - x_1) - \tfrac{1}{2}(x^2 + x_2^2)(x_2 - x) \end{aligned}$$

After expanding and canceling terms, we get

$f(x) = \frac{1}{2}(x_2x_1^2 - x_1x_2^2 - xx_1^2 + x_1x^2 - x_2x^2 + xx_2^2) = \frac{1}{2}[x_1^2(x_2 - x) + x_2^2(x - x_1) + x^2(x_1 - x_2)]$

$f'(x) = \frac{1}{2}[-x_1^2 + x_2^2 + 2x(x_1 - x_2)]$. $f''(x) = \frac{1}{2}[2(x_1 - x_2)] = x_1 - x_2 < 0$ since $x_2 > x_1$.

$f'(x) = 0 \Rightarrow 2x(x_1 - x_2) = x_1^2 - x_2^2 \Rightarrow x_P = \frac{1}{2}(x_1 + x_2)$.

$$\begin{aligned} f(x_P) &= \tfrac{1}{2}\left(x_1^2\left[\tfrac{1}{2}(x_2 - x_1)\right] + x_2^2\left[\tfrac{1}{2}(x_2 - x_1)\right] + \tfrac{1}{4}(x_1 + x_2)^2(x_1 - x_2)\right) \\ &= \tfrac{1}{2}\left[\tfrac{1}{2}(x_2 - x_1)(x_1^2 + x_2^2) - \tfrac{1}{4}(x_2 - x_1)(x_1 + x_2)^2\right] \\ &= \tfrac{1}{8}(x_2 - x_1)\left[2(x_1^2 + x_2^2) - (x_1^2 + 2x_1x_2 + x_2^2)\right] \\ &= \tfrac{1}{8}(x_2 - x_1)(x_1^2 - 2x_1x_2 + x_2^2) = \tfrac{1}{8}(x_2 - x_1)(x_1 - x_2)^2 = \tfrac{1}{8}(x_2 - x_1)(x_2 - x_1)^2 \\ &= \tfrac{1}{8}(x_2 - x_1)^3 \end{aligned}$$

To put this in terms of m and b, we solve the system $y = x_1^2$ and $y = mx_1 + b$, giving us $x_1^2 - mx_1 - b = 0 \Rightarrow$ $x_1 = \frac{1}{2}\left(m - \sqrt{m^2 + 4b}\right)$. Similarly, $x_2 = \frac{1}{2}\left(m + \sqrt{m^2 + 4b}\right)$. The area is then $\frac{1}{8}(x_2 - x_1)^3 = \frac{1}{8}\left(\sqrt{m^2 + 4b}\right)^3$, and is attained at the point $P(x_P, x_P^2) = P\left(\frac{1}{2}m, \frac{1}{4}m^2\right)$.

Note: Another way to get an expression for $f(x)$ is to use the formula for an area of a triangle in terms of the coordinates of the vertices: $f(x) = \frac{1}{2}\left[(x_2x_1^2 - x_1x_2^2) + (x_1x^2 - xx_1^2) + (xx_2^2 - x_2x^2)\right]$.

11. $f(x) = (a^2 + a - 6)\cos 2x + (a - 2)x + \cos 1 \Rightarrow f'(x) = -(a^2 + a - 6)\sin 2x\,(2) + (a - 2)$.

The derivative exists for all x, so the only possible critical points will occur where $f'(x) = 0 \Leftrightarrow$ $2(a - 2)(a + 3)\sin 2x = a - 2 \Leftrightarrow$ either $a = 2$ or $2(a + 3)\sin 2x = 1$, with the latter implying that $\sin 2x = \dfrac{1}{2(a + 3)}$. Since the range of $\sin 2x$ is $[-1, 1]$, this equation has no solution whenever either $\dfrac{1}{2(a + 3)} < -1$ or $\dfrac{1}{2(a + 3)} > 1$. Solving these inequalities, we get $-\frac{7}{2} < a < -\frac{5}{2}$.

