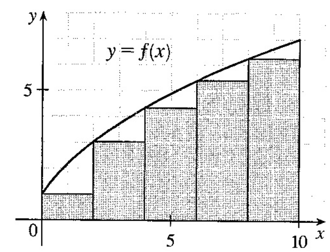
5 ☐ INTEGRALS

5.1 Areas and Distances

1. (a) Since f is *increasing*, we can obtain a *lower* estimate by using *left* endpoints. We are instructed to use five rectangles, so $n = 5$.



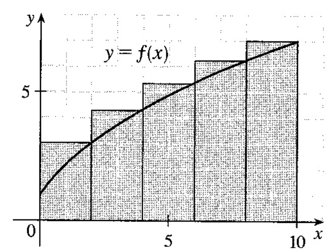
$$L_5 = \sum_{i=1}^{5} f(x_{i-1})\,\Delta x \qquad [\Delta x = \tfrac{b-a}{n} = \tfrac{10-0}{5} = 2]$$

$$= f(x_0)\cdot 2 + f(x_1)\cdot 2 + f(x_2)\cdot 2 + f(x_3)\cdot 2 + f(x_4)\cdot 2$$

$$= 2\,[f(0) + f(2) + f(4) + f(6) + f(8)]$$

$$\approx 2(1 + 3 + 4.3 + 5.4 + 6.3) = 2(20) = 40$$

Since f is *increasing*, we can obtain an *upper* estimate by using *right* endpoints.



$$R_5 = \sum_{i=1}^{5} f(x_i)\,\Delta x$$

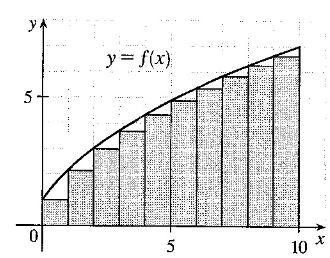
$$= 2\,[f(x_1) + f(x_2) + f(x_3) + f(x_4) + f(x_5)]$$

$$= 2\,[f(2) + f(4) + f(6) + f(8) + f(10)]$$

$$\approx 2(3 + 4.3 + 5.4 + 6.3 + 7) = 2(26) = 52$$

Comparing R_5 to L_5, we see that we have added the area of the rightmost upper rectangle, $f(10)\cdot 2$, to the sum and subtracted the area of the leftmost lower rectangle, $f(0)\cdot 2$, from the sum.

(b) $$L_{10} = \sum_{i=1}^{10} f(x_{i-1})\,\Delta x \qquad [\Delta x = \tfrac{10-0}{10} = 1]$$



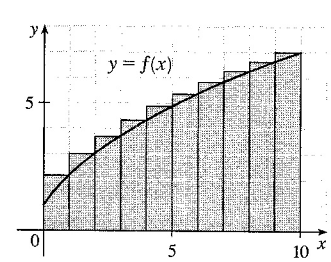
$$= 1\,[f(x_0) + f(x_1) + \cdots + f(x_9)]$$

$$= f(0) + f(1) + \cdots + f(9)$$

$$\approx 1 + 2.1 + 3 + 3.7 + 4.3 + 4.9 + 5.4 + 5.8 + 6.3 + 6.7$$

$$= 43.2$$

$$R_{10} = \sum_{i=1}^{10} f(x_i)\,\Delta x = f(1) + f(2) + \cdots + f(10)$$



$$= L_{10} + 1\cdot f(10) - 1\cdot f(0) \qquad \left[\begin{array}{l}\text{add rightmost upper rectangle,}\\ \text{subtract leftmost lower rectangle}\end{array}\right]$$

$$= 43.2 + 7 - 1 = 49.2$$

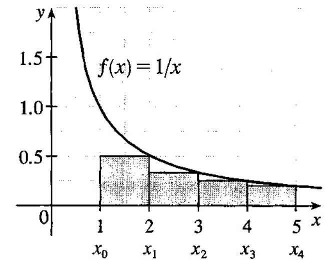
3. (a) $R_4 = \sum\limits_{i=1}^{4} f(x_i)\,\Delta x \quad [\Delta x = \frac{5-1}{4} = 1]$

$= f(x_1)\cdot 1 + f(x_2)\cdot 1 + f(x_3)\cdot 1 + f(x_4)\cdot 1$

$= f(2) + f(3) + f(4) + f(5)$

$= \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} = \frac{77}{60} = 1.28\overline{3}$

Since f is *decreasing* on $[1, 5]$, an *underestimate* is obtained by using the *right* endpoint approximation, R_4.

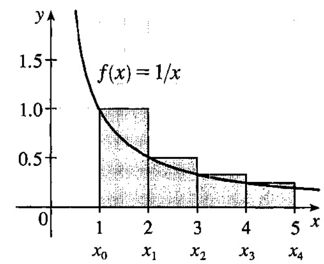


(b) $L_4 = \sum\limits_{i=1}^{4} f(x_{i-1})\,\Delta x$

$= f(1) + f(2) + f(3) + f(4)$

$= 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} = \frac{25}{12} = 2.08\overline{3}$

L_4 is an overestimate. Alternatively, we could just add the area of the leftmost upper rectangle and subtract the area of the rightmost lower rectangle; that is, $L_4 = R_4 + f(1)\cdot 1 - f(5)\cdot 1$.



5. (a) $f(x) = 1 + x^2$ and $\Delta x = \dfrac{2-(-1)}{3} = 1 \Rightarrow$

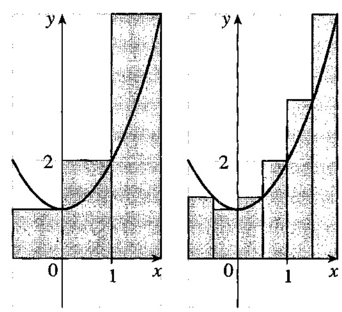
$R_3 = 1\cdot f(0) + 1\cdot f(1) + 1\cdot f(2) = 1\cdot 1 + 1\cdot 2 + 1\cdot 5 = 8.$

$\Delta x = \dfrac{2-(-1)}{6} = 0.5 \Rightarrow$

$R_6 = 0.5[f(-0.5) + f(0) + f(0.5) + f(1) + f(1.5) + f(2)]$

$= 0.5(1.25 + 1 + 1.25 + 2 + 3.25 + 5)$

$= 0.5(13.75) = 6.875$

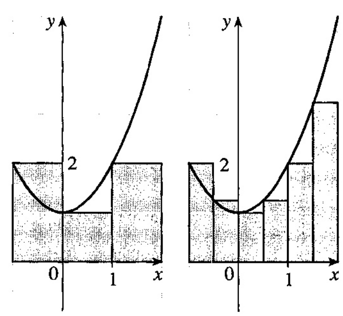


(b) $L_3 = 1\cdot f(-1) + 1\cdot f(0) + 1\cdot f(1) = 1\cdot 2 + 1\cdot 1 + 1\cdot 2 = 5$

$L_6 = 0.5[f(-1) + f(-0.5) + f(0) + f(0.5) + f(1) + f(1.5)]$

$= 0.5(2 + 1.25 + 1 + 1.25 + 2 + 3.25)$

$= 0.5(10.75) = 5.375$



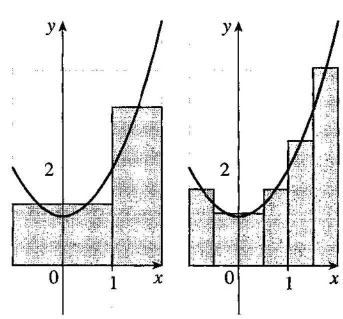
(c) $M_3 = 1\cdot f(-0.5) + 1\cdot f(0.5) + 1\cdot f(1.5)$

$= 1\cdot 1.25 + 1\cdot 1.25 + 1\cdot 3.25 = 5.75$

$M_6 = 0.5[f(-0.75) + f(-0.25) + f(0.25) + f(0.75) + f(1.25) + f(1.75)]$

$= 0.5(1.5625 + 1.0625 + 1.0625 + 1.5625 + 2.5625 + 4.0625)$

$= 0.5(11.875) = 5.9375$



(d) M_6 appears to be the best estimate.

7. Here is one possible algorithm (ordered sequence of operations) for calculating the sums:

1 Let SUM = 0, X_MIN = 0, X_MAX = π, N = 10 (or 30 or 50, depending on which sum we are calculating), DELTA_X = (X_MAX − X_MIN)/N, and RIGHT_ENDPOINT = X_MIN + DELTA_X.

2 Repeat steps 2a, 2b in sequence until RIGHT_ENDPOINT > X_MAX.

2a Add sin (RIGHT_ENDPOINT) to SUM.

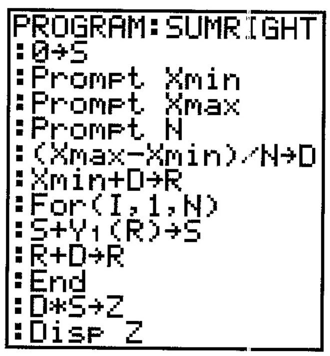
2b Add DELTA_X to RIGHT_ENDPOINT.

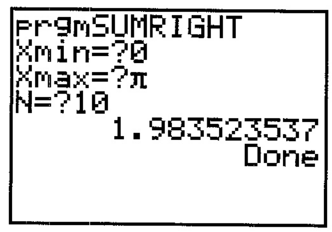
At the end of this procedure, (DELTA_X) · (SUM) is equal to the answer we are looking for. We find that

$$R_{10} = \frac{\pi}{10}\sum_{i=1}^{10}\sin\left(\frac{i\pi}{10}\right) \approx 1.9835,\ R_{30} = \frac{\pi}{30}\sum_{i=1}^{30}\sin\left(\frac{i\pi}{30}\right) \approx 1.9982, \text{ and } R_{50} = \frac{\pi}{50}\sum_{i=1}^{50}\sin\left(\frac{i\pi}{50}\right) \approx 1.9993.$$

It appears that the exact area is 2.

Shown below is program SUMRIGHT and its output from a TI-83 Plus calculator. To generalize the program, we have input (rather than assigned) values for Xmin, Xmax, and N. Also, the function, $\sin x$, is assigned to Y_1, enabling us to evaluate any right sum merely by changing Y_1 and running the program.



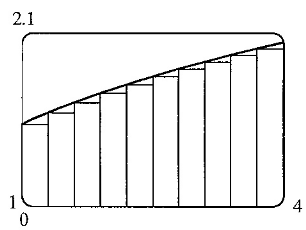


9. In Maple, we have to perform a number of steps before getting a numerical answer. After loading the student package [command: `with(student);`] we use the command `left_sum:=leftsum(x^(1/2),x=1..4,10` [or `30`, or `50`]); which gives us the expression in summation notation. To get a numerical approximation to the sum, we use `evalf(left_sum);`. Mathematica does not have a special command for these sums, so we must type them in manually. For example, the first left sum is given by `(3/10)*Sum[Sqrt[1+3(i-1)/10],{i,1,10}]`, and we use the `N` command on the resulting output to get a numerical approximation.

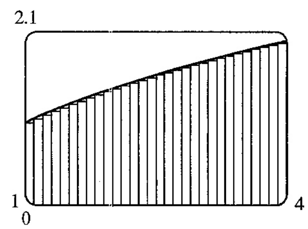
In Derive, we use the `LEFT_RIEMANN` command to get the left sums, but must define the right sums ourselves. (We can define a new function using `LEFT_RIEMANN` with k ranging from 1 to n instead of from 0 to $n-1$.)

(a) With $f(x) = \sqrt{x}$, $1 \le x \le 4$, the left sums are of the form $L_n = \frac{3}{n}\sum_{i=1}^{n}\sqrt{1+\frac{3(i-1)}{n}}$. Specifically, $L_{10} \approx 4.5148$, $L_{30} \approx 4.6165$, and $L_{50} \approx 4.6366$. The right sums are of the form $R_n = \frac{3}{n}\sum_{i=1}^{n}\sqrt{1+\frac{3i}{n}}$. Specifically, $R_{10} \approx 4.8148$, $R_{30} \approx 4.7165$, and $R_{50} \approx 4.6966$.

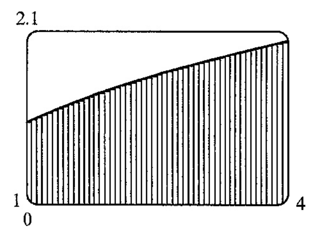
(b) In Maple, we use the `leftbox` and `rightbox` commands (with the same arguments as `leftsum` and `rightsum` above) to generate the graphs.



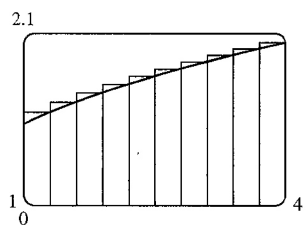
left endpoints, $n = 10$



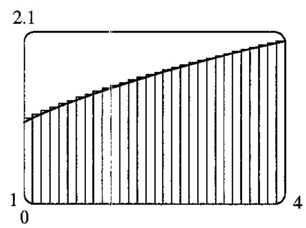
left endpoints, $n = 30$



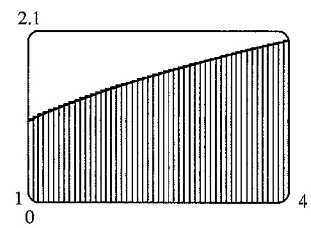
left endpoints, $n = 50$



right endpoints, $n = 10$



right endpoints, $n = 30$



right endpoints, $n = 50$

(c) We know that since $\sqrt{x}$ is an increasing function on $(1, 4)$, all of the left sums are smaller than the actual area, and all of the right sums are larger than the actual area. Since the left sum with $n = 50$ is about $4.637 > 4.6$ and the right sum with $n = 50$ is about $4.697 < 4.7$, we conclude that $4.6 < L_{50} <$ exact area $< R_{50} < 4.7$, so the exact area is between 4.6 and 4.7.

11. Since v is an increasing function, L_6 will give us a lower estimate and R_6 will give us an upper estimate.

$$\begin{aligned} L_6 &= (0 \text{ ft/s})(0.5 \text{ s}) + (6.2)(0.5) + (10.8)(0.5) + (14.9)(0.5) + (18.1)(0.5) + (19.4)(0.5) \\ &= 0.5(69.4) = 34.7 \text{ ft} \end{aligned}$$

$$R_6 = 0.5(6.2 + 10.8 + 14.9 + 18.1 + 19.4 + 20.2) = 0.5(89.6) = 44.8 \text{ ft}$$

13. Lower estimate for oil leakage: $R_5 = (7.6 + 6.8 + 6.2 + 5.7 + 5.3)(2) = (31.6)(2) = 63.2$ L.

Upper estimate for oil leakage: $L_5 = (8.7 + 7.6 + 6.8 + 6.2 + 5.7)(2) = (35)(2) = 70$ L.

15. For a decreasing function, using left endpoints gives us an overestimate and using right endpoints results in an underestimate. We will use M_6 to get an estimate. $\Delta t = 1$, so

$$\begin{aligned} M_6 &= 1[v(0.5) + v(1.5) + v(2.5) + v(3.5) + v(4.5) + v(5.5)] \\ &\approx 55 + 40 + 28 + 18 + 10 + 4 = 155 \text{ ft} \end{aligned}$$

For a very rough check on the above calculation, we can draw a line from $(0, 70)$ to $(6, 0)$ and calculate the area of the triangle: $\frac{1}{2}(70)(6) = 210$. This is clearly an overestimate, so our midpoint estimate of 155 is reasonable.

