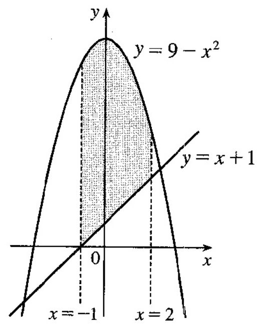
6 ☐ APPLICATIONS OF INTEGRATION

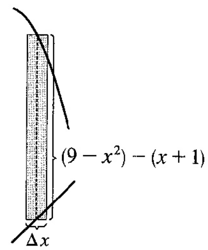
6.1 Areas between Curves

1. $A = \int_{x=0}^{x=4} (y_T - y_B)\,dx = \int_0^4 \left[(5x - x^2) - x\right] dx = \int_0^4 (4x - x^2)\,dx$
$= \left[2x^2 - \frac{1}{3}x^3\right]_0^4 = (32 - \frac{64}{3}) - (0) = \frac{32}{3}$

3. $A = \int_{y=-1}^{y=1} (x_R - x_L)\,dy = \int_{-1}^{1} \left[e^y - (y^2 - 2)\right] dy$
$= \int_{-1}^{1} (e^y - y^2 + 2)\,dy = \left[e^y - \frac{1}{3}y^3 + 2y\right]_{-1}^{1} = \left(e^1 - \frac{1}{3} + 2\right) - \left(e^{-1} + \frac{1}{3} - 2\right) = e - \dfrac{1}{e} + \dfrac{10}{3}$

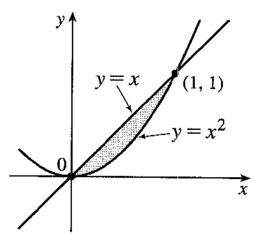
5. $A = \int_{-1}^{2} \left[(9 - x^2) - (x + 1)\right] dx$
$= \int_{-1}^{2} (8 - x - x^2)\,dx$
$= \left[8x - \dfrac{x^2}{2} - \dfrac{x^3}{3}\right]_{-1}^{2}$
$= (16 - 2 - \frac{8}{3}) - (-8 - \frac{1}{2} + \frac{1}{3})$
$= 22 - 3 + \frac{1}{2} = \frac{39}{2}$

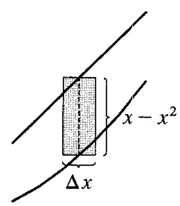




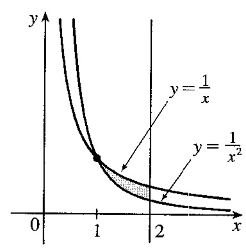
7. The curves intersect when $x = x^2 \;\Rightarrow\; x^2 - x = 0 \;\Leftrightarrow\; x(x - 1) = 0 \;\Leftrightarrow\; x = 0, 1.$

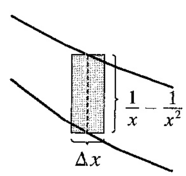
$A = \int_0^1 (x - x^2)\,dx$
$= \left[\frac{1}{2}x^2 - \frac{1}{3}x^3\right]_0^1$
$= \frac{1}{2} - \frac{1}{3}$
$= \frac{1}{6}$





9. $A = \int_1^2 \left(\dfrac{1}{x} - \dfrac{1}{x^2}\right) dx = \left[\ln x + \dfrac{1}{x}\right]_1^2$
$= (\ln 2 + \frac{1}{2}) - (\ln 1 + 1)$
$= \ln 2 - \frac{1}{2} \approx 0.19$