PROBLEMS PLUS

1. Let $y = f(x) = e^{-x^2}$. The area of the rectangle under the curve from $-x$ to x is $A(x) = 2xe^{-x^2}$ where $x \geq 0$. We maximize $A(x)$: $A'(x) = 2e^{-x^2} - 4x^2e^{-x^2} = 2e^{-x^2}(1 - 2x^2) = 0 \Rightarrow x = \frac{1}{\sqrt{2}}$. This gives a maximum since $A'(x) > 0$ for $0 \leq x < \frac{1}{\sqrt{2}}$ and $A'(x) < 0$ for $x > \frac{1}{\sqrt{2}}$. We next determine the points of inflection of $f(x)$. Notice that $f'(x) = -2xe^{-x^2} = -A(x)$. So $f''(x) = -A'(x)$ and hence, $f''(x) < 0$ for $-\frac{1}{\sqrt{2}} < x < \frac{1}{\sqrt{2}}$ and $f''(x) > 0$ for $x < -\frac{1}{\sqrt{2}}$ and $x > \frac{1}{\sqrt{2}}$. So $f(x)$ changes concavity at $x = \pm\frac{1}{\sqrt{2}}$, and the two vertices of the rectangle of largest area are at the inflection points.

3. First, we recognize some symmetry in the inequality: $\dfrac{e^{x+y}}{xy} \geq e^2 \Leftrightarrow \dfrac{e^x}{x} \cdot \dfrac{e^y}{y} \geq e \cdot e$. This suggests that we need to show that $\dfrac{e^x}{x} \geq e$ for $x > 0$. If we can do this, then the inequality $\dfrac{e^y}{y} \geq e$ is true, and the given inequality follows. $f(x) = \dfrac{e^x}{x} \Rightarrow f'(x) = \dfrac{xe^x - e^x}{x^2} = \dfrac{e^x(x-1)}{x^2} = 0 \Rightarrow x = 1$. By the First Derivative Test, we have a minimum of $f(1) = e$, so $e^x/x \geq e$ for all x.

5. First we show that $x(1-x) \leq \frac{1}{4}$ for all x. Let $f(x) = x(1-x) = x - x^2$. Then $f'(x) = 1 - 2x$. This is 0 when $x = \frac{1}{2}$ and $f'(x) > 0$ for $x < \frac{1}{2}$, $f'(x) < 0$ for $x > \frac{1}{2}$, so the absolute maximum of f is $f\left(\frac{1}{2}\right) = \frac{1}{4}$. Thus, $x(1-x) \leq \frac{1}{4}$ for all x.

Now suppose that the given assertion is false, that is, $a(1-b) > \frac{1}{4}$ and $b(1-a) > \frac{1}{4}$. Multiply these inequalities: $a(1-b)b(1-a) > \frac{1}{16} \Rightarrow [a(1-a)]\,[b(1-b)] > \frac{1}{16}$. But we know that $a(1-a) \leq \frac{1}{4}$ and $b(1-b) \leq \frac{1}{4} \Rightarrow [a(1-a)]\,[b(1-b)] \leq \frac{1}{16}$. Thus, we have a contradiction, so the given assertion is proved.

7. Differentiating $x^2 + xy + y^2 = 12$ implicitly with respect to x gives $2x + y + x\,\dfrac{dy}{dx} + 2y\,\dfrac{dy}{dx} = 0$, so $\dfrac{dy}{dx} = -\dfrac{2x+y}{x+2y}$. At a highest or lowest point, $\dfrac{dy}{dx} = 0 \Leftrightarrow y = -2x$. Substituting $-2x$ for y in the original equation gives $x^2 + x(-2x) + (-2x)^2 = 12$, so $3x^2 = 12$ and $x = \pm 2$. If $x = 2$, then $y = -2x = -4$, and if $x = -2$ then $y = 4$. Thus, the highest and lowest points are $(-2, 4)$ and $(2, -4)$.