17. $f(x) = \sqrt[4]{x}$, $1 \le x \le 16$. $\Delta x = (16 - 1)/n = 15/n$ and $x_i = 1 + i\,\Delta x = 1 + 15i/n$.

$$A = \lim_{n\to\infty} R_n = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\,\Delta x = \lim_{n\to\infty} \sum_{i=1}^{n} \sqrt[4]{1 + \tfrac{15i}{n}} \cdot \tfrac{15}{n}.$$

19. $f(x) = x\cos x$, $0 \le x \le \frac{\pi}{2}$. $\Delta x = (\frac{\pi}{2} - 0)/n = \frac{\pi}{2}/n$ and $x_i = 0 + i\,\Delta x = \frac{\pi}{2}i/n$.

$$A = \lim_{n\to\infty} R_n = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\,\Delta x = \lim_{n\to\infty} \sum_{i=1}^{n} \frac{i\pi}{2n} \cos\left(\frac{i\pi}{2n}\right) \cdot \frac{\pi}{2n}.$$

21. $\displaystyle\lim_{n\to\infty} \sum_{i=1}^{n} \frac{\pi}{4n} \tan\frac{i\pi}{4n}$ can be interpreted as the area of the region lying under the graph of $y = \tan x$ on the interval $\left[0, \frac{\pi}{4}\right]$, since for $y = \tan x$ on $\left[0, \frac{\pi}{4}\right]$ with $\Delta x = \dfrac{\pi/4 - 0}{n} = \dfrac{\pi}{4n}$, $x_i = 0 + i\,\Delta x = \dfrac{i\pi}{4n}$, and $x_i^* = x_i$, the expression for the area is $\displaystyle A = \lim_{n\to\infty} \sum_{i=1}^{n} f\left(x_i^*\right)\Delta x = \lim_{n\to\infty} \sum_{i=1}^{n} \tan\left(\frac{i\pi}{4n}\right)\frac{\pi}{4n}$. Note that this answer is not unique, since the expression for the area is the same for the function $y = \tan(x - k\pi)$ on the interval $\left[k\pi, k\pi + \frac{\pi}{4}\right]$, where k is any integer.

23. (a) $y = f(x) = x^5$. $\Delta x = \dfrac{2-0}{n} = \dfrac{2}{n}$ and $x_i = 0 + i\,\Delta x = \dfrac{2i}{n}$.

$$A = \lim_{n\to\infty} R_n = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\,\Delta x = \lim_{n\to\infty} \sum_{i=1}^{n} \left(\frac{2i}{n}\right)^5 \cdot \frac{2}{n} = \lim_{n\to\infty} \sum_{i=1}^{n} \frac{32i^5}{n^5} \cdot \frac{2}{n} = \lim_{n\to\infty} \frac{64}{n^6} \sum_{i=1}^{n} i^5.$$

(b) $\displaystyle\sum_{i=1}^{n} i^5 \overset{\text{CAS}}{=} \frac{n^2(n+1)^2(2n^2+2n-1)}{12}$

(c)
$$\lim_{n\to\infty} \frac{64}{n^6} \cdot \frac{n^2(n+1)^2(2n^2+2n-1)}{12} = \frac{64}{12} \lim_{n\to\infty} \frac{(n^2+2n+1)(2n^2+2n-1)}{n^2 \cdot n^2}$$
$$= \frac{16}{3} \lim_{n\to\infty} \left(1 + \frac{2}{n} + \frac{1}{n^2}\right)\left(2 + \frac{2}{n} - \frac{1}{n^2}\right) = \tfrac{16}{3} \cdot 1 \cdot 2 = \tfrac{32}{3}$$

25. $y = f(x) = \cos x$. $\Delta x = \dfrac{b-0}{n} = \dfrac{b}{n}$ and $x_i = 0 + i\,\Delta x = \dfrac{bi}{n}$.

$$A = \lim_{n\to\infty} R_n = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\,\Delta x = \lim_{n\to\infty} \sum_{i=1}^{n} \cos\left(\frac{bi}{n}\right) \cdot \frac{b}{n} \overset{\text{CAS}}{=} \lim_{n\to\infty} \left[\frac{b\sin\left(b\left(\frac{1}{2n} + 1\right)\right)}{2n\sin\left(\frac{b}{2n}\right)} - \frac{b}{2n}\right] \overset{\text{CAS}}{=} \sin b$$

If $b = \frac{\pi}{2}$, then $A = \sin\frac{\pi}{2} = 1$.

5.2 The Definite Integral

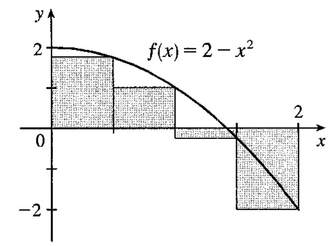
1. $R_4 = \sum_{i=1}^{4} f(x_i)\,\Delta x \quad [x_i^* = x_i \text{ is a right endpoint and } \Delta x = 0.5]$

$= 0.5\,[f(0.5) + f(1) + f(1.5) + f(2)] \quad [f(x) = 2 - x^2]$

$= 0.5\,[1.75 + 1 + (-0.25) + (-2)]$

$= 0.5(0.5) = 0.25$

The Riemann sum represents the sum of the areas of the two rectangles above the x-axis minus the sum of the areas of the two rectangles below the x-axis; that is, the *net area* of the rectangles with respect to the x-axis.

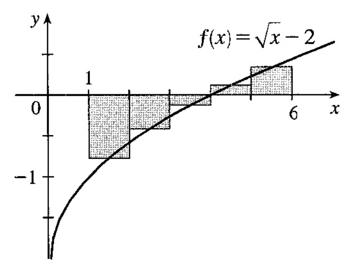


3. $M_5 = \sum_{i=1}^{5} f(\overline{x}_i)\,\Delta x \quad [x_i^* = \overline{x}_i = \frac{1}{2}(x_{i-1} + x_i) \text{ is a midpoint and } \Delta x = 1]$

$= 1\,[f(1.5) + f(2.5) + f(3.5) + f(4.5) + f(5.5)] \quad [f(x) = \sqrt{x} - 2]$

≈ -0.856759

The Riemann sum represents the sum of the areas of the two rectangles above the x-axis minus the sum of the areas of the three rectangles below the x-axis.



5. $\Delta x = (b - a)/n = (8 - 0)/4 = 8/4 = 2.$

(a) Using the right endpoints to approximate $\int_0^8 f(x)\,dx$, we have

$$\sum_{i=1}^{4} f(x_i)\,\Delta x = 2[f(2) + f(4) + f(6) + f(8)] \approx 2[1 + 2 + (-2) + 1] = 4.$$

(b) Using the left endpoints to approximate $\int_0^8 f(x)\,dx$, we have

$$\sum_{i=1}^{4} f(x_{i-1})\,\Delta x = 2[f(0) + f(2) + f(4) + f(6)] \approx 2[2 + 1 + 2 + (-2)] = 6.$$

(c) Using the midpoint of each subinterval to approximate $\int_0^8 f(x)\,dx$, we have

$$\sum_{i=1}^{4} f(\overline{x}_i)\,\Delta x = 2[f(1) + f(3) + f(5) + f(7)] \approx 2[3 + 2 + 1 + (-1)] = 10.$$

7. Since f is increasing, $L_5 \le \int_0^{25} f(x)\,dx \le R_5$.

$$\text{Lower estimate} = L_5 = \sum_{i=1}^{5} f(x_{i-1})\,\Delta x = 5[f(0) + f(5) + f(10) + f(15) + f(20)]$$
$$= 5(-42 - 37 - 25 - 6 + 15) = 5(-95) = -475$$

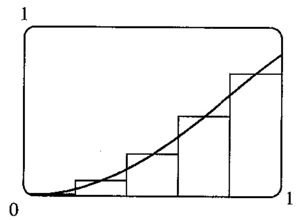
$$\text{Upper estimate} = R_5 = \sum_{i=1}^{5} f(x_i)\,\Delta x = 5[f(5) + f(10) + f(15) + f(20) + f(25)]$$
$$= 5(-37 - 25 - 6 + 15 + 36) = 5(-17) = -85$$

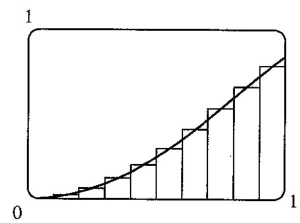
9. $\Delta x = (10 - 2)/4 = 2$, so the endpoints are 2, 4, 6, 8, and 10, and the midpoints are 3, 5, 7, and 9. The Midpoint Rule gives $\int_2^{10} \sqrt{x^3 + 1}\,dx \approx \sum_{i=1}^{4} f(\overline{x}_i)\,\Delta x = 2\left(\sqrt{3^3 + 1} + \sqrt{5^3 + 1} + \sqrt{7^3 + 1} + \sqrt{9^3 + 1}\right) \approx 124.1644.$

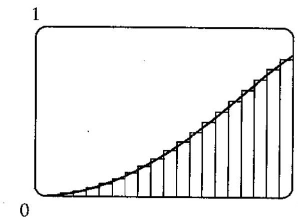
11. $\Delta x = (1-0)/5 = 0.2$, so the endpoints are 0, 0.2, 0.4, 0.6, 0.8, and 1, and the midpoints are 0.1, 0.3, 0.5, 0.7, and 0.9. The Midpoint Rule gives

$$\int_0^1 \sin(x^2)\,dx \approx \sum_{i=1}^{5} f(\overline{x}_i)\,\Delta x = 0.2\left[\sin(0.1)^2 + \sin(0.3)^2 + \sin(0.5)^2 + \sin(0.7)^2 + \sin(0.9)^2\right] \approx 0.3084.$$

13. In Maple, we use the command `with(student);` to load the sum and box commands, then `m:=middlesum(sin(x^2),x=0..1,5);` which gives us the sum in summation notation, then `M:=evalf(m);` which gives $M_5 \approx 0.30843908$, confirming the result of Exercise 11. The command `middlebox(sin(x^2),x=0..1,5)` generates the graph. Repeating for $n = 10$ and $n = 20$ gives $M_{10} \approx 0.30981629$ and $M_{20} \approx 0.31015563$.







15. We'll create the table of values to approximate $\int_0^\pi \sin x\,dx$ by using the program in the solution to Exercise 5.1.7 with $Y_1 = \sin x$, Xmin $= 0$, Xmax $= \pi$, and $n = 5$, 10, 50, and 100.

n	R_n
5	1.933766
10	1.983524
50	1.999342
100	1.999836

The values of R_n appear to be approaching 2.

17. On $[0, \pi]$, $\lim\limits_{n\to\infty} \sum\limits_{i=1}^{n} x_i \sin x_i\,\Delta x = \int_0^\pi x \sin x\,dx$.

19. On $[1, 8]$, $\lim\limits_{n\to\infty} \sum\limits_{i=1}^{n} \sqrt{2x_i^* + (x_i^*)^2}\,\Delta x = \int_1^8 \sqrt{2x + x^2}\,dx$.

21. Note that $\Delta x = \dfrac{5-(-1)}{n} = \dfrac{6}{n}$ and $x_i = -1 + i\,\Delta x = -1 + \dfrac{6i}{n}$.

$$\int_{-1}^{5} (1+3x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\,\Delta x = \lim_{n\to\infty} \sum_{i=1}^{n} \left[1 + 3\left(-1 + \frac{6i}{n}\right)\right]\frac{6}{n}$$

$$= \lim_{n\to\infty} \frac{6}{n} \sum_{i=1}^{n} \left[-2 + \frac{18i}{n}\right] = \lim_{n\to\infty} \frac{6}{n}\left[\sum_{i=1}^{n}(-2) + \sum_{i=1}^{n} \frac{18i}{n}\right]$$

$$= \lim_{n\to\infty} \frac{6}{n}\left[-2n + \frac{18}{n}\sum_{i=1}^{n} i\right] = \lim_{n\to\infty} \frac{6}{n}\left[-2n + \frac{18}{n} \cdot \frac{n(n+1)}{2}\right]$$

$$= \lim_{n\to\infty} \left[-12 + \frac{108}{n^2} \cdot \frac{n(n+1)}{2}\right] = \lim_{n\to\infty} \left[-12 + 54\frac{n+1}{n}\right]$$

$$= \lim_{n\to\infty} \left[-12 + 54\left(1 + \frac{1}{n}\right)\right] = -12 + 54 \cdot 1 = 42$$

23. Note that $\Delta x = \dfrac{2-0}{n} = \dfrac{2}{n}$ and $x_i = 0 + i\,\Delta x = \dfrac{2i}{n}$.

$$\int_0^2 (2-x^2)\,dx = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\,\Delta x = \lim_{n\to\infty}\sum_{i=1}^{n}\left(2-\frac{4i^2}{n^2}\right)\left(\frac{2}{n}\right)$$

$$= \lim_{n\to\infty}\frac{2}{n}\left[\sum_{i=1}^{n} 2 - \frac{4}{n^2}\sum_{i=1}^{n} i^2\right] = \lim_{n\to\infty}\frac{2}{n}\left(2n - \frac{4}{n^2}\sum_{i=1}^{n} i^2\right)$$

$$= \lim_{n\to\infty}\left[4 - \frac{8}{n^3}\cdot\frac{n(n+1)(2n+1)}{6}\right] = \lim_{n\to\infty}\left(4 - \frac{4}{3}\cdot\frac{n+1}{n}\cdot\frac{2n+1}{n}\right)$$

$$= \lim_{n\to\infty}\left[4 - \frac{4}{3}\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right)\right] = 4 - \tfrac{4}{3}\cdot 1\cdot 2 = \tfrac{4}{3}$$

25. Note that $\Delta x = \dfrac{2-1}{n} = \dfrac{1}{n}$ and $x_i = 1 + i\,\Delta x = 1 + i(1/n) = 1 + i/n$.

$$\int_1^2 x^3\,dx = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\,\Delta x = \lim_{n\to\infty}\sum_{i=1}^{n}\left(1+\frac{i}{n}\right)^3\left(\frac{1}{n}\right) = \lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}\left(\frac{n+i}{n}\right)^3$$

$$= \lim_{n\to\infty}\frac{1}{n^4}\sum_{i=1}^{n}(n^3+3n^2i+3ni^2+i^3) = \lim_{n\to\infty}\frac{1}{n^4}\left[\sum_{i=1}^{n} n^3 + \sum_{i=1}^{n} 3n^2 i + \sum_{i=1}^{n} 3ni^2 + \sum_{i=1}^{n} i^3\right]$$

$$= \lim_{n\to\infty}\frac{1}{n^4}\left[n\cdot n^3 + 3n^2\sum_{i=1}^{n} i + 3n\sum_{i=1}^{n} i^2 + \sum_{i=1}^{n} i^3\right]$$

$$= \lim_{n\to\infty}\left[1 + \frac{3}{n^2}\cdot\frac{n(n+1)}{2} + \frac{3}{n^3}\cdot\frac{n(n+1)(2n+1)}{6} + \frac{1}{n^4}\cdot\frac{n^2(n+1)^2}{4}\right]$$

$$= \lim_{n\to\infty}\left[1 + \frac{3}{2}\cdot\frac{n+1}{n} + \frac{1}{2}\cdot\frac{n+1}{n}\cdot\frac{2n+1}{n} + \frac{1}{4}\cdot\frac{(n+1)^2}{n^2}\right]$$

$$= \lim_{n\to\infty}\left[1 + \frac{3}{2}\left(1+\frac{1}{n}\right) + \frac{1}{2}\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right) + \frac{1}{4}\left(1+\frac{1}{n}\right)^2\right] = 1 + \frac{3}{2} + \frac{1}{2}\cdot 2 + \frac{1}{4} = 3.75$$

27. $$\int_a^b x\,dx = \lim_{n\to\infty}\frac{b-a}{n}\sum_{i=1}^{n}\left[a + \frac{b-a}{n}i\right] = \lim_{n\to\infty}\left[\frac{a(b-a)}{n}\sum_{i=1}^{n} 1 + \frac{(b-a)^2}{n^2}\sum_{i=1}^{n} i\right]$$

$$= \lim_{n\to\infty}\left[\frac{a(b-a)}{n}n + \frac{(b-a)^2}{n^2}\cdot\frac{n(n+1)}{2}\right] = a(b-a) + \lim_{n\to\infty}\frac{(b-a)^2}{2}\left(1+\frac{1}{n}\right)$$

$$= a(b-a) + \tfrac{1}{2}(b-a)^2 = (b-a)(a + \tfrac{1}{2}b - \tfrac{1}{2}a) = (b-a)\tfrac{1}{2}(b+a) = \tfrac{1}{2}(b^2-a^2)$$

29. $f(x) = \dfrac{x}{1+x^5}$, $a = 2$, $b = 6$, and $\Delta x = \dfrac{6-2}{n} = \dfrac{4}{n}$. Using Equation 3, we get $x_i^* = x_i = 2 + i\,\Delta x = 2 + \dfrac{4i}{n}$,

so $$\int_2^6 \frac{x}{1+x^5}\,dx = \lim_{n\to\infty} R_n = \lim_{n\to\infty}\sum_{i=1}^{n}\frac{2+\dfrac{4i}{n}}{1+\left(2+\dfrac{4i}{n}\right)^5}\cdot\frac{4}{n}.$$

31. $\Delta x = (\pi - 0)/n = \pi/n$ and $x_i^* = x_i = \pi i/n$.

$$\int_0^\pi \sin 5x\,dx = \lim_{n\to\infty}\sum_{i=1}^{n}(\sin 5x_i)\left(\frac{\pi}{n}\right) = \lim_{n\to\infty}\sum_{i=1}^{n}\left(\sin\frac{5\pi i}{n}\right)\frac{\pi}{n} \overset{\text{CAS}}{=} \pi\lim_{n\to\infty}\frac{1}{n}\cot\left(\frac{5\pi}{2n}\right) \overset{\text{CAS}}{=} \pi\left(\frac{2}{5\pi}\right) = \frac{2}{5}$$

33. (a) Think of $\int_0^2 f(x)\,dx$ as the area of a trapezoid with bases 1 and 3 and height 2. The area of a trapezoid is $A = \frac{1}{2}(b+B)h$, so $\int_0^2 f(x)\,dx = \frac{1}{2}(1+3)2 = 4$.