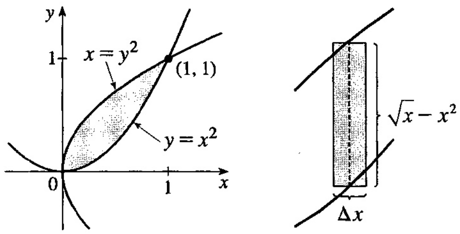


11. $A = \int_0^1 \left(\sqrt{x} - x^2\right) dx$

$= \left[\frac{2}{3}x^{3/2} - \frac{1}{3}x^3\right]_0^1$

$= \frac{2}{3} - \frac{1}{3}$

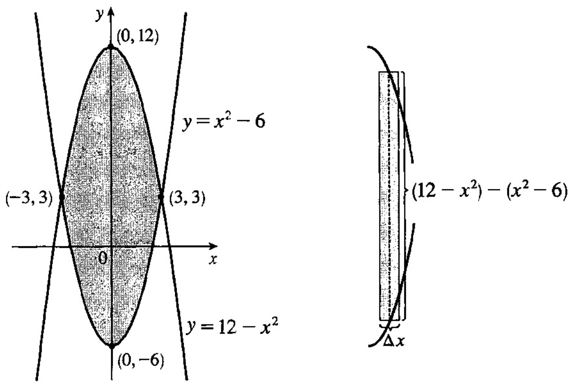
$= \frac{1}{3}$



13. $12 - x^2 = x^2 - 6 \quad \Leftrightarrow \quad 2x^2 = 18 \quad \Leftrightarrow \quad x^2 = 9 \quad \Leftrightarrow \quad x = \pm 3$, so

$$A = \int_{-3}^{3} \left[\left(12 - x^2\right) - \left(x^2 - 6\right)\right] dx = 2\int_0^3 \left(18 - 2x^2\right) dx \quad \text{[by symmetry]}$$

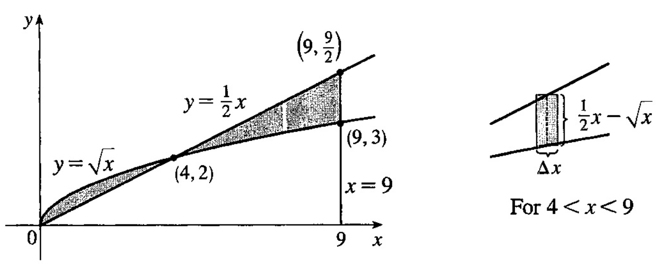
$$= 2\left[18x - \tfrac{2}{3}x^3\right]_0^3 = 2\left[(54 - 18) - 0\right] = 2(36) = 72$$



15. $\frac{1}{2}x = \sqrt{x} \quad \Rightarrow \quad \frac{1}{4}x^2 = x \quad \Rightarrow \quad x^2 - 4x = 0 \quad \Rightarrow \quad x(x-4) = 0 \quad \Rightarrow \quad x = 0$ or 4, so

$$A = \int_0^4 \left(\sqrt{x} - \tfrac{1}{2}x\right) dx + \int_4^9 \left(\tfrac{1}{2}x - \sqrt{x}\right) dx = \left[\tfrac{2}{3}x^{3/2} - \tfrac{1}{4}x^2\right]_0^4 + \left[\tfrac{1}{4}x^2 - \tfrac{2}{3}x^{3/2}\right]_4^9$$

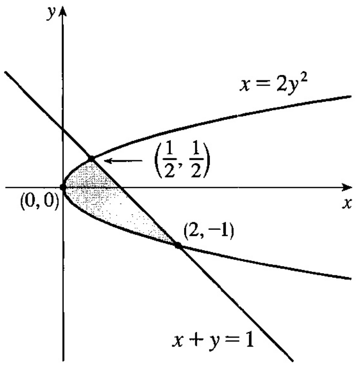
$$= \left[\left(\tfrac{16}{3} - 4\right) - 0\right] + \left[\left(\tfrac{81}{4} - 18\right) - \left(4 - \tfrac{16}{3}\right)\right] = \tfrac{81}{4} + \tfrac{32}{3} - 26 = \tfrac{59}{12}$$

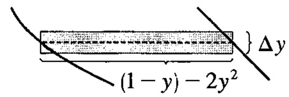


17. $2y^2 = 1 - y \quad \Leftrightarrow \quad 2y^2 + y - 1 = 0 \quad \Leftrightarrow \quad (2y-1)(y+1) = 0 \quad \Leftrightarrow \quad y = \frac{1}{2}$ or -1, so $x = \frac{1}{2}$ or 2 and

$$A = \int_{-1}^{1/2} \left[(1-y) - 2y^2\right] dy = \int_{-1}^{1/2} \left(1 - y - 2y^2\right) dy = \left[y - \tfrac{1}{2}y^2 - \tfrac{2}{3}y^3\right]_{-1}^{1/2}$$

$$= \left(\tfrac{1}{2} - \tfrac{1}{8} - \tfrac{1}{12}\right) - \left(-1 - \tfrac{1}{2} + \tfrac{2}{3}\right) = \tfrac{7}{24} - \left(-\tfrac{5}{6}\right) = \tfrac{7}{24} + \tfrac{20}{24} = \tfrac{27}{24} = \tfrac{9}{8}$$