17. Note that $f(0) = 0$, so for $x \neq 0$, $\left|\dfrac{f(x) - f(0)}{x - 0}\right| = \left|\dfrac{f(x)}{x}\right| = \dfrac{|f(x)|}{|x|} \leq \dfrac{|\sin x|}{|x|} = \dfrac{\sin x}{x}$.

Therefore, $|f'(0)| = \left|\lim\limits_{x\to 0} \dfrac{f(x) - f(0)}{x - 0}\right| = \lim\limits_{x\to 0}\left|\dfrac{f(x) - f(0)}{x - 0}\right| \leq \lim\limits_{x\to 0} \dfrac{\sin x}{x} = 1$. But

$f'(x) = a_1 \cos x + 2a_2 \cos 2x + \cdots + na_n \cos nx$, so $|f'(0)| = |a_1 + 2a_2 + \cdots + na_n| \leq 1$.

Another solution: We are given that $\left|\sum_{k=1}^{n} a_k \sin kx\right| \leq |\sin x|$. So for x close to 0, and $x \neq 0$, we have

$\left|\sum\limits_{k=1}^{n} a_k \dfrac{\sin kx}{\sin x}\right| \leq 1 \;\Rightarrow\; \lim\limits_{x\to 0}\left|\sum\limits_{k=1}^{n} a_k \dfrac{\sin kx}{\sin x}\right| \leq 1 \;\Rightarrow\; \left|\sum\limits_{k=1}^{n} a_k \lim\limits_{x\to 0} \dfrac{\sin kx}{\sin x}\right| \leq 1$. But by l'Hospital's Rule,

$\lim\limits_{x\to 0} \dfrac{\sin kx}{\sin x} = \lim\limits_{x\to 0} \dfrac{k\cos kx}{\cos x} = k$, so $\left|\sum\limits_{k=1}^{n} ka_k\right| \leq 1$.

19. (a) Distance = rate × time, so time = distance/rate. $T_1 = \dfrac{D}{c_1}$,

$$T_2 = \frac{2\,|PR|}{c_1} + \frac{|RS|}{c_2} = \frac{2h\sec\theta}{c_1} + \frac{D - 2h\tan\theta}{c_2},\quad T_3 = \frac{2\sqrt{h^2 + D^2/4}}{c_1} = \frac{\sqrt{4h^2 + D^2}}{c_1}.$$

(b) $\dfrac{dT_2}{d\theta} = \dfrac{2h}{c_1}\cdot \sec\theta\,\tan\theta - \dfrac{2h}{c_2}\sec^2\theta = 0$ when $2h\sec\theta\left(\dfrac{1}{c_1}\tan\theta - \dfrac{1}{c_2}\sec\theta\right) = 0 \;\Rightarrow$

$\dfrac{1}{c_1}\dfrac{\sin\theta}{\cos\theta} - \dfrac{1}{c_2}\dfrac{1}{\cos\theta} = 0 \;\Rightarrow\; \dfrac{\sin\theta}{c_1\cos\theta} = \dfrac{1}{c_2\cos\theta} \;\Rightarrow\; \sin\theta = \dfrac{c_1}{c_2}$. The First Derivative Test shows that this gives a minimum.

(c) Using part (a) with $D = 1$ and $T_1 = 0.26$, we have $T_1 = \dfrac{D}{c_1} \;\Rightarrow$

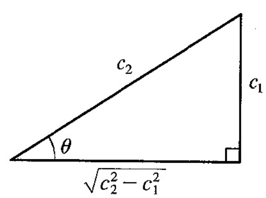
$c_1 = \frac{1}{0.26} \approx 3.85$ km/s. $T_3 = \dfrac{\sqrt{4h^2 + D^2}}{c_1} \;\Rightarrow\; 4h^2 + D^2 = T_3^2 c_1^2 \;\Rightarrow$