(b) $\int_0^5 f(x)\,dx = \int_0^2 f(x)\,dx + \int_2^3 f(x)\,dx + \int_3^5 f(x)\,dx$

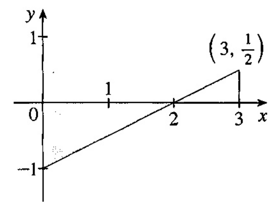
trapezoid rectangle triangle

$= \frac{1}{2}(1+3)2 + 3\cdot 1 + \frac{1}{2}\cdot 2\cdot 3 = 4+3+3 = 10$

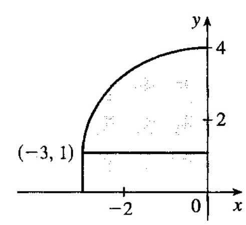
(c) $\int_5^7 f(x)\,dx$ is the negative of the area of the triangle with base 2 and height 3. $\int_5^7 f(x)\,dx = -\frac{1}{2}\cdot 2\cdot 3 = -3$.

(d) $\int_7^9 f(x)\,dx$ is the negative of the area of a trapezoid with bases 3 and 2 and height 2, so it equals $-\frac{1}{2}(B+b)h = -\frac{1}{2}(3+2)2 = -5$. Thus,
$\int_0^9 f(x)\,dx = \int_0^5 f(x)\,dx + \int_5^7 f(x)\,dx + \int_7^9 f(x)\,dx = 10 + (-3) + (-5) = 2$.

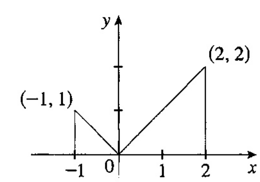
35. $\int_0^3 \left(\frac{1}{2}x - 1\right) dx$ can be interpreted as the area of the triangle above the x-axis minus the area of the triangle below the x-axis; that is,
$\frac{1}{2}(1)\left(\frac{1}{2}\right) - \frac{1}{2}(2)(1) = \frac{1}{4} - 1 = -\frac{3}{4}$.



37. $\int_{-3}^0 \left(1 + \sqrt{9-x^2}\right) dx$ can be interpreted as the area under the graph of $f(x) = 1 + \sqrt{9-x^2}$ between $x=-3$ and $x=0$. This is equal to one-quarter the area of the circle with radius 3, plus the area of the rectangle, so
$\int_{-3}^0 \left(1 + \sqrt{9-x^2}\right) dx = \frac{1}{4}\pi\cdot 3^2 + 1\cdot 3 = 3 + \frac{9}{4}\pi$.



39. $\int_{-1}^2 |x|\,dx$ can be interpreted as the sum of the areas of the two shaded triangles; that is, $\frac{1}{2}(1)(1) + \frac{1}{2}(2)(2) = \frac{1}{2} + \frac{4}{2} = \frac{5}{2}$.



41. $\int_9^4 \sqrt{t}\,dt = -\int_4^9 \sqrt{t}\,dt$ [because we reversed the limits of integration]

$= -\int_4^9 \sqrt{x}\,dx$ [we can use any letter without changing the value of the integral]

$= -\frac{38}{3}$

43. $\int_0^1 (5 - 6x^2)\,dx = \int_0^1 5\,dx - 6\int_0^1 x^2\,dx = 5(1-0) - 6\left(\frac{1}{3}\right) = 5 - 2 = 3$

45. $\int_1^3 e^{x+2}\,dx = \int_1^3 e^x \cdot e^2\,dx = e^2 \int_1^3 e^x\,dx = e^2(e^3 - e) = e^5 - e^3$

47. $\int_{-2}^2 f(x)\,dx + \int_2^5 f(x)\,dx - \int_{-2}^{-1} f(x)\,dx = \int_{-2}^5 f(x)\,dx + \int_{-1}^{-2} f(x)\,dx$ [by Property 5 and reversing limits]

$= \int_{-1}^5 f(x)\,dx$ [Property 5]

49. $\int_0^9 [2f(x) + 3g(x)]\,dx = 2\int_0^9 f(x)\,dx + 3\int_0^9 g(x)\,dx = 2(37) + 3(16) = 122$

51. $0 \le \sin x < 1$ on $\left[0, \frac{\pi}{4}\right]$, so $\sin^3 x \le \sin^2 x$ on $\left[0, \frac{\pi}{4}\right]$. Hence, $\int_0^{\pi/4} \sin^3 x\,dx \le \int_0^{\pi/4} \sin^2 x\,dx$ (Property 7).

53. If $-1 \le x \le 1$, then $0 \le x^2 \le 1$ and $1 \le 1 + x^2 \le 2$, so $1 \le \sqrt{1+x^2} \le \sqrt{2}$ and $1[1-(-1)] \le \int_{-1}^{1} \sqrt{1+x^2}\,dx \le \sqrt{2}\,[1-(-1)]$ [Property 8]; that is, $2 \le \int_{-1}^{1} \sqrt{1+x^2}\,dx \le 2\sqrt{2}$.

55. If $1 \le x \le 2$, then $\frac{1}{2} \le \frac{1}{x} \le 1$, so $\frac{1}{2}(2-1) \le \int_1^2 \frac{1}{x}\,dx \le 1(2-1)$ or $\frac{1}{2} \le \int_1^2 \frac{1}{x}\,dx \le 1$.

57. If $\frac{\pi}{4} \le x \le \frac{\pi}{3}$, then $1 \le \tan x \le \sqrt{3}$, so $1\left(\frac{\pi}{3} - \frac{\pi}{4}\right) \le \int_{\pi/4}^{\pi/3} \tan x\,dx \le \sqrt{3}\left(\frac{\pi}{3} - \frac{\pi}{4}\right)$ or $\frac{\pi}{12} \le \int_{\pi/4}^{\pi/3} \tan x\,dx \le \frac{\pi}{12}\sqrt{3}$.

59. The only critical number of $f(x) = xe^{-x}$ on $[0, 2]$ is $x = 1$. Since $f(0) = 0$, $f(1) = e^{-1} \approx 0.368$, and $f(2) = 2e^{-2} \approx 0.271$, we know that the absolute minimum value of f on $[0, 2]$ is 0, and the absolute maximum is e^{-1}. By Property 8, $0 \le xe^{-x} \le e^{-1}$ for $0 \le x \le 2$ $\Rightarrow$ $0(2-0) \le \int_0^2 xe^{-x}\,dx \le e^{-1}(2-0)$ $\Rightarrow$ $0 \le \int_0^2 xe^{-x}\,dx \le 2/e$.

61. $\sqrt{x^4+1} \ge \sqrt{x^4} = x^2$, so $\int_1^3 \sqrt{x^4+1}\,dx \ge \int_1^3 x^2\,dx = \frac{1}{3}\left(3^3 - 1^3\right) = \frac{26}{3}$.

63. Using a regular partition and right endpoints as in the proof of Property 2, we calculate

$\int_a^b cf(x)\,dx = \lim\limits_{n\to\infty} \sum\limits_{i=1}^{n} cf(x_i)\,\Delta x_i = \lim\limits_{n\to\infty} c\sum\limits_{i=1}^{n} f(x_i)\,\Delta x_i = c\lim\limits_{n\to\infty} \sum\limits_{i=1}^{n} f(x_i)\,\Delta x_i = c\int_a^b f(x)\,dx.$

65. Since $-|f(x)| \le f(x) \le |f(x)|$, it follows from Property 7 that

$$-\int_a^b |f(x)|\,dx \le \int_a^b f(x)\,dx \le \int_a^b |f(x)|\,dx \quad\Rightarrow\quad \left|\int_a^b f(x)\,dx\right| \le \int_a^b |f(x)|\,dx$$

Note that the definite integral is a real number, and so the following property applies: $-a \le b \le a \Rightarrow |b| \le a$ for all real numbers b and nonnegative numbers a.

67. $\displaystyle \lim_{n\to\infty} \sum_{i=1}^{n} \frac{i^4}{n^5} = \lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} \left(\frac{i}{n}\right)^4 = \int_0^1 x^4\,dx$

69. Choose $x_i = 1 + \dfrac{i}{n}$ and $x_i^* = \sqrt{x_{i-1}x_i} = \sqrt{\left(1 + \dfrac{i-1}{n}\right)\left(1 + \dfrac{i}{n}\right)}$. Then

$$\begin{aligned}
\int_1^2 x^{-2}dx &= \lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} \frac{1}{\left(1+\frac{i-1}{n}\right)\left(1+\frac{i}{n}\right)} \\
&= \lim_{n\to\infty} n\sum_{i=1}^{n} \frac{1}{(n+i-1)(n+i)} \\
&= \lim_{n\to\infty} n\sum_{i=1}^{n} \left(\frac{1}{n+i-1} - \frac{1}{n+i}\right) \qquad \text{[by the hint]} \\
&= \lim_{n\to\infty} n\left(\sum_{i=0}^{n-1} \frac{1}{n+i} - \sum_{i=1}^{n} \frac{1}{n+i}\right) \\
&= \lim_{n\to\infty} n\left(\left[\frac{1}{n} + \frac{1}{n+1} + \cdots + \frac{1}{2n-1}\right] - \left[\frac{1}{n+1} + \cdots + \frac{1}{2n-1} + \frac{1}{2n}\right]\right) \\
&= \lim_{n\to\infty} n\left(\frac{1}{n} - \frac{1}{2n}\right) = \lim_{n\to\infty}\left(1 - \tfrac{1}{2}\right) = \tfrac{1}{2}
\end{aligned}$$

21. $\int_2^8 (4x+3)\,dx = \left[\frac{4}{2}x^2 + 3x\right]_2^8 = (2\cdot 8^2 + 3\cdot 8) - (2\cdot 2^2 + 3\cdot 2) = 152 - 14 = 138$

23. $\int_0^1 x^{4/5}\,dx = \left[\frac{5}{9}x^{9/5}\right]_0^1 = \frac{5}{9} - 0 = \frac{5}{9}$

25. $\displaystyle\int_1^2 \frac{3}{t^4}\,dt = 3\int_1^2 t^{-4}\,dt = 3\left[\frac{t^{-3}}{-3}\right]_1^2 = \frac{3}{-3}\left[\frac{1}{t^3}\right]_1^2 = -1\left(\frac{1}{8} - 1\right) = \frac{7}{8}$

27. $\displaystyle\int_{-5}^5 \frac{2}{x^3}\,dx$ does not exist because the function $f(x) = \dfrac{2}{x^3}$ has an infinite discontinuity at $x = 0$; that is, f is discontinuous on the interval $[-5, 5]$.

29. $\int_0^2 x(2+x^5)\,dx = \int_0^2 (2x + x^6)\,dx = \left[x^2 + \frac{1}{7}x^7\right]_0^2 = \left(4 + \frac{128}{7}\right) - (0+0) = \frac{156}{7}$

31. $\int_0^{\pi/4} \sec^2 t\,dt = [\tan t]_0^{\pi/4} = \tan\frac{\pi}{4} - \tan 0 = 1 - 0 = 1$

33. $\int_\pi^{2\pi} \csc^2\theta\,d\theta$ does not exist because the function $f(\theta) = \csc^2\theta$ has infinite discontinuities at $\theta = \pi$ and $\theta = 2\pi$; that is, f is discontinuous on the interval $[\pi, 2\pi]$.

35. $\displaystyle\int_1^9 \frac{1}{2x}\,dx = \frac{1}{2}\int_1^9 \frac{1}{x}\,dx = \frac{1}{2}\Big[\ln|x|\Big]_1^9 = \frac{1}{2}(\ln 9 - \ln 1) = \frac{1}{2}\ln 9 - 0 = \ln 9^{1/2} = \ln 3$

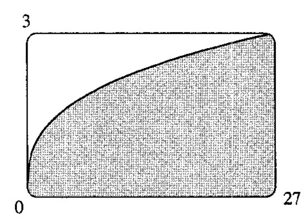
37. $\displaystyle\int_{1/2}^{\sqrt{3}/2} \frac{6}{\sqrt{1-t^2}}\,dt = 6\int_{1/2}^{\sqrt{3}/2} \frac{1}{\sqrt{1-t^2}}\,dt = 6\left[\sin^{-1} t\right]_{1/2}^{\sqrt{3}/2} = 6\left[\sin^{-1}\left(\frac{\sqrt{3}}{2}\right) - \sin^{-1}\left(\frac{1}{2}\right)\right]$

$= 6\left(\frac{\pi}{3} - \frac{\pi}{6}\right) = 6\left(\frac{\pi}{6}\right) = \pi$

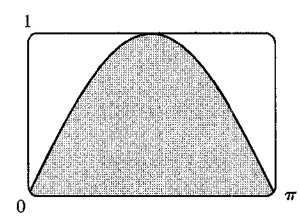
39. $\int_{-1}^1 e^{u+1}\,du = \left[e^{u+1}\right]_{-1}^1 = e^2 - e^0 = e^2 - 1$ [or start with $e^{u+1} = e^u e^1$]

41. $\int_0^2 f(x)\,dx = \int_0^1 x^4\,dx + \int_1^2 x^5\,dx = \left[\frac{1}{5}x^5\right]_0^1 + \left[\frac{1}{6}x^6\right]_1^2 = \left(\frac{1}{5} - 0\right) + \left(\frac{64}{6} - \frac{1}{6}\right) = 10.7$

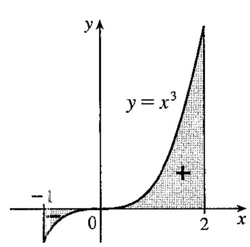
43. From the graph, it appears that the area is about 60. The actual area is $\int_0^{27} x^{1/3}dx = \left[\frac{3}{4}x^{4/3}\right]_0^{27} = \frac{3}{4}\cdot 81 - 0 = \frac{243}{4} = 60.75$. This is $\frac{3}{4}$ of the area of the viewing rectangle.



45. It appears that the area under the graph is about $\frac{2}{3}$ of the area of the viewing rectangle, or about $\frac{2}{3}\pi \approx 2.1$. The actual area is $\int_0^\pi \sin x\,dx = [-\cos x]_0^\pi = (-\cos\pi) - (-\cos 0) = -(-1) + 1 = 2$.