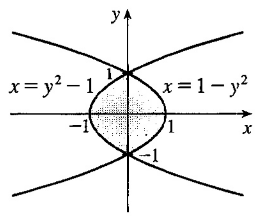
19. The curves intersect when $1 - y^2 = y^2 - 1 \quad \Leftrightarrow \quad 2 = 2y^2 \quad \Leftrightarrow \quad y^2 = 1 \quad \Leftrightarrow \quad y = \pm 1$.

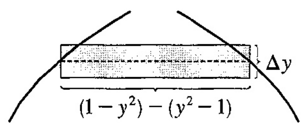
$$A = \int_{-1}^{1} \left[\left(1 - y^2\right) - \left(y^2 - 1\right)\right] dy$$

$$= \int_{-1}^{1} 2\left(1 - y^2\right) dy$$

$$= 2 \cdot 2 \int_{0}^{1} \left(1 - y^2\right) dy$$

$$= 4\left[y - \tfrac{1}{3}y^3\right]_0^1 = 4\left(1 - \tfrac{1}{3}\right) = \tfrac{8}{3}$$

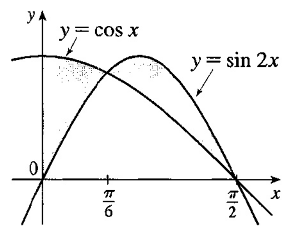




21. Notice that $\cos x = \sin 2x = 2\sin x \cos x \quad \Leftrightarrow$

$2\sin x \cos x - \cos x = 0 \quad \Leftrightarrow \quad \cos x\,(2\sin x - 1) = 0 \quad \Leftrightarrow$

$2\sin x = 1$ or $\cos x = 0 \quad \Leftrightarrow \quad x = \frac{\pi}{6}$ or $\frac{\pi}{2}$.



$$A = \int_0^{\pi/6} (\cos x - \sin 2x)\, dx + \int_{\pi/6}^{\pi/2} (\sin 2x - \cos x)\, dx$$

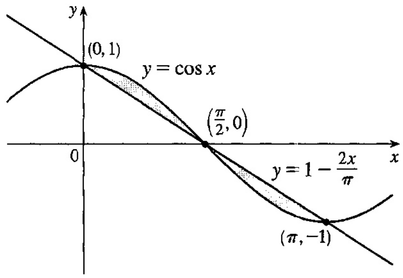
$$= \left[\sin x + \tfrac{1}{2}\cos 2x\right]_0^{\pi/6} + \left[-\tfrac{1}{2}\cos 2x - \sin x\right]_{\pi/6}^{\pi/2}$$

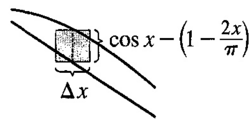
$$= \tfrac{1}{2} + \tfrac{1}{2} \cdot \tfrac{1}{2} - \left(0 + \tfrac{1}{2} \cdot 1\right) + \left(\tfrac{1}{2} - 1\right) - \left(-\tfrac{1}{2} \cdot \tfrac{1}{2} - \tfrac{1}{2}\right) = \tfrac{1}{2}$$

23. From the graph, we see that the curves intersect at $x = 0$, $x = \frac{\pi}{2}$, and $x = \pi$. By symmetry,

$$A = \int_0^{\pi} \left| \cos x - \left(1 - \frac{2x}{\pi}\right) \right| dx = 2\int_0^{\pi/2} \left[\cos x - \left(1 - \frac{2x}{\pi}\right)\right] dx = 2\int_0^{\pi/2} \left(\cos x - 1 + \frac{2x}{\pi}\right) dx$$

$$= 2\left[\sin x - x + \tfrac{1}{\pi}x^2\right]_0^{\pi/2} = 2\left[\left(1 - \tfrac{\pi}{2} + \tfrac{1}{\pi}\cdot\tfrac{\pi^2}{4}\right) - 0\right] = 2\left(1 - \tfrac{\pi}{2} + \tfrac{\pi}{4}\right) = 2 - \tfrac{\pi}{2}$$