$h = \frac{1}{2}\sqrt{T_3^2 c_1^2 - D^2} = \frac{1}{2}\sqrt{(0.34)^2(1/0.26)^2 - 1^2} \approx 0.42$ km. To find c_2, we use $\sin\theta = \dfrac{c_1}{c_2}$ from part (b)

and $T_2 = \dfrac{2h\sec\theta}{c_1} + \dfrac{D - 2h\tan\theta}{c_2}$ from part (a). From the figure,

$\sin\theta = \dfrac{c_1}{c_2} \;\Rightarrow\; \sec\theta = \dfrac{c_2}{\sqrt{c_2^2 - c_1^2}}$ and $\tan\theta = \dfrac{c_1}{\sqrt{c_2^2 - c_1^2}}$, so

$$T_2 = \frac{2hc_2}{c_1\sqrt{c_2^2 - c_1^2}} + \frac{D\sqrt{c_2^2 - c_1^2} - 2hc_1}{c_2\sqrt{c_2^2 - c_1^2}}.$$

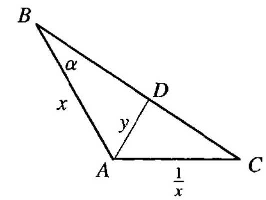


Using the values for T_2 [given as 0.32], h, c_1, and D, we can graph

$Y_1 = T_2$ and $Y_2 = \dfrac{2hc_2}{c_1\sqrt{c_2^2 - c_1^2}} + \dfrac{D\sqrt{c_2^2 - c_1^2} - 2hc_1}{c_2\sqrt{c_2^2 - c_1^2}}$ and find their intersection points. Doing so gives us

$c_2 \approx 4.10$ and 7.66, but if $c_2 = 4.10$, then $\theta = \arcsin(c_1/c_2) \approx 69.6°$, which implies that point S is to the left of point R in the diagram. So $c_2 = 7.66$ km/s.

13. (a) Let $y = |AD|$, $x = |AB|$, and $1/x = |AC|$, so that $|AB| \cdot |AC| = 1$.



We compute the area $\mathcal{A}$ of $\triangle ABC$ in two ways. First,

$\mathcal{A} = \frac{1}{2}\,|AB|\,|AC| \sin\frac{2\pi}{3} = \frac{1}{2} \cdot 1 \cdot \frac{\sqrt{3}}{2} = \frac{\sqrt{3}}{4}$. Second,

$$\begin{aligned}\mathcal{A} &= (\text{area of } \triangle ABD) + (\text{area of } \triangle ACD) \\ &= \tfrac{1}{2}\,|AB|\,|AD| \sin\tfrac{\pi}{3} + \tfrac{1}{2}\,|AD|\,|AC| \sin\tfrac{\pi}{3} \\ &= \tfrac{1}{2}xy\tfrac{\sqrt{3}}{2} + \tfrac{1}{2}y(1/x)\tfrac{\sqrt{3}}{2} = \tfrac{\sqrt{3}}{4}y(x + 1/x)\end{aligned}$$

Equating the two expressions for the area, we get $\frac{\sqrt{3}}{4}y\left(x + \frac{1}{x}\right) = \frac{\sqrt{3}}{4} \quad\Leftrightarrow\quad y = \frac{1}{x + 1/x} = \frac{x}{x^2 + 1}$, $x > 0$.

Another method: Use the Law of Sines on the triangles ABD and ABC. In $\triangle ABD$, we have

$\angle A + \angle B + \angle D = 180° \quad\Leftrightarrow\quad 60° + \alpha + \angle D = 180° \quad\Leftrightarrow\quad \angle D = 120° - \alpha$. Thus,

$$\frac{x}{y} = \frac{\sin(120° - \alpha)}{\sin\alpha} = \frac{\sin 120° \cos\alpha - \cos 120° \sin\alpha}{\sin\alpha} = \frac{\frac{\sqrt{3}}{2}\cos\alpha + \frac{1}{2}\sin\alpha}{\sin\alpha} \quad\Rightarrow\quad \frac{x}{y} = \frac{\sqrt{3}}{2}\cot\alpha + \frac{1}{2},$$

and by a similar argument with $\triangle ABC$, $\frac{\sqrt{3}}{2}\cot\alpha = x^2 + \frac{1}{2}$. Eliminating $\cot\alpha$ gives $\frac{x}{y} = \left(x^2 + \frac{1}{2}\right) + \frac{1}{2} \quad\Rightarrow$

$y = \dfrac{x}{x^2 + 1}$, $x > 0$.