47. $\int_{-1}^2 x^3\,dx = \left[\frac{1}{4}x^4\right]_{-1}^2 = 4 - \frac{1}{4} = \frac{15}{4} = 3.75$



49. $g(x) = \int_{2x}^{3x} \frac{u^2-1}{u^2+1}\,du = \int_{2x}^{0} \frac{u^2-1}{u^2+1}\,du + \int_{0}^{3x} \frac{u^2-1}{u^2+1}\,du = -\int_{0}^{2x} \frac{u^2-1}{u^2+1}\,du + \int_{0}^{3x} \frac{u^2-1}{u^2+1}\,du \Rightarrow$

$g'(x) = -\frac{(2x)^2-1}{(2x)^2+1} \cdot \frac{d}{dx}(2x) + \frac{(3x)^2-1}{(3x)^2+1} \cdot \frac{d}{dx}(3x) = -2 \cdot \frac{4x^2-1}{4x^2+1} + 3 \cdot \frac{9x^2-1}{9x^2+1}$

51. $y = \int_{\sqrt{x}}^{x^3} \sqrt{t}\sin t\,dt = \int_{\sqrt{x}}^{1} \sqrt{t}\sin t\,dt + \int_{1}^{x^3} \sqrt{t}\sin t\,dt = -\int_{1}^{\sqrt{x}} \sqrt{t}\sin t\,dt + \int_{1}^{x^3} \sqrt{t}\sin t\,dt \Rightarrow$

$y' = -\sqrt[4]{x}\,(\sin\sqrt{x}) \cdot \frac{d}{dx}(\sqrt{x}) + x^{3/2}\sin(x^3) \cdot \frac{d}{dx}(x^3) = -\frac{\sqrt[4]{x}\sin\sqrt{x}}{2\sqrt{x}} + x^{3/2}\sin(x^3)\,(3x^2)$

$= 3x^{7/2}\sin(x^3) - \frac{\sin\sqrt{x}}{2\sqrt[4]{x}}$

53. $F(x) = \int_1^x f(t)\,dt \Rightarrow F'(x) = f(x) = \int_1^{x^2} \frac{\sqrt{1+u^4}}{u}\,du \left[\text{since } f(t) = \int_1^{t^2} \frac{\sqrt{1+u^4}}{u}\,du\right] \Rightarrow$

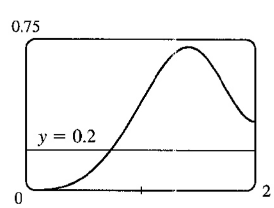
$F''(x) = f'(x) = \frac{\sqrt{1+(x^2)^4}}{x^2} \cdot \frac{d}{dx}(x^2) = \frac{\sqrt{1+x^8}}{x^2} \cdot 2x = \frac{2\sqrt{1+x^8}}{x}$. So $F''(2) = \sqrt{1+2^8} = \sqrt{257}$.

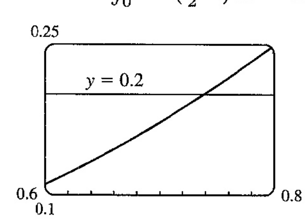
55. By FTC2, $\int_1^4 f'(x)\,dx = f(4) - f(1)$, so $17 = f(4) - 12 \Rightarrow f(4) = 17 + 12 = 29$.

57. (a) The Fresnel function $S(x) = \int_0^x \sin(\frac{\pi}{2}t^2)\,dt$ has local maximum values where $0 = S'(x) = \sin(\frac{\pi}{2}x^2)$ and S' changes from positive to negative. For $x > 0$, this happens when $\frac{\pi}{2}x^2 = (2n-1)\pi$ [odd multiples of π] $\Leftrightarrow$ $x^2 = 2(2n-1) \Leftrightarrow x = \sqrt{4n-2}$, n any positive integer. For $x < 0$, S' changes from positive to negative where $\frac{\pi}{2}x^2 = 2n\pi$ [even multiples of π] $\Leftrightarrow x^2 = 4n \Leftrightarrow x = -2\sqrt{n}$. S' does not change sign at $x = 0$.

(b) S is concave upward on those intervals where $S''(x) > 0$. Differentiating our expression for $S'(x)$, we get $S''(x) = \cos(\frac{\pi}{2}x^2)(2\frac{\pi}{2}x) = \pi x\cos(\frac{\pi}{2}x^2)$. For $x > 0$, $S''(x) > 0$ where $\cos(\frac{\pi}{2}x^2) > 0 \Leftrightarrow 0 < \frac{\pi}{2}x^2 < \frac{\pi}{2}$ or $(2n - \frac{1}{2})\pi < \frac{\pi}{2}x^2 < (2n + \frac{1}{2})\pi$, n any integer $\Leftrightarrow 0 < x < 1$ or $\sqrt{4n-1} < x < \sqrt{4n+1}$, n any positive integer. For $x < 0$, $S''(x) > 0$ where $\cos(\frac{\pi}{2}x^2) < 0 \Leftrightarrow (2n - \frac{3}{2})\pi < \frac{\pi}{2}x^2 < (2n - \frac{1}{2})\pi$, n any integer $\Leftrightarrow 4n - 3 < x^2 < 4n - 1 \Leftrightarrow \sqrt{4n-3} < |x| < \sqrt{4n-1} \Rightarrow \sqrt{4n-3} < -x < \sqrt{4n-1}$ $\Rightarrow -\sqrt{4n-3} > x > -\sqrt{4n-1}$, so the intervals of upward concavity for $x < 0$ are $(-\sqrt{4n-1}, -\sqrt{4n-3})$, n any positive integer. To summarize: S is concave upward on the intervals $(0,1)$, $(-\sqrt{3},-1)$, $(\sqrt{3},\sqrt{5})$, $(-\sqrt{7},-\sqrt{5})$, $(\sqrt{7},3)$, ...

(c) In Maple, we use `plot({int(sin(Pi*t^2/2),t=0..x),0.2},x=0..2);`. Note that Maple recognizes the Fresnel function, calling it `FresnelS(x)`. In Mathematica, we use `Plot[{Integrate[Sin[Pi*t^2/2],{t,0,x}],0.2},{x,0,2}]`. In Derive, we load the utility file FRESNEL and plot `FRESNEL_SIN(x)`. From the graphs, we see that $\int_0^x \sin(\frac{\pi}{2}t^2)\,dt = 0.2$ at $x \approx 0.74$.



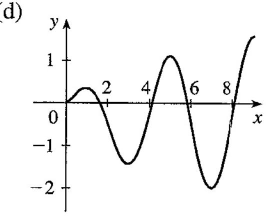


59. (a) By FTC1, $g'(x) = f(x)$. So $g'(x) = f(x) = 0$ at $x = 1, 3, 5, 7$, and 9. g has local maxima at $x = 1$ and 5 (since $f = g'$ changes from positive to negative there) and local minima at $x = 3$ and 7. There is no local maximum or minimum at $x = 9$, since f is not defined for $x > 9$.

(b) We can see from the graph that $\left|\int_0^1 f\,dt\right| < \left|\int_1^3 f\,dt\right| < \left|\int_3^5 f\,dt\right| < \left|\int_5^7 f\,dt\right| < \left|\int_7^9 f\,dt\right|$. So $g(1) = \left|\int_0^1 f\,dt\right|$, $g(5) = \int_0^5 f\,dt = g(1) - \left|\int_1^3 f\,dt\right| + \left|\int_3^5 f\,dt\right|$, and $g(9) = \int_0^9 f\,dt = g(5) - \left|\int_5^7 f\,dt\right| + \left|\int_7^9 f\,dt\right|$. Thus, $g(1) < g(5) < g(9)$, and so the absolute maximum of $g(x)$ occurs at $x = 9$.

(c) g is concave downward on those intervals where $g'' < 0$. But $g'(x) = f(x)$, so $g''(x) = f'(x)$, which is negative on (approximately) $\left(\frac{1}{2}, 2\right)$, $(4, 6)$ and $(8, 9)$. So g is concave downward on these intervals.

(d)



61. $\displaystyle \lim_{n\to\infty} \sum_{i=1}^{n} \frac{i^3}{n^4} = \lim_{n\to\infty} \frac{1-0}{n} \sum_{i=1}^{n} \left(\frac{i}{n}\right)^3 = \int_0^1 x^3\,dx = \left[\frac{x^4}{4}\right]_0^1 = \frac{1}{4}$

63. Suppose $h < 0$. Since f is continuous on $[x + h, x]$, the Extreme Value Theorem says that there are numbers u and v in $[x + h, x]$ such that $f(u) = m$ and $f(v) = M$, where m and M are the absolute minimum and maximum values of f on $[x + h, x]$. By Property 8 of integrals, $m(-h) \le \int_{x+h}^{x} f(t)\,dt \le M(-h)$; that is, $f(u)(-h) \le -\int_x^{x+h} f(t)\,dt \le f(v)(-h)$. Since $-h > 0$, we can divide this inequality by $-h$: $f(u) \le \dfrac{1}{h}\displaystyle\int_x^{x+h} f(t)\,dt \le f(v)$. By Equation 2, $\dfrac{g(x+h) - g(x)}{h} = \dfrac{1}{h}\displaystyle\int_x^{x+h} f(t)\,dt$ for $h \ne 0$, and hence $f(u) \le \dfrac{g(x+h) - g(x)}{h} \le f(v)$, which is Equation 3 in the case where $h < 0$.

65. (a) Let $f(x) = \sqrt{x} \;\Rightarrow\; f'(x) = 1/(2\sqrt{x}) > 0$ for $x > 0 \;\Rightarrow\; f$ is increasing on $(0, \infty)$. If $x \ge 0$, then $x^3 \ge 0$, so $1 + x^3 \ge 1$ and since f is increasing, this means that $f(1 + x^3) \ge f(1) \;\Rightarrow\; \sqrt{1 + x^3} \ge 1$ for $x \ge 0$. Next let $g(t) = t^2 - t \;\Rightarrow\; g'(t) = 2t - 1 \;\Rightarrow\; g'(t) > 0$ when $t \ge 1$. Thus, g is increasing on $(1, \infty)$. And since $g(1) = 0$, $g(t) \ge 0$ when $t \ge 1$. Now let $t = \sqrt{1 + x^3}$, where $x \ge 0$. $\sqrt{1 + x^3} \ge 1$ (from above) $\Rightarrow\; t \ge 1 \;\Rightarrow\; g(t) \ge 0 \;\Rightarrow\; (1 + x^3) - \sqrt{1 + x^3} \ge 0$ for $x \ge 0$. Therefore, $1 \le \sqrt{1 + x^3} \le 1 + x^3$ for $x \ge 0$.

(b) From part (a) and Property 7: $\int_0^1 1\,dx \le \int_0^1 \sqrt{1 + x^3}\,dx \le \int_0^1 (1 + x^3)\,dx \;\Leftrightarrow$ $[x]_0^1 \le \int_0^1 \sqrt{1 + x^3}\,dx \le \left[x + \frac{1}{4}x^4\right]_0^1 \;\Leftrightarrow\; 1 \le \int_0^1 \sqrt{1 + x^3}\,dx \le 1 + \frac{1}{4} = 1.25$.

67. Using FTC1, we differentiate both sides of $6 + \displaystyle\int_a^x \frac{f(t)}{t^2}\,dt = 2\sqrt{x}$ to get $\dfrac{f(x)}{x^2} = 2\,\dfrac{1}{2\sqrt{x}} \;\Rightarrow\; f(x) = x^{3/2}$.

To find a, we substitute $x = a$ in the original equation to obtain $6 + \displaystyle\int_a^a \frac{f(t)}{t^2}\,dt = 2\sqrt{a} \;\Rightarrow\; 6 + 0 = 2\sqrt{a} \;\Rightarrow$ $3 = \sqrt{a} \;\Rightarrow\; a = 9$.

69. (a) Let $F(t) = \int_0^t f(s)\,ds$. Then, by FTC1, $F'(t) = f(t)$ = rate of depreciation, so $F(t)$ represents the loss in value over the interval $[0, t]$.

(b) $C(t) = \dfrac{1}{t}\left[A + \displaystyle\int_0^t f(s)\,ds\right] = \dfrac{A + F(t)}{t}$ represents the average expenditure per unit of t during the interval $[0, t]$, assuming that there has been only one overhaul during that time period. The company wants to minimize average expenditure.

(c) $C(t) = \dfrac{1}{t}\left[A + \displaystyle\int_0^t f(s)\,ds\right]$. Using FTC1, we have $C'(t) = -\dfrac{1}{t^2}\left[A + \displaystyle\int_0^t f(s)\,ds\right] + \dfrac{1}{t} f(t)$.

$$C'(t) = 0 \quad\Rightarrow\quad t\,f(t) = A + \int_0^t f(s)\,ds \quad\Rightarrow\quad f(t) = \frac{1}{t}\left[A + \int_0^t f(s)\,ds\right] = C(t).$$

5.4 Indefinite Integrals and the Net Change Theorem

1. $\dfrac{d}{dx}\left[\sqrt{x^2+1} + C\right] = \dfrac{d}{dx}\left[(x^2+1)^{1/2} + C\right] = \frac{1}{2}(x^2+1)^{-1/2} \cdot 2x = \dfrac{x}{\sqrt{x^2+1}}$

3. $\dfrac{d}{dx}\left[\dfrac{x}{a^2\sqrt{a^2-x^2}} + C\right] = \dfrac{1}{a^2}\dfrac{\sqrt{a^2-x^2} - x(-x)/\sqrt{a^2-x^2}}{a^2-x^2} = \dfrac{1}{a^2}\dfrac{(a^2-x^2)+x^2}{(a^2-x^2)^{3/2}} = \dfrac{1}{\sqrt{(a^2-x^2)^3}}$

5. $\displaystyle\int x^{-3/4}\,dx = \frac{x^{-3/4+1}}{-3/4+1} + C = \frac{x^{1/4}}{1/4} + C = 4x^{1/4} + C$

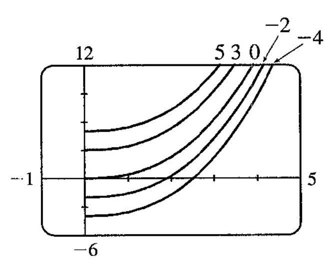
7. $\displaystyle\int (x^3 + 6x + 1)\,dx = \frac{x^4}{4} + 6\frac{x^2}{2} + x + C = \tfrac{1}{4}x^4 + 3x^2 + x + C$

9. $\displaystyle\int (1-t)(2+t^2)\,dt = \int (2 - 2t + t^2 - t^3)\,dt = 2t - 2\frac{t^2}{2} + \frac{t^3}{3} - \frac{t^4}{4} + C = 2t - t^2 + \tfrac{1}{3}t^3 - \tfrac{1}{4}t^4 + C$

11. $\displaystyle\int \left(2 - \sqrt{x}\right)^2 dx = \int \left(4 - 4\sqrt{x} + x\right) dx = 4x - 4\frac{x^{3/2}}{3/2} + \frac{x^2}{2} + C = 4x - \tfrac{8}{3}x^{3/2} + \frac{1}{2}x^2 + C$

13. $\displaystyle\int \frac{\sin x}{1 - \sin^2 x}\,dx = \int \frac{\sin x}{\cos^2 x}\,dx = \int \frac{1}{\cos x} \cdot \frac{\sin x}{\cos x}\,dx = \int \sec x\,\tan x\,dx = \sec x + C$

15. $\int x\sqrt{x}\,dx = \int x^{3/2}\,dx = \frac{2}{5}x^{5/2} + C$.
The members of the family in the figure correspond to $C = 5, 3, 0, -2$, and -4.