25. The curves intersect when $x^2 = \dfrac{2}{x^2+1} \;\Leftrightarrow\; x^4 + x^2 = 2 \;\Leftrightarrow\; x^4 + x^2 - 2 = 0 \;\Leftrightarrow$

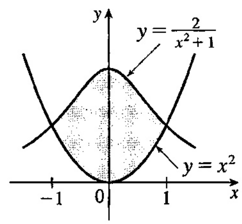
$(x^2+2)(x^2-1) = 0 \;\Leftrightarrow\; x^2 = 1 \;\Leftrightarrow\; x = \pm 1.$

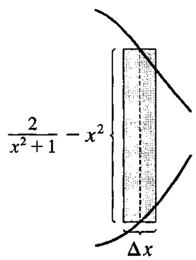
$$A = \int_{-1}^{1} \left(\frac{2}{x^2+1} - x^2\right) dx$$

$$= 2\int_0^1 \left(\frac{2}{x^2+1} - x^2\right) dx$$

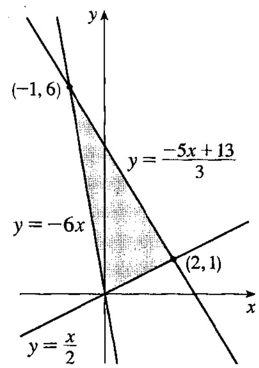
$$= 2\left[2\tan^{-1} x - \tfrac{1}{3}x^3\right]_0^1 = 2\left(2\cdot\tfrac{\pi}{4} - \tfrac{1}{3}\right)$$

$$= \pi - \tfrac{2}{3} \approx 2.47$$





27. An equation of the line through $(0,0)$ and $(2,1)$ is $y = \frac{1}{2}x$; through $(0,0)$ and $(-1,6)$ is $y = -6x$; through $(2,1)$ and $(-1,6)$ is $y = -\frac{5}{3}x + \frac{13}{3}$.



$$A = \int_{-1}^{0} \left[\left(-\tfrac{5}{3}x + \tfrac{13}{3}\right) - (-6x)\right] dx + \int_0^2 \left[\left(-\tfrac{5}{3}x + \tfrac{13}{3}\right) - \tfrac{1}{2}x\right] dx$$

$$= \int_{-1}^{0} \left(\tfrac{13}{3}x + \tfrac{13}{3}\right) dx + \int_0^2 \left(-\tfrac{13}{6}x + \tfrac{13}{3}\right) dx$$

$$= \tfrac{13}{3}\int_{-1}^{0} (x+1)\, dx + \tfrac{13}{3}\int_0^2 \left(-\tfrac{1}{2}x + 1\right) dx$$

$$= \tfrac{13}{3}\left[\tfrac{1}{2}x^2 + x\right]_{-1}^{0} + \tfrac{13}{3}\left[-\tfrac{1}{4}x^2 + x\right]_0^2$$

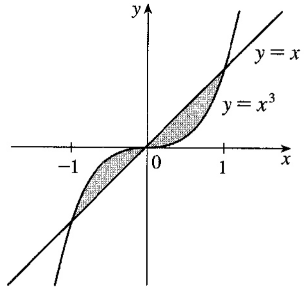
$$= \tfrac{13}{3}\left[0 - \left(\tfrac{1}{2} - 1\right)\right] + \tfrac{13}{3}\left[(-1+2) - 0\right] = \tfrac{13}{3}\cdot\tfrac{1}{2} + \tfrac{13}{3}\cdot 1 = \tfrac{13}{2}$$

29. $A = \int_{-1}^{1} |x^3 - x|\, dx$

$= 2\int_0^1 (x - x^3)\, dx$ [by symmetry]