(b) We differentiate our expression for y with respect to x to find the maximum:

$\dfrac{dy}{dx} = \dfrac{(x^2 + 1) - x(2x)}{(x^2 + 1)^2} = \dfrac{1 - x^2}{(x^2 + 1)^2} = 0$ when $x = 1$. This indicates a maximum by the First Derivative

Test, since $y'(x) > 0$ for $0 < x < 1$ and $y'(x) < 0$ for $x > 1$, so the maximum value of y is $y(1) = \frac{1}{2}$.

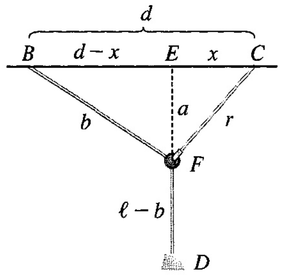
15. Suppose that the curve $y = a^x$ intersects the line $y = x$. Then $a^{x_0} = x_0$ for some $x_0 > 0$, and hence $a = x_0^{1/x_0}$.

We find the maximum value of $g(x) = x^{1/x}$, > 0, because if a is larger than the maximum value of this function, then the curve $y = a^x$ does not intersect the line $y = x$.

$g'(x) = e^{(1/x)\ln x}\left(-\dfrac{1}{x^2}\ln x + \dfrac{1}{x} \cdot \dfrac{1}{x}\right) = x^{1/x}\left(\dfrac{1}{x^2}\right)(1 - \ln x)$. This is 0 only where $x = e$, and for $0 < x < e$,

$f'(x) > 0$, while for $x > e$, $f'(x) < 0$, so g has an absolute maximum of $g(e) = e^{1/e}$. So if $y = a^x$ intersects $y = x$, we must have $0 < a \le e^{1/e}$. Conversely, suppose that $0 < a \le e^{1/e}$. Then $a^e \le e$, so the graph of $y = a^x$ lies below or touches the graph of $y = x$ at $x = e$. Also $a^0 = 1 > 0$, so the graph of $y = a^x$ lies above that of $y = x$ at $x = 0$. Therefore, by the Intermediate Value Theorem, the graphs of $y = a^x$ and $y = x$ must intersect somewhere between $x = 0$ and $x = e$.

21.



Let $a = |EF|$ and $b = |BF|$ as shown in the figure.
Since $\ell = |BF| + |FD|$, $|FD| = \ell - b$. Now

$$\begin{aligned} |ED| &= |EF| + |FD| = a + \ell - b \\ &= \sqrt{r^2 - x^2} + \ell - \sqrt{(d-x)^2 + a^2} \\ &= \sqrt{r^2 - x^2} + \ell - \sqrt{(d-x)^2 + \left(\sqrt{r^2 - x^2}\right)^2} \\ &= \sqrt{r^2 - x^2} + \ell - \sqrt{d^2 - 2dx + x^2 + r^2 - x^2} \end{aligned}$$

Let $f(x) = \sqrt{r^2 - x^2} + \ell - \sqrt{d^2 + r^2 - 2dx}$.

$f'(x) = \frac{1}{2}\left(r^2 - x^2\right)^{-1/2}(-2x) - \frac{1}{2}\left(d^2 + r^2 - 2dx\right)^{-1/2}(-2d) = \dfrac{-x}{\sqrt{r^2 - x^2}} + \dfrac{d}{\sqrt{d^2 + r^2 - 2dx}}$.