17. $\int_0^2 (6x^2 - 4x + 5)\,dx = \left[6 \cdot \tfrac{1}{3}x^3 - 4 \cdot \tfrac{1}{2}x^2 + 5x\right]_0^2 = \left[2x^3 - 2x^2 + 5x\right]_0^2 = (16 - 8 + 10) - 0 = 18$

19. $\int_{-1}^0 (2x - e^x)\,dx = \left[x^2 - e^x\right]_{-1}^0 = (0 - 1) - \left(1 - e^{-1}\right) = -2 + 1/e$

21. $\int_{-2}^{2}(3u+1)^2\,du = \int_{-2}^{2}(9u^2+6u+1)\,du = \left[9\cdot\frac13 u^3+6\cdot\frac12u^2+u\right]_{-2}^{2} = \left[3u^3+3u^2+u\right]_{-2}^{2}$

$= (24+12+2)-(-24+12-2) = 38-(-14)=52$

23. $\int_1^4\sqrt{t}\,(1+t)\,dt = \int_1^4(t^{1/2}+t^{3/2})\,dt = \left[\frac23 t^{3/2}+\frac25t^{5/2}\right]_1^4 = \left(\frac{16}{3}+\frac{64}{5}\right)-\left(\frac23+\frac25\right)=\frac{14}{3}+\frac{62}{5}=\frac{256}{15}$

25. $$\int_{-2}^{-1}\left(4y^3+\frac{2}{y^3}\right)dy = \left[4\cdot\frac14y^4+2\cdot\frac{1}{-2}y^{-2}\right]_{-2}^{-1} = \left[y^4-\frac{1}{y^2}\right]_{-2}^{-1} = (1-1)-\left(16-\frac14\right)=-\frac{63}{4}$$

27. $\int_0^1 x(\sqrt[3]{x}+\sqrt[4]{x})\,dx = \int_0^1(x^{4/3}+x^{5/4})\,dx = \left[\frac37x^{7/3}+\frac49x^{9/4}\right]_0^1 = \left(\frac37+\frac49\right)-0=\frac{55}{63}$

29. $\int_1^4\sqrt{5/x}\,dx = \sqrt5\int_1^4 x^{-1/2}\,dx = \sqrt5\,[2\sqrt{x}\,]_1^4 = \sqrt5\,(2\cdot2-2\cdot1)=2\sqrt5$

31. $\int_0^\pi(4\sin\theta-3\cos\theta)\,d\theta = [-4\cos\theta-3\sin\theta]_0^\pi = (4-0)-(-4-0)=8$

33. $$\int_0^{\pi/4}\frac{1+\cos^2\theta}{\cos^2\theta}\,d\theta = \int_0^{\pi/4}\left(\frac{1}{\cos^2\theta}+\frac{\cos^2\theta}{\cos^2\theta}\right)d\theta = \int_0^{\pi/4}\left(\sec^2\theta+1\right)d\theta$$

$$= [\tan\theta+\theta]_0^{\pi/4} = \left(\tan\frac{\pi}{4}+\frac{\pi}{4}\right)-(0+0)=1+\frac{\pi}{4}$$

35. $$\int_1^{64}\frac{1+\sqrt[3]{x}}{\sqrt{x}}\,dx = \int_1^{64}\left(\frac{1}{x^{1/2}}+\frac{x^{1/3}}{x^{1/2}}\right)dx = \int_1^{64}\left(x^{-1/2}+x^{(1/3)-(1/2)}\right)dx=\int_1^{64}\left(x^{-1/2}+x^{-1/6}\right)dx$$

$$=\left[2x^{1/2}+\frac65x^{5/6}\right]_1^{64} = \left(16+\frac{192}{5}\right)-\left(2+\frac65\right)=14+\frac{186}{5}=\frac{256}{5}$$

37. $$\int_1^e\frac{x^2+x+1}{x}\,dx = \int_1^e\left(x+1+\frac1x\right)dx = \left[\frac12x^2+x+\ln|x|\right]_1^e$$

$$=\left(\frac12e^2+e+\ln e\right)-\left(\frac12+1+\ln 1\right)=\frac12e^2+e-\frac12$$

39. $\int_{-1}^2(x-2|x|)\,dx = \int_{-1}^0[x-2(-x)]\,dx+\int_0^2[x-2(x)]\,dx = \int_{-1}^0 3x\,dx+\int_0^2(-x)\,dx = 3\left[\frac12x^2\right]_{-1}^0-\left[\frac12x^2\right]_0^2$

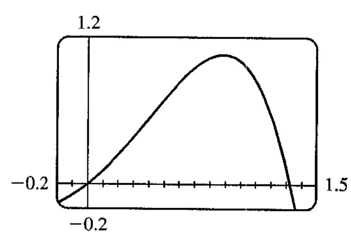
$=3\left(0-\frac12\right)-(2-0)=-\frac72=-3.5$

41. The graph shows that $y=x+x^2-x^4$ has x-intercepts at $x=0$ and at $x=a\approx 1.32$. So the area of the region that lies under the curve and above the x-axis is

$\int_0^a(x+x^2-x^4)\,dx = \left[\frac12x^2+\frac13x^3-\frac15x^5\right]_0^a$

$=\left(\frac12a^2+\frac13a^3-\frac15a^5\right)-0$

≈ 0.84



43. $A=\int_0^2(2y-y^2)\,dy = \left[y^2-\frac13y^3\right]_0^2=\left(4-\frac83\right)-0=\frac43$

45. If $w'(t)$ is the rate of change of weight in pounds per year, then $w(t)$ represents the weight in pounds of the child at age t. We know from the Net Change Theorem that $\int_5^{10}w'(t)\,dt = w(10)-w(5)$, so the integral represents the increase in the child's weight (in pounds) between the ages of 5 and 10.

47. Since $r(t)$ is the rate at which oil leaks, we can write $r(t)=-V'(t)$, where $V(t)$ is the volume of oil at time t. [Note that the minus sign is needed because V is decreasing, so $V'(t)$ is negative, but $r(t)$ is positive.] Thus, by the Net Change Theorem, $\int_0^{120}r(t)\,dt = -\int_0^{120}V'(t)\,dt = -[V(120)-V(0)] = V(0)-V(120)$, which is the number of gallons of oil that leaked from the tank in the first two hours (120 minutes).

49. By the Net Change Theorem, $\int_{1000}^{5000} R'(x)\,dx = R(5000) - R(1000)$, so it represents the increase in revenue when production is increased from 1000 units to 5000 units.

51. In general, the unit of measurement for $\int_a^b f(x)\,dx$ is the product of the unit for $f(x)$ and the unit for x. Since $f(x)$ is measured in newtons and x is measured in meters, the units for $\int_0^{100} f(x)\,dx$ are newton-meters. (A newton-meter is abbreviated N-m and is called a joule.)

53. (a) displacement $= \int_0^3 (3t-5)\,dt = \left[\frac{3}{2}t^2 - 5t\right]_0^3 = \frac{27}{2} - 15 = -\frac{3}{2}$ m

(b) distance traveled $= \int_0^3 |3t-5|\,dt = \int_0^{5/3}(5-3t)\,dt + \int_{5/3}^3 (3t-5)\,dt$

$$= \left[5t - \tfrac{3}{2}t^2\right]_0^{5/3} + \left[\tfrac{3}{2}t^2 - 5t\right]_{5/3}^3 = \tfrac{25}{3} - \tfrac{3}{2}\cdot\tfrac{25}{9} + \tfrac{27}{2} - 15 - \left(\tfrac{3}{2}\cdot\tfrac{25}{9} - \tfrac{25}{3}\right) = \tfrac{41}{6}\text{ m}$$

55. (a) $v'(t) = a(t) = t + 4 \;\Rightarrow\; v(t) = \frac{1}{2}t^2 + 4t + C \;\Rightarrow\; v(0) = C = 5 \;\Rightarrow\; v(t) = \frac{1}{2}t^2 + 4t + 5$ m/s

(b) distance traveled $= \int_0^{10} |v(t)|\,dt = \int_0^{10} \left|\frac{1}{2}t^2 + 4t + 5\right|dt = \int_0^{10}\left(\frac{1}{2}t^2 + 4t + 5\right)dt$

$$= \left[\tfrac{1}{6}t^3 + 2t^2 + 5t\right]_0^{10} = \tfrac{500}{3} + 200 + 50 = 416\tfrac{2}{3}\text{ m}$$

57. Since $m'(x) = \rho(x)$, $m = \int_0^4 \rho(x)\,dx = \int_0^4 (9 + 2\sqrt{x})\,dx = \left[9x + \frac{4}{3}x^{3/2}\right]_0^4 = 36 + \frac{32}{3} - 0 = \frac{140}{3} = 46\frac{2}{3}$ kg.

59. Let s be the position of the car. We know from Equation 2 that $s(100) - s(0) = \int_0^{100} v(t)\,dt$. We use the Midpoint Rule for $0 \le t \le 100$ with $n = 5$. Note that the length of each of the five time intervals is 20 seconds $= \frac{20}{3600}$ hour $= \frac{1}{180}$ hour. So the distance traveled is

$$\begin{aligned}\int_0^{100} v(t)\,dt &\approx \tfrac{1}{180}[v(10) + v(30) + v(50) + v(70) + v(90)]\\ &= \tfrac{1}{180}(38 + 58 + 51 + 53 + 47)\\ &= \tfrac{247}{180} \approx 1.4 \text{ miles}\end{aligned}$$

61. From the Net Change Theorem, the increase in cost if the production level is raised from 2000 yards to 4000 yards is $C(4000) - C(2000) = \int_{2000}^{4000} C'(x)\,dx$.

$$\begin{aligned}\int_{2000}^{4000} C'(x)\,dx &= \int_{2000}^{4000}\left(3 - 0.01x + 0.000006x^2\right)dx\\ &= \left[3x - 0.005x^2 + 0.000002x^3\right]_{2000}^{4000} = 60{,}000 - 2{,}000 = \$58{,}000\end{aligned}$$

63. (a) We can find the area between the Lorenz curve and the line $y = x$ by subtracting the area under $y = L(x)$ from the area under $y = x$. Thus,

$$\begin{aligned}\text{coefficient of inequality} &= \frac{\text{area between Lorenz curve and line } y = x}{\text{area under line } y = x} = \frac{\int_0^1 [x - L(x)]\,dx}{\int_0^1 x\,dx}\\ &= \frac{\int_0^1 [x - L(x)]\,dx}{\left[x^2/2\right]_0^1} = \frac{\int_0^1 [x - L(x)]\,dx}{1/2} = 2\int_0^1 [x - L(x)]\,dx\end{aligned}$$

(b) $L(x) = \frac{5}{12}x^2 + \frac{7}{12}x \;\Rightarrow\; L(50\%) = L\left(\frac{1}{2}\right) = \frac{5}{48} + \frac{7}{24} = \frac{19}{48} = 0.3958\overline{3}$, so the bottom 50% of the households receive at most about 40% of the income. Using the result in part (a),

$$\begin{aligned}\text{coefficient of inequality} &= 2\int_0^1 [x - L(x)]\,dx = 2\int_0^1 \left(x - \tfrac{5}{12}x^2 - \tfrac{7}{12}x\right)dx\\ &= 2\int_0^1 \left(\tfrac{5}{12}x - \tfrac{5}{12}x^2\right)dx = 2\int_0^1 \tfrac{5}{12}\left(x - x^2\right)dx\\ &= \tfrac{5}{6}\left[\tfrac{1}{2}x^2 - \tfrac{1}{3}x^3\right]_0^1 = \tfrac{5}{6}\left(\tfrac{1}{2} - \tfrac{1}{3}\right) = \tfrac{5}{6}\left(\tfrac{1}{6}\right) = \tfrac{5}{36}\end{aligned}$$

5.5 The Substitution Rule

1. Let $u = 3x$. Then $du = 3\,dx$, so $dx = \frac{1}{3}du$. Thus,
$\int \cos 3x\,dx = \int \cos u(\frac{1}{3}\,du) = \frac{1}{3}\int \cos u\,du = \frac{1}{3}\sin u + C = \frac{1}{3}\sin 3x + C$. Don't forget that it is often very easy to check an indefinite integration by differentiating your answer. In this case,
$\frac{d}{dx}\left(\frac{1}{3}\sin 3x + C\right) = \frac{1}{3}(\cos 3x)\cdot 3 = \cos 3x$, the desired result.

3. Let $u = x^3 + 1$. Then $du = 3x^2\,dx$ and $x^2\,dx = \frac{1}{3}\,du$, so
$\int x^2\sqrt{x^3+1}\,dx = \int \sqrt{u}\,(\frac{1}{3}\,du) = \frac{1}{3}\frac{u^{3/2}}{3/2} + C = \frac{1}{3}\cdot\frac{2}{3}u^{3/2} + C = \frac{2}{9}(x^3+1)^{3/2} + C.$

5. Let $u = 1 + 2x$. Then $du = 2\,dx$ and $dx = \frac{1}{2}\,du$, so
$\int \frac{4}{(1+2x)^3}\,dx = 4\int u^{-3}(\frac{1}{2}\,du) = 2\frac{u^{-2}}{-2} + C = -\frac{1}{u^2} + C = -\frac{1}{(1+2x)^2} + C.$

7. Let $u = x^2 + 3$. Then $du = 2x\,dx$, so $\int 2x(x^2+3)^4\,dx = \int u^4\,du = \frac{1}{5}u^5 + C = \frac{1}{5}(x^2+3)^5 + C$.

9. Let $u = 3x - 2$. Then $du = 3\,dx$ and $dx = \frac{1}{3}\,du$, so
$\int (3x-2)^{20}\,dx = \int u^{20}\,(\frac{1}{3}\,du) = \frac{1}{3}\cdot\frac{1}{21}u^{21} + C = \frac{1}{63}(3x-2)^{21} + C.$

11. Let $u = 1 + x + 2x^2$. Then $du = (1 + 4x)\,dx$, so
$\int \frac{1+4x}{\sqrt{1+x+2x^2}}\,dx = \int \frac{du}{\sqrt{u}} = \int u^{-1/2}\,du = \frac{u^{1/2}}{1/2} + C = 2\sqrt{1+x+2x^2} + C.$

13. Let $u = 5 - 3x$. Then $du = -3\,dx$ and $dx = -\frac{1}{3}\,du$, so
$\int \frac{dx}{5-3x} = \int \frac{1}{u}\,(-\frac{1}{3}\,du) = -\frac{1}{3}\ln|u| + C = -\frac{1}{3}\ln|5-3x| + C.$

15. Let $u = 2y + 1$. Then $du = 2\,dy$ and $dy = \frac{1}{2}\,du$, so
$\int \frac{3}{(2y+1)^5}\,dy = \int 3u^{-5}(\frac{1}{2}\,du) = \frac{3}{2}\cdot\frac{1}{-4}u^{-4} + C = \frac{-3}{8(2y+1)^4} + C.$

17. Let $u = 4 - t$. Then $du = -dt$ and $dt = -du$, so
$\int \sqrt{4-t}\,dt = \int u^{1/2}\,(-du) = -\frac{2}{3}u^{3/2} + C = -\frac{2}{3}(4-t)^{3/2} + C.$

19. Let $u = \pi t$. Then $du = \pi\,dt$ and $dt = \frac{1}{\pi}\,du$, so
$\int \sin \pi t\,dt = \int \sin u\,(\frac{1}{\pi}\,du) = \frac{1}{\pi}(-\cos u) + C = -\frac{1}{\pi}\cos \pi t + C.$

21. Let $u = \ln x$. Then $du = \frac{dx}{x}$, so $\int \frac{(\ln x)^2}{x}\,dx = \int u^2\,du = \frac{1}{3}u^3 + C = \frac{1}{3}(\ln x)^3 + C$.

23. Let $u = \sqrt{t}$. Then $du = \frac{dt}{2\sqrt{t}}$ and $\frac{1}{\sqrt{t}}\,dt = 2\,du$, so
$\int \frac{\cos\sqrt{t}}{\sqrt{t}}\,dt = \int \cos u\,(2\,du) = 2\sin u + C = 2\sin\sqrt{t} + C.$

25. Let $u = \sin\theta$. Then $du = \cos\theta\,d\theta$, so $\int \cos\theta\,\sin^6\theta\,d\theta = \int u^6\,du = \frac{1}{7}u^7 + C = \frac{1}{7}\sin^7\theta + C$.