$= 2\left[\frac{1}{2}x^2 - \frac{1}{4}x^4\right]_0^1$

$= 2\left(\frac{1}{2} - \frac{1}{4}\right) = \frac{1}{2}$



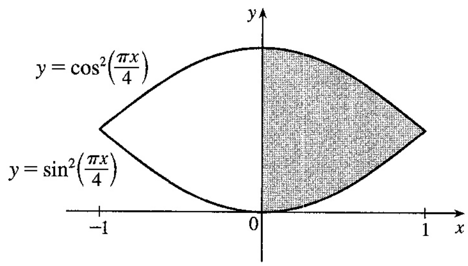
31. Let $f(x) = \cos^2\left(\frac{\pi x}{4}\right) - \sin^2\left(\frac{\pi x}{4}\right)$ and $\Delta x = \frac{1-0}{4}$.

The shaded area is given by

$A = \int_0^1 f(x)\, dx \approx M_4$

$= \frac{1}{4}\left[f\left(\frac{1}{8}\right) + f\left(\frac{3}{8}\right) + f\left(\frac{5}{8}\right) + f\left(\frac{7}{8}\right)\right]$

≈ 0.6407



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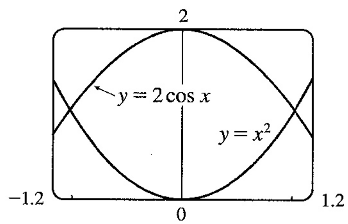
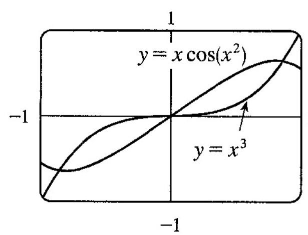


From the graph, we see that the curves intersect at $x = \pm a \approx \pm 1.02$, with $2\cos x > x^2$ on $(-a, a)$. So the area of the region bounded by the curves is

$$A = \int_{-a}^{a} (2\cos x - x^2)\, dx = 2\int_0^a (2\cos x - x^2)\, dx$$

$$= 2\left[2\sin x - \frac{1}{3}x^3\right]_0^a \approx 2.70$$

35.

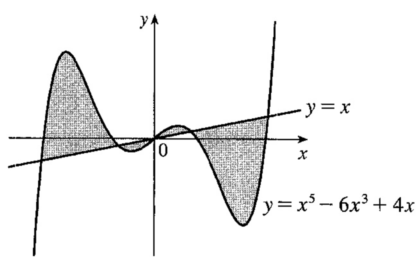


From the graph, we see that the curves intersect at $x = \pm a \approx \pm 0.86$. So the area of the region bounded by the curves is

$$A = 2\int_0^a \left[x\cos(x^2) - x^3\right] dx = 2\left[\frac{1}{2}\sin(x^2) - \frac{1}{4}x^4\right]_0^a$$

$$\approx 0.40$$

37. As the figure illustrates, the curves $y = x$ and $y = x^5 - 6x^3 + 4x$ enclose a four-part region symmetric about the origin (since $x^5 - 6x^3 + 4x$ and x are odd functions of x). The curves intersect at values of x where $x^5 - 6x^3 + 4x = x$; that is, where $x(x^4 - 6x^2 + 3) = 0$. That happens at $x = 0$ and where



$x^2 = \dfrac{6 \pm \sqrt{36-12}}{2} = 3 \pm \sqrt{6}$; that is, at $x = -\sqrt{3+\sqrt{6}}, -\sqrt{3-\sqrt{6}}, 0, \sqrt{3-\sqrt{6}}$, and $\sqrt{3+\sqrt{6}}$. The exact area is

$$\begin{aligned} 2\int_0^{\sqrt{3+\sqrt{6}}} \left|(x^5 - 6x^3 + 4x) - x\right| dx &= 2\int_0^{\sqrt{3+\sqrt{6}}} \left|x^5 - 6x^3 + 3x\right| dx \\ &= 2\int_0^{\sqrt{3-\sqrt{6}}} (x^5 - 6x^3 + 3x)\, dx + 2\int_{\sqrt{3-\sqrt{6}}}^{\sqrt{3+\sqrt{6}}} (-x^5 + 6x^3 - 3x)\, dx \\ &\overset{\text{CAS}}{=} 12\sqrt{6} - 9 \end{aligned}$$

39. 1 second $= \frac{1}{3600}$ hour, so 10 s $= \frac{1}{360}$ h. With the given data, we can take $n = 5$ to use the Midpoint Rule.