$f'(x) = 0 \quad \Rightarrow \quad \dfrac{x}{\sqrt{r^2 - x^2}} = \dfrac{d}{\sqrt{d^2 + r^2 - 2dx}} \quad \Rightarrow \quad \dfrac{x^2}{r^2 - x^2} = \dfrac{d^2}{d^2 + r^2 - 2dx} \quad \Rightarrow$

$d^2x^2 + r^2x^2 - 2dx^3 = d^2r^2 - d^2x^2 \quad \Rightarrow \quad 0 = 2dx^3 - 2d^2x^2 - r^2x^2 + d^2r^2 \quad \Rightarrow$

$0 = 2dx^2(x - d) - r^2\left(x^2 - d^2\right) \quad \Rightarrow \quad 0 = 2dx^2(x - d) - r^2(x + d)(x - d) \quad \Rightarrow$

$0 = (x - d)\left[2dx^2 - r^2(x + d)\right]$

But $d > r > x$, so $x \neq d$. Thus, we solve $2dx^2 - r^2x - dr^2 = 0$ for x:

$x = \dfrac{-(-r^2) \pm \sqrt{(-r^2)^2 - 4(2d)(-dr^2)}}{2(2d)} = \dfrac{r^2 \pm \sqrt{r^4 + 8d^2r^2}}{4d}$. Because $\sqrt{r^4 + 8d^2r^2} > r^2$, the "negative" can be discarded. Thus,

$$\begin{aligned} x &= \frac{r^2 + \sqrt{r^2}\,\sqrt{r^2 + 8d^2}}{4d} = \frac{r^2 + r\sqrt{r^2 + 8d^2}}{4d} \qquad (r > 0) \\ &= \frac{r}{4d}\left(r + \sqrt{r^2 + 8d^2}\right) \end{aligned}$$

The maximum value of $|ED|$ occurs at this value of x.

23. $V = \frac{4}{3}\pi r^3 \quad \Leftrightarrow \quad \dfrac{dV}{dt} = 4\pi r^2\,\dfrac{dr}{dt}$. But $\dfrac{dV}{dt}$ is proportional to the surface area, so $\dfrac{dV}{dt} = k \cdot 4\pi r^2$ for some constant k. Therefore, $4\pi r^2\,\dfrac{dr}{dt} = k \cdot 4\pi r^2 \quad \Leftrightarrow \quad \dfrac{dr}{dt} = k =$ constant. An antiderivative of k with respect to t is kt, so $r = kt + C$. When $t = 0$, the radius r must equal the original radius r_0, so $C = r_0$, and $r = kt + r_0$. To find k we use the fact that when $t = 3$, $r = 3k + r_0$ and $V = \frac{1}{2}V_0 \quad \Rightarrow \quad \frac{4}{3}\pi(3k + r_0)^3 = \frac{1}{2} \cdot \frac{4}{3}\pi r_0^3 \quad \Rightarrow$

$(3k + r_0)^3 = \frac{1}{2}r_0^3 \quad \Rightarrow \quad 3k + r_0 = \frac{1}{\sqrt[3]{2}}r_0 \quad \Rightarrow \quad k = \frac{1}{3}r_0\left(\frac{1}{\sqrt[3]{2}} - 1\right)$. Since $r = kt + r_0$,

$r = \frac{1}{3}r_0\left(\frac{1}{\sqrt[3]{2}} - 1\right)t + r_0$. When the snowball has melted completely we have $r = 0 \quad \Rightarrow$

$\frac{1}{3}r_0\left(\frac{1}{\sqrt[3]{2}} - 1\right)t + r_0 = 0$ which gives $t = \dfrac{3\sqrt[3]{2}}{\sqrt[3]{2} - 1}$. Hence, it takes $\dfrac{3\sqrt[3]{2}}{\sqrt[3]{2} - 1} - 3 = \dfrac{3}{\sqrt[3]{2} - 1} \approx 11$ h 33 min longer.