27. Let $u = 1 + e^x$. Then $du = e^x\,dx$, so $\int e^x\sqrt{1+e^x}\,dx = \int \sqrt{u}\,du = \frac{2}{3}u^{3/2} + C = \frac{2}{3}(1+e^x)^{3/2} + C$.
Or: Let $u = \sqrt{1+e^x}$. Then $u^2 = 1 + e^x$ and $2u\,du = e^x\,dx$, so
$\int e^x\sqrt{1+e^x}\,dx = \int u\cdot 2u\,du = \frac{2}{3}u^3 + C = \frac{2}{3}(1+e^x)^{3/2} + C.$

29. Let $u = 1 + z^3$. Then $du = 3z^2\,dz$ and $z^2\,dz = \frac{1}{3}\,du$, so

$$\int \frac{z^2}{\sqrt[3]{1+z^3}}\,dz = \int u^{-1/3}\left(\tfrac{1}{3}\,du\right) = \tfrac{1}{3}\cdot\tfrac{3}{2}u^{2/3} + C = \tfrac{1}{2}(1+z^3)^{2/3} + C.$$

31. Let $u = \ln x$. Then $du = \dfrac{dx}{x}$, so $\displaystyle\int \frac{dx}{x\ln x} = \int \frac{du}{u} = \ln|u| + C = \ln|\ln x| + C$.

33. Let $u = \cot x$. Then $du = -\csc^2 x\,dx$ and $\csc^2 x\,dx = -du$, so

$$\int \sqrt{\cot x}\csc^2 x\,dx = \int \sqrt{u}\,(-du) = -\frac{u^{3/2}}{3/2} + C = -\tfrac{2}{3}(\cot x)^{3/2} + C.$$

35. $\displaystyle\int \cot x\,dx = \int \frac{\cos x}{\sin x}\,dx$. Let $u = \sin x$. Then $du = \cos x\,dx$, so

$\int \cot x\,dx = \int \frac{1}{u}\,du = \ln|u| + C = \ln|\sin x| + C$.

37. Let $u = \sec x$. Then $du = \sec x\,\tan x\,dx$, so

$\int \sec^3 x\,\tan x\,dx = \int \sec^2 x\,(\sec x\,\tan x)\,dx = \int u^2\,du = \frac{1}{3}u^3 + C = \frac{1}{3}\sec^3 x + C$.

39. Let $u = b + cx^{a+1}$. Then $du = (a+1)cx^a\,dx$, so

$$\int x^a\sqrt{b+cx^{a+1}}\,dx = \int u^{1/2}\frac{1}{(a+1)c}\,du = \frac{1}{(a+1)c}\left(\tfrac{2}{3}u^{3/2}\right) + C = \frac{2}{3c(a+1)}\left(b+cx^{a+1}\right)^{3/2} + C.$$

41. Let $u = 1 + x^2$. Then $du = 2x\,dx$, so

$$\int \frac{1+x}{1+x^2}\,dx = \int \frac{1}{1+x^2}\,dx + \int \frac{x}{1+x^2}\,dx = \tan^{-1}x + \int \frac{\frac{1}{2}du}{u} = \tan^{-1}x + \tfrac{1}{2}\ln|u| + C$$

$$= \tan^{-1}x + \tfrac{1}{2}\ln\left|1+x^2\right| + C = \tan^{-1}x + \tfrac{1}{2}\ln\left(1+x^2\right) + C \quad [\text{since } 1+x^2 > 0].$$

43. Let $u = x + 2$. Then $du = dx$, so

$$\int \frac{x}{\sqrt[4]{x+2}}\,dx = \int \frac{u-2}{\sqrt[4]{u}}\,du = \int \left(u^{3/4} - 2u^{-1/4}\right)du = \tfrac{4}{7}u^{7/4} - 2\cdot\tfrac{4}{3}u^{3/4} + C$$

$$= \tfrac{4}{7}(x+2)^{7/4} - \tfrac{8}{3}(x+2)^{3/4} + C$$

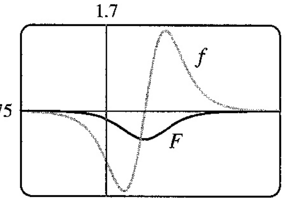
In Exercises 45–48, let $f(x)$ denote the integrand and $F(x)$ its antiderivative (with $C = 0$).

45. $f(x) = \dfrac{3x-1}{(3x^2-2x+1)^4}$.

$u = 3x^2 - 2x + 1 \quad\Rightarrow\quad du = (6x-2)\,dx = 2(3x-1)\,dx$, so

$$\int \frac{3x-1}{(3x^2-2x+1)^4}\,dx = \int \frac{1}{u^4}\left(\frac{1}{2}\,du\right) = \frac{1}{2}\int u^{-4}\,du$$

$$= -\frac{1}{6}u^{-3} + C = -\frac{1}{6\,(3x^2-2x+1)^3} + C$$

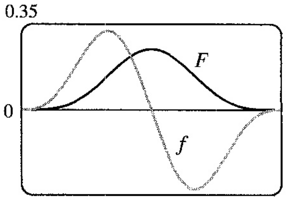


Notice that at $x = \frac{1}{3}$, f changes from negative to positive, and F has a local minimum.

47. $f(x) = \sin^3 x\,\cos x$. $u = \sin x \quad\Rightarrow\quad du = \cos x\,dx$, so

$$\int \sin^3 x\,\cos x\,dx = \int u^3\,du = \tfrac{1}{4}u^4 + C = \tfrac{1}{4}\sin^4 x + C$$

Note that at $x = \frac{\pi}{2}$, f changes from positive to negative and F has a local maximum. Also, both f and F are periodic with period π, so at $x = 0$ and at $x = \pi$, f changes from negative to positive and F has local minima.



49. Let $u = x - 1$, so $du = dx$. When $x = 0$, $u = -1$; when $x = 2$, $u = 1$. Thus, $\int_0^2 (x-1)^{25}\,dx = \int_{-1}^1 u^{25}\,du = 0$ by Theorem 7(b), since $f(u) = u^{25}$ is an odd function.

51. Let $u = 1 + 2x^3$, so $du = 6x^2\,dx$. When $x = 0$, $u = 1$; when $x = 1$, $u = 3$. Thus,

$$\int_0^1 x^2(1+2x^3)^5\,dx = \int_1^3 u^5\left(\tfrac{1}{6}\,du\right) = \tfrac{1}{6}\left[\tfrac{1}{6}u^6\right]_1^3 = \tfrac{1}{36}(3^6 - 1^6) = \tfrac{1}{36}(729 - 1) = \tfrac{728}{36} = \tfrac{182}{9}$$

53. Let $u = t/4$, so $du = \frac{1}{4}\,dt$. When $t = 0$, $u = 0$; when $t = \pi$, $u = \pi/4$. Thus,

$$\int_0^\pi \sec^2(t/4)\,dt = \int_0^{\pi/4} \sec^2 u\,(4\,du) = 4\big[\tan u\big]_0^{\pi/4} = 4\left(\tan\tfrac{\pi}{4} - \tan 0\right) = 4(1-0) = 4.$$

55. $\int_{-\pi/6}^{\pi/6} \tan^3\theta\,d\theta = 0$ by Theorem 7(b), since $f(\theta) = \tan^3\theta$ is an odd function.

57. Let $u = 1/x$, so $du = -1/x^2\,dx$. When $x = 1$, $u = 1$; when $x = 2$, $u = \frac{1}{2}$. Thus,

$$\int_1^2 \frac{e^{1/x}}{x^2}\,dx = \int_1^{1/2} e^u\,(-du) = -\left[e^u\right]_1^{1/2} = -(e^{1/2} - e) = e - \sqrt{e}.$$

59. Let $u = \cos\theta$, so $du = -\sin\theta\,d\theta$. When $\theta = 0$, $u = 1$; when $\theta = \frac{\pi}{3}$, $u = \frac{1}{2}$. Thus,

$$\int_0^{\pi/3} \frac{\sin\theta}{\cos^2\theta}\,d\theta = \int_1^{1/2} \frac{-du}{u^2} = \int_{1/2}^1 u^{-2}\,du = \left[-\frac{1}{u}\right]_{1/2}^1 = -1 - (-2) = 1.$$

61. Let $u = 1 + 2x$, so $du = 2\,dx$. When $x = 0$, $u = 1$; when $x = 13$, $u = 27$. Thus,

$$\int_0^{13} \frac{dx}{\sqrt[3]{(1+2x)^2}} = \int_1^{27} u^{-2/3}\left(\tfrac{1}{2}\,du\right) = \left[\tfrac{1}{2}\cdot 3u^{1/3}\right]_1^{27} = \tfrac{3}{2}(3-1) = 3.$$

63. Let $u = x - 1$, so $u + 1 = x$ and $du = dx$. When $x = 1$, $u = 0$; when $x = 2$, $u = 1$. Thus,

$$\int_1^2 x\sqrt{x-1}\,dx = \int_0^1 (u+1)\sqrt{u}\,du = \int_0^1 (u^{3/2} + u^{1/2})\,du = \left[\tfrac{2}{5}u^{5/2} + \tfrac{2}{3}u^{3/2}\right]_0^1 = \tfrac{2}{5} + \tfrac{2}{3} = \tfrac{16}{15}.$$

65. Let $u = \ln x$, so $du = \dfrac{dx}{x}$. When $x = e$, $u = 1$; when $x = e^4$; $u = 4$. Thus,

$$\int_e^{e^4} \frac{dx}{x\sqrt{\ln x}} = \int_1^4 u^{-1/2}\,du = 2\left[u^{1/2}\right]_1^4 = 2(2-1) = 2.$$

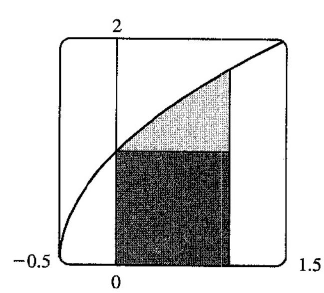
67. $\displaystyle\int_0^4 \frac{dx}{(x-2)^3}$ does not exist since $f(x) = \dfrac{1}{(x-2)^3}$ has an infinite discontinuity at $x = 2$.

69. Let $u = x^2 + a^2$, so $du = 2x\,dx$ and $x\,dx = \frac{1}{2}\,du$. When $x = 0$, $u = a^2$; when $x = a$, $u = 2a^2$. Thus,

$$\int_0^a x\sqrt{x^2+a^2}\,dx = \int_{a^2}^{2a^2} u^{1/2}\left(\tfrac{1}{2}\,du\right) = \tfrac{1}{2}\left[\tfrac{2}{3}u^{3/2}\right]_{a^2}^{2a^2} = \left[\tfrac{1}{3}u^{3/2}\right]_{a^2}^{2a^2}$$
$$= \tfrac{1}{3}\left[(2a^2)^{3/2} - (a^2)^{3/2}\right] = \tfrac{1}{3}\left(2\sqrt{2} - 1\right)a^3$$

71. From the graph, it appears that the area under the curve is about $1 + \left(\text{a little more than } \frac{1}{2}\cdot 1\cdot 0.7\right)$, or about 1.4. The exact area is given by $A = \int_0^1 \sqrt{2x+1}\,dx$. Let $u = 2x + 1$, so $du = 2\,dx$. The limits change to $2\cdot 0 + 1 = 1$ and $2\cdot 1 + 1 = 3$, and

$$A = \int_1^3 \sqrt{u}\left(\tfrac{1}{2}\,du\right) = \tfrac{1}{2}\left[\tfrac{2}{3}u^{3/2}\right]_1^3 = \tfrac{1}{3}\left(3\sqrt{3} - 1\right) = \sqrt{3} - \tfrac{1}{3} \approx 1.399.$$



73. First write the integral as a sum of two integrals:

$I = \int_{-2}^{2}(x+3)\sqrt{4-x^2}\,dx = I_1 + I_2 = \int_{-2}^{2} x\sqrt{4-x^2}\,dx + \int_{-2}^{2} 3\sqrt{4-x^2}\,dx$. $I_1 = 0$ by Theorem 7(b), since $f(x) = x\sqrt{4-x^2}$ is an odd function and we are integrating from $x=-2$ to $x=2$. We interpret I_2 as three times the area of a semicircle with radius 2, so $I = 0 + 3\cdot\frac{1}{2}(\pi\cdot 2^2) = 6\pi$.

75. First Figure Let $u=\sqrt{x}$, so $x=u^2$ and $dx = 2u\,du$. When $x=0$, $u=0$; when $x=1$, $u=1$. Thus,

$A_1 = \int_0^1 e^{\sqrt{x}}\,dx = \int_0^1 e^u(2u\,du) = 2\int_0^1 ue^u\,du.$

Second Figure $A_2 = \int_0^1 2xe^x\,dx = 2\int_0^1 ue^u\,du.$

Third Figure Let $u=\sin x$, so $du = \cos x\,dx$. When $x=0$, $u=0$; when $x=\frac{\pi}{2}$, $u=1$. Thus,

$A_3 = \int_0^{\pi/2} e^{\sin x}\sin 2x\,dx = \int_0^{\pi/2} e^{\sin x}(2\sin x\,\cos x)\,dx = \int_0^1 e^u(2u\,du) = 2\int_0^1 ue^u\,du.$

Since $A_1 = A_2 = A_3$, all three areas are equal.

77. The volume of inhaled air in the lungs at time t is

$$V(t) = \int_0^t f(u)\,du = \int_0^t \tfrac{1}{2}\sin\left(\tfrac{2\pi}{5}u\right)du = \int_0^{2\pi t/5}\tfrac{1}{2}\sin v\left(\tfrac{5}{2\pi}\,dv\right) \quad \left[\text{substitute } v=\tfrac{2\pi}{5}u,\ dv=\tfrac{2\pi}{5}\,du\right]$$

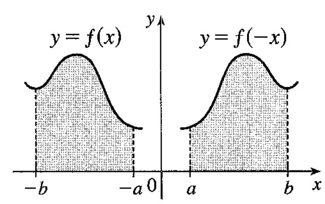
$$= \tfrac{5}{4\pi}\left[-\cos v\right]_0^{2\pi t/5} = \tfrac{5}{4\pi}\left[-\cos\left(\tfrac{2\pi}{5}t\right)+1\right] = \tfrac{5}{4\pi}\left[1-\cos\left(\tfrac{2\pi}{5}t\right)\right] \text{ liters}$$

79. Let $u=2x$. Then $du = 2\,dx$, so $\int_0^2 f(2x)\,dx = \int_0^4 f(u)\left(\frac{1}{2}\,du\right) = \frac{1}{2}\int_0^4 f(u)\,du = \frac{1}{2}(10) = 5$.

81. Let $u=-x$. Then $du = -dx$, so

$\int_a^b f(-x)\,dx = \int_{-a}^{-b} f(u)(-du) = \int_{-b}^{-a} f(u)\,du = \int_{-b}^{-a} f(x)\,dx$

From the diagram, we see that the equality follows from the fact that we are reflecting the graph of f, and the limits of integration, about the y-axis.



83. Let $u = 1-x$. Then $x = 1-u$ and $dx = -du$, so

$\int_0^1 x^a(1-x)^b\,dx = \int_1^0 (1-u)^a u^b(-du) = \int_0^1 u^b(1-u)^a\,du = \int_0^1 x^b(1-x)^a\,dx.$

85. $\dfrac{x\sin x}{1+\cos^2 x} = x\cdot\dfrac{\sin x}{2-\sin^2 x} = xf(\sin x)$, where $f(t) = \dfrac{t}{2-t^2}$. By Exercise 84,

$$\int_0^\pi \frac{x\sin x}{1+\cos^2 x}\,dx = \int_0^\pi xf(\sin x)\,dx = \frac{\pi}{2}\int_0^\pi f(\sin x)\,dx = \frac{\pi}{2}\int_0^\pi \frac{\sin x}{1+\cos^2 x}\,dx$$

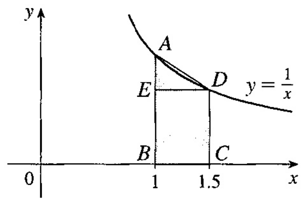
Let $u=\cos x$. Then $du = -\sin x\,dx$. When $x=\pi$, $u=-1$ and when $x=0$, $u=1$. So

$$\frac{\pi}{2}\int_0^\pi \frac{\sin x}{1+\cos^2 x}\,dx = -\frac{\pi}{2}\int_1^{-1}\frac{du}{1+u^2} = \frac{\pi}{2}\int_{-1}^{1}\frac{du}{1+u^2} = \frac{\pi}{2}\left[\tan^{-1}u\right]_{-1}^{1}$$

$$= \frac{\pi}{2}\left[\tan^{-1}1 - \tan^{-1}(-1)\right] = \frac{\pi}{2}\left[\frac{\pi}{4} - \left(-\frac{\pi}{4}\right)\right] = \frac{\pi^2}{4}$$

5.6 The Logarithm Defined as an Integral

1. (a)

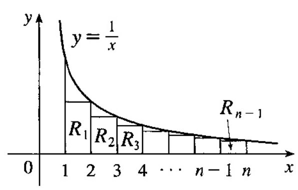


We interpret $\ln 1.5$ as the area under the curve $y = 1/x$ from $x = 1$ to $x = 1.5$. The area of the rectangle $BCDE$ is $\frac{1}{2} \cdot \frac{2}{3} = \frac{1}{3}$. The area of the trapezoid $ABCD$ is $\frac{1}{2} \cdot \frac{1}{2}\left(1 + \frac{2}{3}\right) = \frac{5}{12}$. Thus, by comparing areas, we observe that $\frac{1}{3} < \ln 1.5 < \frac{5}{12}$.