$\Delta t = \frac{1/360 - 0}{5} = \frac{1}{1800}$, so

$$\begin{aligned} \text{distance}_{\text{Kelly}} - \text{distance}_{\text{Chris}} &= \int_0^{1/360} v_K\, dt - \int_0^{1/360} v_C\, dt = \int_0^{1/360} (v_K - v_C)\, dt \\ &\approx M_5 = \tfrac{1}{1800}\left[(v_K - v_C)(1) + (v_K - v_C)(3) + (v_K - v_C)(5) \right. \\ &\qquad \left. + (v_K - v_C)(7) + (v_K - v_C)(9)\right] \\ &= \tfrac{1}{1800}[(22-20) + (52-46) + (71-62) + (86-75) + (98-86)] \\ &= \tfrac{1}{1800}(2+6+9+11+12) = \tfrac{1}{1800}(40) = \tfrac{1}{45} \text{ mile, or } 117\tfrac{1}{3} \text{ feet} \end{aligned}$$

41. We know that the area under curve A between $t = 0$ and $t = x$ is $\int_0^x v_A(t)\, dt = s_A(x)$, where $v_A(t)$ is the velocity of car A and s_A is its displacement. Similarly, the area under curve B between $t = 0$ and $t = x$ is $\int_0^x v_B(t)\, dt = s_B(x)$.

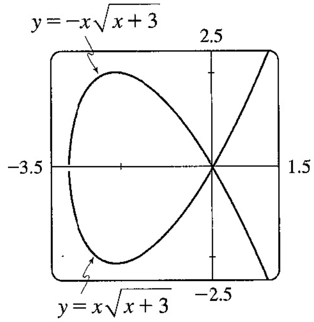
(a) After one minute, the area under curve A is greater than the area under curve B. So car A is ahead after one minute.

(b) The area of the shaded region has numerical value $s_A(1) - s_B(1)$, which is the distance by which A is ahead of B after 1 minute.

(c) After two minutes, car B is traveling faster than car A and has gained some ground, but the area under curve A from $t = 0$ to $t = 2$ is still greater than the corresponding area for curve B, so car A is still ahead.

(d) From the graph, it appears that the area between curves A and B for $0 \le t \le 1$ (when car A is going faster), which corresponds to the distance by which car A is ahead, seems to be about 3 squares. Therefore, the cars will be side by side at the time x where the area between the curves for $1 \le t \le x$ (when car B is going faster) is the same as the area for $0 \le t \le 1$. From the graph, it appears that this time is $x \approx 2.2$. So the cars are side by side when $t \approx 2.2$ minutes.

43.

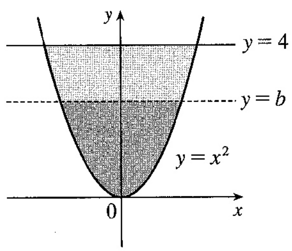


To graph this function, we must first express it as a combination of explicit functions of y; namely, $y = \pm x\sqrt{x+3}$. We can see from the graph that the loop extends from $x = -3$ to $x = 0$, and that by symmetry, the area we seek is just twice the area under the top half of the curve on this interval, the equation of the top half being $y = -x\sqrt{x+3}$. So the area is $A = 2\int_{-3}^{0}\left(-x\sqrt{x+3}\right)dx$. We substitute $u = x+3$, so $du = dx$ and the limits change to 0 and 3, and we get

$$A = -2\int_0^3 \left[(u-3)\sqrt{u}\right] du = -2\int_0^3 \left(u^{3/2} - 3u^{1/2}\right) du$$

$$= -2\left[\tfrac{2}{5}u^{5/2} - 2u^{3/2}\right]_0^3 = -2\left[\tfrac{2}{5}\left(3^2\sqrt{3}\right) - 2\left(3\sqrt{3}\right)\right] = \tfrac{24}{5}\sqrt{3}$$

45.