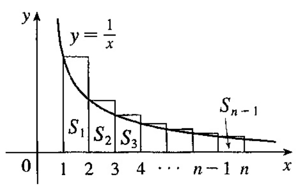
(b) With $f(t) = 1/t$, $n = 10$, and $\Delta x = 0.05$, we have

$$\ln 1.5 = \int_1^{1.5} (1/t)\, dt \approx (0.05)[f(1.025) + f(1.075) + \cdots + f(1.475)]$$
$$= (0.05)\left[\tfrac{1}{1.025} + \tfrac{1}{1.075} + \cdots + \tfrac{1}{1.475}\right] \approx 0.4054$$

3.



The area of R_i is $\dfrac{1}{i+1}$ and so $\dfrac{1}{2} + \dfrac{1}{3} + \cdots + \dfrac{1}{n} < \displaystyle\int_1^n \frac{1}{t}\, dt = \ln n$.



The area of S_i is $\dfrac{1}{i}$ and so $1 + \dfrac{1}{2} + \cdots + \dfrac{1}{n-1} > \displaystyle\int_1^n \frac{1}{t}\, dt = \ln n$.

Thus, $\dfrac{1}{2} + \dfrac{1}{3} + \cdots + \dfrac{1}{n} < \ln n < 1 + \dfrac{1}{2} + \cdots + \dfrac{1}{n-1}$.

5. If $f(x) = \ln(x^r)$, then $f'(x) = (1/x^r)(rx^{r-1}) = r/x$. But if $g(x) = r\ln x$, then $g'(x) = r/x$. So f and g must differ by a constant: $\ln(x^r) = r\ln x + C$. Put $x = 1$: $\ln(1^r) = r\ln 1 + C \;\Rightarrow\; C = 0$, so $\ln(x^r) = r\ln x$.

7. Using the third law of logarithms and Equation 10, we have $\ln e^{rx} = rx = r\ln e^x = \ln(e^x)^r$. Since ln is a one-to-one function, it follows that $e^{rx} = (e^x)^r$.

9. Using Definition 13, the first law of logarithms, and the first law of exponents for e^x, we have

$$(ab)^x = e^{x\ln(ab)} = e^{x(\ln a + \ln b)} = e^{x\ln a + x\ln b} = e^{x\ln a}e^{x\ln b} = a^x b^x.$$

5 Review

CONCEPT CHECK

1. (a) $\sum_{i=1}^{n} f(x_i^*)\,\Delta x$ is an expression for a Riemann sum of a function f.
x_i^* is a point in the ith subinterval $[x_{i-1}, x_i]$ and Δx is the length of the subintervals.

(b) See Figure 1 in Section 5.2.

(c) In Section 5.2, see Figure 3 and the paragraph beside it.

2. (a) See Definition 5.2.2.

(b) See Figure 2 in Section 5.2.

(c) In Section 5.2, see Figure 4 and the paragraph above it.

3. See the Fundamental Theorem of Calculus after Example 8 in Section 5.3.

4. (a) See the Net Change Theorem after Example 5 in Section 5.4.

(b) $\int_{t_1}^{t_2} r(t)\,dt$ represents the change in the amount of water in the reservoir between time t_1 and time t_2.

5. (a) $\int_{60}^{120} v(t)\,dt$ represents the change in position of the particle from $t = 60$ to $t = 120$ seconds.

(b) $\int_{60}^{120} |v(t)|\,dt$ represents the total distance traveled by the particle from $t = 60$ to 120 seconds.

(c) $\int_{60}^{120} a(t)\,dt$ represents the change in the velocity of the particle from $t = 60$ to $t = 120$ seconds.

6. (a) $\int f(x)\,dx$ is the family of functions $\{F \mid F' = f\}$. Any two such functions differ by a constant.

(b) The connection is given by the Evaluation Theorem: $\int_a^b f(x)\,dx = \left[\int f(x)\,dx\right]_a^b$ if f is continuous.

7. The precise version of this statement is given by the Fundamental Theorem of Calculus. See the statement of this theorem and the paragraph that follows it at the end of Section 5.3.

8. See the Substitution Rule (5.5.4). This says that it is permissible to operate with the dx after an integral sign as if it were a differential.

TRUE-FALSE QUIZ

1. True by Property 2 of the Integral in Section 5.2.

3. True by Property 3 of the Integral in Section 5.2.

5. False. For example, let $f(x) = x^2$. Then $\int_0^1 \sqrt{x^2}\,dx = \int_0^1 x\,dx = \frac{1}{2}$, but $\sqrt{\int_0^1 x^2\,dx} = \sqrt{\frac{1}{3}} = \frac{1}{\sqrt{3}}$.

7. True by Comparison Property 7 of the Integral in Section 5.2.

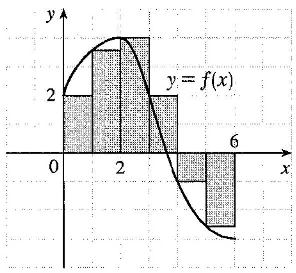
9. True. The integrand is an odd function that is continuous on $[-1, 1]$, so the result follows from Theorem 5.5.6(b).

11. False. The function $f(x) = 1/x^4$ is not bounded on the interval $[-2, 1]$. It has an infinite discontinuity at $x = 0$, so it is not integrable on the interval. (If the integral were to exist, a positive value would be expected, by Comparison Property 6 of Integrals.)

13. False. For example, the function $y = |x|$ is continuous on $\mathbb{R}$, but has no derivative at $x = 0$.

EXERCISES

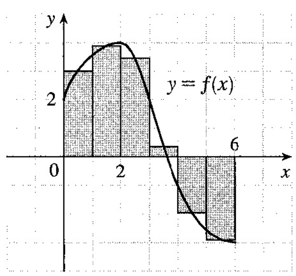
1. (a)



$$L_6 = \sum_{i=1}^{6} f(x_{i-1})\,\Delta x \quad [\Delta x = \tfrac{6-0}{6} = 1]$$
$$= f(x_0)\cdot 1 + f(x_1)\cdot 1 + f(x_2)\cdot 1 + f(x_3)\cdot 1 + f(x_4)\cdot 1 + f(x_5)\cdot 1$$
$$\approx 2 + 3.5 + 4 + 2 + (-1) + (-2.5) = 8$$

The Riemann sum represents the sum of the areas of the four rectangles above the x-axis minus the sum of the areas of the two rectangles below the x-axis.

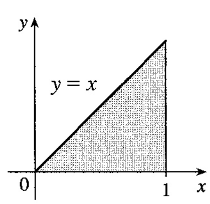
(b)

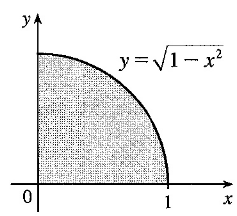


$$M_6 = \sum_{i=1}^{6} f(\overline{x}_i)\,\Delta x \quad [\Delta x = \tfrac{6-0}{6} = 1]$$
$$= f(\overline{x}_1)\cdot 1 + f(\overline{x}_2)\cdot 1 + f(\overline{x}_3)\cdot 1 + f(\overline{x}_4)\cdot 1 + f(\overline{x}_5)\cdot 1 + f(\overline{x}_6)\cdot 1$$
$$= f(0.5) + f(1.5) + f(2.5) + f(3.5) + f(4.5) + f(5.5)$$
$$\approx 3 + 3.9 + 3.4 + 0.3 + (-2) + (-2.9) = 5.7$$

The Riemann sum represents the sum of the areas of the four rectangles above the x-axis minus the sum of the areas of the two rectangles below the x-axis.

3. $\int_0^1 \left(x + \sqrt{1-x^2}\right)dx = \int_0^1 x\,dx + \int_0^1 \sqrt{1-x^2}\,dx = I_1 + I_2$.





I_1 can be interpreted as the area of the triangle shown in the figure and I_2 can be interpreted as the area of the quarter-circle. Area $= \frac{1}{2}(1)(1) + \frac{1}{4}(\pi)(1)^2 = \frac{1}{2} + \frac{\pi}{4}$.

5. $\int_0^6 f(x)\,dx = \int_0^4 f(x)\,dx + \int_4^6 f(x)\,dx \;\Rightarrow\; 10 = 7 + \int_4^6 f(x)\,dx \;\Rightarrow\; \int_4^6 f(x)\,dx = 10 - 7 = 3$

7. First note that either a or b must be the graph of $\int_0^x f(t)\,dt$, since $\int_0^0 f(t)\,dt = 0$, and $c(0) \neq 0$. Now notice that $b > 0$ when c is increasing, and that $c > 0$ when a is increasing. It follows that c is the graph of $f(x)$, b is the graph of $f'(x)$, and a is the graph of $\int_0^x f(t)\,dt$.

9. $\int_1^2 (8x^3 + 3x^2)\,dx = \left[8\cdot\frac{1}{4}x^4 + 3\cdot\frac{1}{3}x^3\right]_1^2 = \left[2x^4 + x^3\right]_1^2 = (2\cdot 2^4 + 2^3) - (2+1) = 40 - 3 = 37$

11. $\int_0^1 (1 - x^9)\,dx = \left[x - \frac{1}{10}x^{10}\right]_0^1 = \left(1 - \frac{1}{10}\right) - 0 = \frac{9}{10}$

13. $\int_1^9 \frac{\sqrt{u}-2u^2}{u}\,du = \int_1^9 (u^{-1/2}-2u)\,du = \left[2u^{1/2}-u^2\right]_1^9 = (6-81)-(2-1) = -76$

15. Let $u = y^2+1$, so $du = 2y\,dy$ and $y\,dy = \frac{1}{2}\,du$. When $y=0$, $u=1$; when $y=1$, $u=2$. Thus,

$\int_0^1 y(y^2+1)^5\,dy = \int_1^2 u^5\left(\frac{1}{2}\,du\right) = \frac{1}{2}\left[\frac{1}{6}u^6\right]_1^2 = \frac{1}{12}(64-1) = \frac{63}{12} = \frac{21}{4}$.

17. $\int_1^5 \frac{dt}{(t-4)^2}$ does not exist because the function $f(t) = \frac{1}{(t-4)^2}$ has an infinite discontinuity at $t=4$; that is, f is discontinuous on the interval $[1,5]$.

19. Let $u = v^3$, so $du = 3v^2\,dv$. When $v=0$, $u=0$; when $v=1$, $u=1$. Thus,

$\int_0^1 v^2\cos(v^3)\,dv = \int_0^1 \cos u\left(\frac{1}{3}\,du\right) = \frac{1}{3}\left[\sin u\right]_0^1 = \frac{1}{3}(\sin 1 - 0) = \frac{1}{3}\sin 1$.

21. $\int_0^1 e^{\pi t}\,dt = \left[\frac{1}{\pi}e^{\pi t}\right]_0^1 = \frac{1}{\pi}(e^{\pi}-1)$

23. $\int_2^4 \frac{1+x-x^2}{x^2}\,dx = \int_2^4 \left(\frac{1}{x^2}+\frac{x}{x^2}-\frac{x^2}{x^2}\right)dx = \int_2^4 \left(x^{-2}+\frac{1}{x}-1\right)dx = \left[-\frac{1}{x}+\ln|x|-x\right]_2^4$

$= \left(-\frac{1}{4}+\ln 4-4\right)-\left(-\frac{1}{2}+\ln 2-2\right) = \ln 2-\frac{7}{4}$

25. Let $u = x^2+4x$. Then $du = (2x+4)\,dx = 2(x+2)\,dx$, so

$\int \frac{x+2}{\sqrt{x^2+4x}}\,dx = \int u^{-1/2}\left(\frac{1}{2}\,du\right) = \frac{1}{2}\cdot 2u^{1/2}+C = \sqrt{u}+C = \sqrt{x^2+4x}+C$.

27. Let $u = \sin\pi t$. Then $du = \pi\cos\pi t\,dt$, so $\int \sin\pi t\cos\pi t\,dt = \int u\left(\frac{1}{\pi}\,du\right) = \frac{1}{\pi}\cdot\frac{1}{2}u^2+C = \frac{1}{2\pi}(\sin\pi t)^2+C$.

29. Let $u = \sqrt{x}$. Then $du = \frac{dx}{2\sqrt{x}}$, so $\int \frac{e^{\sqrt{x}}}{\sqrt{x}}\,dx = 2\int e^u\,du = 2e^u+C = 2e^{\sqrt{x}}+C$.

31. Let $u = \ln(\cos x)$. Then $du = \frac{-\sin x}{\cos x}\,dx = -\tan x\,dx$, so

$\int \tan x\ln(\cos x)\,dx = -\int u\,du = -\frac{1}{2}u^2+C = -\frac{1}{2}[\ln(\cos x)]^2+C$.

33. Let $u = 1+x^4$. Then $du = 4x^3\,dx$, so $\int \frac{x^3}{1+x^4}\,dx = \frac{1}{4}\int\frac{1}{u}\,du = \frac{1}{4}\ln|u|+C = \frac{1}{4}\ln(1+x^4)+C$.

35. Let $u = 1+\sec\theta$. Then $du = \sec\theta\tan\theta\,d\theta$, so

$\int \frac{\sec\theta\tan\theta}{1+\sec\theta}\,d\theta = \int \frac{1}{1+\sec\theta}(\sec\theta\tan\theta\,d\theta) = \int\frac{1}{u}\,du = \ln|u|+C = \ln|1+\sec\theta|+C$.

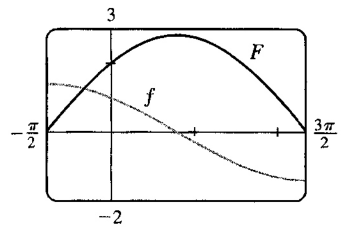
37. Since $x^2-4<0$ for $0\le x<2$ and $x^2-4>0$ for $2<x\le 3$, we have $|x^2-4| = -(x^2-4) = 4-x^2$ for $0\le x<2$ and $|x^2-4| = x^2-4$ for $2<x\le 3$. Thus,

$$\int_0^3 |x^2-4|\,dx = \int_0^2 (4-x^2)\,dx + \int_2^3 (x^2-4)\,dx = \left[4x-\frac{x^3}{3}\right]_0^2 + \left[\frac{x^3}{3}-4x\right]_2^3$$
$$= \left(8-\frac{8}{3}\right)-0+(9-12)-\left(\frac{8}{3}-8\right) = \frac{16}{3}-3+\frac{16}{3} = \frac{32}{3}-\frac{9}{3} = \frac{23}{3}$$

In Exercises 39 and 40, let $f(x)$ denote the integrand and $F(x)$ its antiderivative (with $C = 0$).