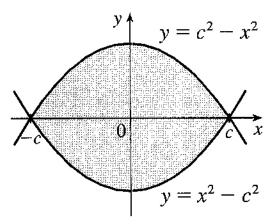
By the symmetry of the problem, we consider only the first quadrant, where $y = x^2 \;\Rightarrow\; x = \sqrt{y}$. We are looking for a number b such that $\int_0^b \sqrt{y}\,dy = \int_b^4 \sqrt{y}\,dy \;\Rightarrow\; \tfrac{2}{3}\left[y^{3/2}\right]_0^b = \tfrac{2}{3}\left[y^{3/2}\right]_b^4 \;\Rightarrow$ $b^{3/2} = 4^{3/2} - b^{3/2} \;\Rightarrow\; 2b^{3/2} = 8 \;\Rightarrow\; b^{3/2} = 4 \;\Rightarrow$ $b = 4^{2/3} \approx 2.52$.

47. We first assume that $c > 0$, since c can be replaced by $-c$ in both equations without changing the graphs, and if $c = 0$ the curves do not enclose a region. We see from the graph that the enclosed area A lies between $x = -c$ and $x = c$, and by symmetry, it is equal to four times the area in the first quadrant.

The enclosed area is

$$A = 4\int_0^c \left(c^2 - x^2\right) dx = 4\left[c^2 x - \tfrac{1}{3}x^3\right]_0^c$$

$$= 4\left(c^3 - \tfrac{1}{3}c^3\right) = 4\left(\tfrac{2}{3}c^3\right) = \tfrac{8}{3}c^3$$



So $A = 576 \;\Leftrightarrow\; \tfrac{8}{3}c^3 = 576 \;\Leftrightarrow\; c^3 = 216 \;\Leftrightarrow\; c = \sqrt[3]{216} = 6$.

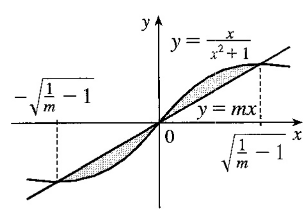
Note that $c = -6$ is another solution, since the graphs are the same.

49. The curve and the line will determine a region when they intersect at two or more points. So we solve the equation

$x/\left(x^2+1\right) = mx \;\Rightarrow\; x = x\left(mx^2 + m\right) \;\Rightarrow$

$x\left(mx^2 + m\right) - x = 0 \;\Rightarrow\; x\left(mx^2 + m - 1\right) = 0 \;\Rightarrow$

$x = 0$ or $mx^2 + m - 1 = 0 \;\Rightarrow\; x = 0$ or $x^2 = \dfrac{1-m}{m} \;\Rightarrow$



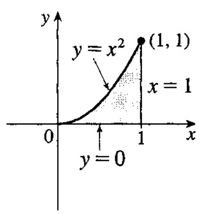
$x = 0$ or $x = \pm\sqrt{\dfrac{1}{m} - 1}$. Note that if $m = 1$, this has only the solution $x = 0$, and no region is determined. But if $1/m - 1 > 0 \Leftrightarrow 1/m > 1 \Leftrightarrow 0 < m < 1$, then there are two solutions. [Another way of seeing this is to observe that the slope of the tangent to $y = x/(x^2 + 1)$ at the origin is $y' = 1$ and therefore we must have $0 < m < 1$.] Note that we cannot just integrate between the positive and negative roots, since the curve and the line cross at the origin. Since mx and $x/(x^2 + 1)$ are both odd functions, the total area is twice the area between the curves on the interval $\left[0, \sqrt{1/m - 1}\right]$. So the total area enclosed is

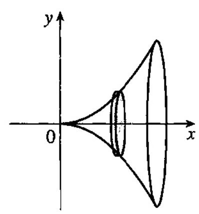
$$2\int_0^{\sqrt{1/m-1}}\left[\frac{x}{x^2+1} - mx\right]dx = 2\left[\tfrac{1}{2}\ln(x^2+1) - \tfrac{1}{2}mx^2\right]_0^{\sqrt{1/m-1}}$$
$$= [\ln(1/m - 1 + 1) - m(1/m - 1)] - (\ln 1 - 0)$$
$$= \ln(1/m) - 1 + m = m - \ln m - 1$$