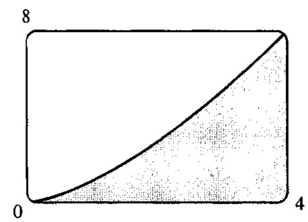
39. Let $u = 1 + \sin x$. Then $du = \cos x\, dx$, so

$$\int \frac{\cos x\, dx}{\sqrt{1+\sin x}} = \int u^{-1/2}\, du = 2u^{1/2} + C = 2\sqrt{1+\sin x} + C.$$



41. From the graph, it appears that the area under the curve $y = x\sqrt{x}$ between $x = 0$ and $x = 4$ is somewhat less than half the area of an 8×4 rectangle, so perhaps about 13 or 14. To find the exact value, we evaluate

$\int_0^4 x\sqrt{x}\, dx = \int_0^4 x^{3/2}\, dx = \left[\frac{2}{5}x^{5/2}\right]_0^4 = \frac{2}{5}(4)^{5/2} = \frac{64}{5} = 12.8.$



43. By FTC1, $F(x) = \int_1^x \sqrt{1+t^4}\, dt \quad\Rightarrow\quad F'(x) = \sqrt{1+x^4}$.

45. $g(x) = \displaystyle\int_0^{x^3} \frac{t\, dt}{\sqrt{1+t^3}}$. Let $y = g(u)$ and $u = x^3$.

Then $g'(x) = \dfrac{dy}{dx} = \dfrac{dy}{du}\dfrac{du}{dx} = \dfrac{u}{\sqrt{1+u^3}}3x^2 = \dfrac{x^3}{\sqrt{1+x^9}}3x^2 = \dfrac{3x^5}{\sqrt{1+x^9}}$.

47. $y = \displaystyle\int_{\sqrt{x}}^{x} \frac{e^t}{t}\, dt = \int_{\sqrt{x}}^{1} \frac{e^t}{t}\, dt + \int_1^x \frac{e^t}{t}\, dt = -\int_1^{\sqrt{x}} \frac{e^t}{t}\, dt + \int_1^x \frac{e^t}{t}\, dt \quad\Rightarrow$

$\dfrac{dy}{dx} = -\dfrac{d}{dx}\left(\displaystyle\int_1^{\sqrt{x}} \frac{e^t}{t}\, dt\right) + \dfrac{d}{dx}\left(\displaystyle\int_1^{x} \frac{e^t}{t}\, dt\right)$. Let $u = \sqrt{x}$. Then

$$\frac{d}{dx}\int_1^{\sqrt{x}} \frac{e^t}{t}\, dt = \frac{d}{dx}\int_1^{u} \frac{e^t}{t}\, dt = \frac{d}{du}\left(\int_1^{u} \frac{e^t}{t}\, dt\right)\frac{du}{dx} = \frac{e^u}{u}\cdot\frac{1}{2\sqrt{x}} = \frac{e^{\sqrt{x}}}{\sqrt{x}}\cdot\frac{1}{2\sqrt{x}} = \frac{e^{\sqrt{x}}}{2x},$$

so $\dfrac{dy}{dx} = -\dfrac{e^{\sqrt{x}}}{2x} + \dfrac{e^x}{x}$.

49. If $1 \le x \le 3$, then $\sqrt{1^2+3} \le \sqrt{x^2+3} \le \sqrt{3^2+3} \quad\Rightarrow\quad 2 \le \sqrt{x^2+3} \le 2\sqrt{3}$, so $2(3-1) \le \int_1^3 \sqrt{x^2+3}\, dx \le 2\sqrt{3}(3-1)$; that is, $4 \le \int_1^3 \sqrt{x^2+3}\, dx \le 4\sqrt{3}$.

51. $0 \le x \le 1 \quad\Rightarrow\quad 0 \le \cos x \le 1 \quad\Rightarrow\quad x^2 \cos x \le x^2 \quad\Rightarrow$

$\int_0^1 x^2 \cos x\, dx \le \int_0^1 x^2\, dx = \frac{1}{3}\left[x^3\right]_0^1 = \frac{1}{3}$ [Property 7].

53. $\cos x \le 1 \quad\Rightarrow\quad e^x \cos x \le e^x \quad\Rightarrow\quad \int_0^1 e^x \cos x\, dx \le \int_0^1 e^x\, dx = \left[e^x\right]_0^1 = e - 1$

55. Let $f(x) = \sqrt{1+x^3}$ on $[0,1]$. The Midpoint Rule with $n = 5$ gives

$$\int_0^1 \sqrt{1+x^3}\,dx \approx \tfrac{1}{5}[f(0.1)+f(0.3)+f(0.5)+f(0.7)+f(0.9)]$$

$$= \tfrac{1}{5}\left[\sqrt{1+(0.1)^3}+\sqrt{1+(0.3)^3}+\cdots+\sqrt{1+(0.9)^3}\right] \approx 1.110$$

57. Note that $r(t) = b'(t)$, where $b(t) =$ the number of barrels of oil consumed up to time t. So, by the Net Change Theorem, $\int_0^3 r(t)\,dt = b(3) - b(0)$ represents the number of barrels of oil consumed from Jan. 1, 2000, through Jan. 1, 2003.

59. We use the Midpoint Rule with $n = 6$ and $\Delta t = \frac{24-0}{6} = 4$. The increase in the bee population was

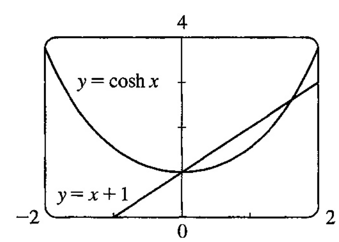
$$\int_0^{24} r(t)\,dt \approx M_6 = 4[r(2)+r(6)+r(10)+r(14)+r(18)+r(22)]$$

$$\approx 4[50+1000+7000+8550+1350+150]$$

$$= 4(18{,}100) = 72{,}400$$

61. By the Fundamental Theorem of Calculus, we know that $F(x) = \int_a^x t^2\sin(t^2)\,dt$ is an antiderivative of $f(x) = x^2\sin(x^2)$. This integral cannot be expressed in any simpler form. Since $\int_a^a f\,dt = 0$ for any a, we can take $a = 1$, and then $F(1) = 0$, as required. So $F(x) = \int_1^x t^2\sin(t^2)\,dt$ is the desired function.

63. Area under the curve $y = \sinh cx$ between $x = 0$ and $x = 1$ is equal to 1 $\Rightarrow$ $\int_0^1 \sinh cx\,dx = 1$ $\Rightarrow$ $\frac{1}{c}[\cosh cx]_0^1 = 1$ $\Rightarrow$ $\frac{1}{c}(\cosh c - 1) = 1$ $\Rightarrow$ $\cosh c - 1 = c$ $\Rightarrow$ $\cosh c = c + 1$. From the graph, we get $c = 0$ and $c \approx 1.6161$, but $c = 0$ isn't a solution for this problem since the curve $y = \sinh cx$ becomes $y = 0$ and the area under it is 0. Thus, $c \approx 1.6161$.



65. Using FTC1, we differentiate both sides of the given equation, $\int_0^x f(t)\,dt = xe^{2x} + \int_0^x e^{-t}f(t)\,dt$, and get

$$f(x) = e^{2x} + 2xe^{2x} + e^{-x}f(x) \quad\Rightarrow\quad f(x)(1-e^{-x}) = e^{2x} + 2xe^{2x} \quad\Rightarrow\quad f(x) = \frac{e^{2x}(1+2x)}{1-e^{-x}}.$$

67. Let $u = f(x)$ and $du = f'(x)\,dx$. So $2\int_a^b f(x)f'(x)\,dx = 2\int_{f(a)}^{f(b)} u\,du = [u^2]_{f(a)}^{f(b)} = [f(b)]^2 - [f(a)]^2$.

69. Let $u = 1 - x$. Then $du = -dx$, so $\int_0^1 f(1-x)\,dx = \int_1^0 f(u)(-du) = \int_0^1 f(u)\,du = \int_0^1 f(x)\,dx$.

□ PROBLEMS PLUS

1. Differentiating both sides of the equation $x\sin\pi x = \int_0^{x^2} f(t)\,dt$ (using FTC1 and the Chain Rule for the right side) gives $\sin\pi x + \pi x\cos\pi x = 2xf(x^2)$. Letting $x = 2$ so that $f(x^2) = f(4)$, we obtain $\sin 2\pi + 2\pi\cos 2\pi = 4f(4)$, so $f(4) = \frac{1}{4}(0 + 2\pi\cdot 1) = \frac{\pi}{2}$.

3. For $1 \le x \le 2$, we have $x^4 \le 2^4 = 16$, so $1 + x^4 \le 17$ and $\dfrac{1}{1+x^4} \ge \dfrac{1}{17}$. Thus,

$\displaystyle\int_1^2 \frac{1}{1+x^4}\,dx \ge \int_1^2 \frac{1}{17}\,dx = \frac{1}{17}$. Also $1 + x^4 > x^4$ for $1 \le x \le 2$, so $\dfrac{1}{1+x^4} < \dfrac{1}{x^4}$ and

$\displaystyle\int_1^2 \frac{1}{1+x^4}\,dx < \int_1^2 x^{-4}\,dx = \left[\frac{x^{-3}}{-3}\right]_1^2 = -\frac{1}{24} + \frac{1}{3} = \frac{7}{24}$. Thus, we have the estimate

$$\frac{1}{17} \le \int_1^2 \frac{1}{1+x^4}\,dx \le \frac{7}{24}.$$

5. $f(x) = \displaystyle\int_0^{g(x)} \frac{1}{\sqrt{1+t^3}}\,dt$, where $g(x) = \int_0^{\cos x} \left[1 + \sin(t^2)\right] dt$. Using FTC1 and the Chain Rule (twice) we have $f'(x) = \dfrac{1}{\sqrt{1+[g(x)]^3}}g'(x) = \dfrac{1}{\sqrt{1+[g(x)]^3}}\left[1 + \sin(\cos^2 x)\right](-\sin x)$. Now

$g(\frac{\pi}{2}) = \int_0^0 \left[1 + \sin(t^2)\right] dt = 0$, so $f'(\frac{\pi}{2}) = \frac{1}{\sqrt{1+0}}(1 + \sin 0)(-1) = 1\cdot 1\cdot(-1) = -1$.

7. By l'Hospital's Rule and the Fundamental Theorem, using the notation $\exp(y) = e^y$,

$$\lim_{x\to 0}\frac{\int_0^x (1-\tan 2t)^{1/t}\,dt}{x} \overset{\text{H}}{=} \lim_{x\to 0}\frac{(1-\tan 2x)^{1/x}}{1} = \exp\left(\lim_{x\to 0}\frac{\ln(1-\tan 2x)}{x}\right)$$

$$\overset{\text{H}}{=} \exp\left(\lim_{x\to 0}\frac{-2\sec^2 2x}{1-\tan 2x}\right) = \exp\left(\frac{-2\cdot 1^2}{1-0}\right) = e^{-2}.$$

9. $f(x) = 2 + x - x^2 = (-x+2)(x+1) = 0 \iff x = 2$ or $x = -1$. $f(x) \ge 0$ for $x \in [-1, 2]$ and $f(x) < 0$ everywhere else. The integral $\int_a^b (2 + x - x^2)\,dx$ has a maximum on the interval where the integrand is positive, which is $[-1, 2]$. So $a = -1$, $b = 2$. (Any larger interval gives a smaller integral since $f(x) < 0$ outside $[-1, 2]$. Any smaller interval also gives a smaller integral since $f(x) \ge 0$ in $[-1, 2]$.)

11. (a) We can split the integral $\int_0^n [\![x]\!]\,dx$ into the sum $\sum\limits_{i=1}^{n}\left[\int_{i-1}^{i} [\![x]\!]\,dx\right]$. But on each of the intervals $[i-1, i)$ of integration, $[\![x]\!]$ is a constant function, namely $i - 1$. So the ith integral in the sum is equal to $(i-1)[i-(i-1)] = (i-1)$. So the original integral is equal to $\sum\limits_{i=1}^{n}(i-1) = \sum\limits_{i=1}^{n-1} i = \dfrac{(n-1)n}{2}$.

(b) We can write $\int_a^b [\![x]\!]\,dx = \int_0^b [\![x]\!]\,dx - \int_0^a [\![x]\!]\,dx$.

Now $\int_0^b [\![x]\!]\,dx = \int_0^{[\![b]\!]} [\![x]\!]\,dx + \int_{[\![b]\!]}^b [\![x]\!]\,dx$. The first of these integrals is equal to $\frac{1}{2}([\![b]\!]-1)\,[\![b]\!]$, by part (a), and since $[\![x]\!] = [\![b]\!]$ on $[[\![b]\!], b]$, the second integral is just $[\![b]\!]\,(b - [\![b]\!])$. So

$\int_0^b [\![x]\!]\,dx = \frac{1}{2}([\![b]\!]-1)\,[\![b]\!] + [\![b]\!]\,(b - [\![b]\!]) = \frac{1}{2}\,[\![b]\!]\,(2b - [\![b]\!] - 1)$ and similarly

$\int_0^a [\![x]\!]\,dx = \frac{1}{2}\,[\![a]\!]\,(2a - [\![a]\!] - 1)$. Therefore, $\int_a^b [\![x]\!]\,dx = \frac{1}{2}\,[\![b]\!]\,(2b - [\![b]\!] - 1) - \frac{1}{2}\,[\![a]\!]\,(2a - [\![a]\!] - 1)$.

13. Differentiating the equation $\int_0^x f(t)\,dt = [f(x)]^2$ using FTC1 gives $f(x) = 2f(x)f'(x) \;\Rightarrow$ $f(x)[2f'(x) - 1] = 0$, so $f(x) = 0$ or $f'(x) = \frac{1}{2}$. $f'(x) = \frac{1}{2} \;\Rightarrow\; f(x) = \frac{1}{2}x + C$. To find C we substitute into the original equation to get $\int_0^x \left(\frac{1}{2}t + C\right) dt = \left(\frac{1}{2}x + C\right)^2 \;\Leftrightarrow\; \frac{1}{4}x^2 + Cx = \frac{1}{4}x^2 + Cx + C^2$. It follows that $C = 0$, so $f(x) = \frac{1}{2}x$. Therefore, $f(x) = 0$ or $f(x) = \frac{1}{2}x$.

15. Note that $\dfrac{d}{dx}\left(\displaystyle\int_0^x \left[\int_0^u f(t)\,dt\right] du\right) = \displaystyle\int_0^x f(t)\,dt$ by FTC1, while

$$\frac{d}{dx}\left[\int_0^x f(u)(x-u)\,du\right] = \frac{d}{dx}\left[x\int_0^x f(u)\,du\right] - \frac{d}{dx}\left[\int_0^x f(u)u\,du\right]$$

$$= \int_0^x f(u)\,du + xf(x) - f(x)x = \int_0^x f(u)\,du$$

Hence, $\int_0^x f(u)(x-u)\,du = \int_0^x \left[\int_0^u f(t)\,dt\right] du + C$. Setting $x = 0$ gives $C = 0$.

17.
$$\lim_{n\to\infty}\left(\frac{1}{\sqrt{n}\sqrt{n+1}} + \frac{1}{\sqrt{n}\sqrt{n+2}} + \cdots + \frac{1}{\sqrt{n}\sqrt{n+n}}\right)$$
$$= \lim_{n\to\infty}\frac{1}{n}\left(\sqrt{\frac{n}{n+1}} + \sqrt{\frac{n}{n+2}} + \cdots + \sqrt{\frac{n}{n+n}}\right)$$
$$= \lim_{n\to\infty}\frac{1}{n}\left(\frac{1}{\sqrt{1+1/n}} + \frac{1}{\sqrt{1+2/n}} + \cdots + \frac{1}{\sqrt{1+1}}\right)$$
$$= \lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n} f\left(\frac{i}{n}\right) \qquad \left[\text{where } f(x) = \frac{1}{\sqrt{1+x}}\right]$$
$$= \int_0^1 \frac{1}{\sqrt{1+x}}\,dx = \left[2\sqrt{1+x}\right]_0^1 = 2\left(\sqrt{2}-1\right)$$

19. The shaded region has area $\int_0^1 f(x)\,dx = \frac{1}{3}$. The integral $\int_0^1 f^{-1}(y)\,dy$ gives the area of the unshaded region, which we know to be $1 - \frac{1}{3} = \frac{2}{3}$. So $\int_0^1 f^{-1}(y)\,dy = \frac{2}{3}$.