6.2 Volumes

1. A cross-section is circular with radius x^2, so its area is $A(x) = \pi\left(x^2\right)^2$.

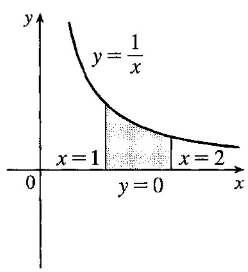
$$V = \int_0^1 A(x)\,dx = \int_0^1 \pi\left(x^2\right)^2 dx = \pi\int_0^1 x^4 dx = \pi\left[\tfrac{1}{5}x^5\right]_0^1 = \tfrac{\pi}{5}$$

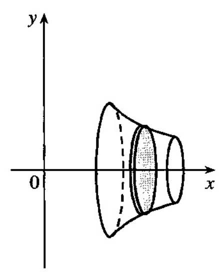




3. A cross-section is a disk with radius $1/x$, so its area is $A(x) = \pi(1/x)^2$.

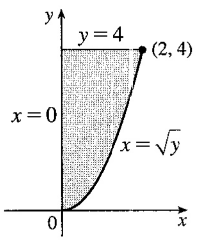
$$V = \int_1^2 A(x)\,dx = \int_1^2 \pi\left(\frac{1}{x}\right)^2 dx = \pi\int_1^2 \frac{1}{x^2}\,dx = \pi\left[-\frac{1}{x}\right]_1^2 = \pi\left[-\tfrac{1}{2} - (-1)\right] = \tfrac{\pi}{2}$$

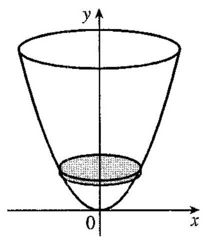




5. A cross-section is a disk with radius $\sqrt{y}$, so its area is $A(y) = \pi\left(\sqrt{y}\right)^2$.

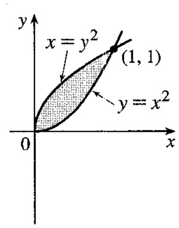
$$V = \int_0^4 A(y)\,dy = \int_0^4 \pi\left(\sqrt{y}\right)^2 dy = \pi \int_0^4 y\,dy = \pi\left[\tfrac{1}{2}y^2\right]_0^4 = 8\pi$$

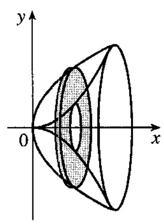




7. A cross-section is a washer (annulus) with inner radius x^2 and outer radius $\sqrt{x}$, so its area is $A(x) = \pi\left(\sqrt{x}\,\right)^2 - \pi\left(x^2\right)^2 = \pi\left(x - x^4\right)$.

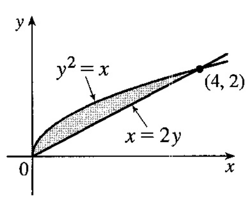
$$V = \int_0^1 A(x)\,dx = \pi \int_0^1 \left(x - x^4\right) dx = \pi\left[\tfrac{1}{2}x^2 - \tfrac{1}{5}x^5\right]_0^1 = \pi\left(\tfrac{1}{2} - \tfrac{1}{5}\right) = \tfrac{3\pi}{10}$$

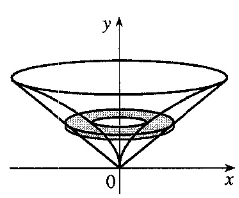




9. A cross-section is a washer with inner radius y^2 and outer radius $2y$, so its area is $A(y) = \pi(2y)^2 - \pi\left(y^2\right)^2 = \pi\left(4y^2 - y^4\right)$.

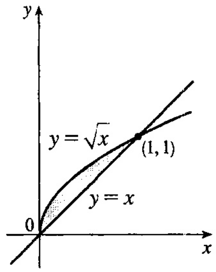
$$V = \int_0^2 A(y)\,dy = \pi \int_0^2 \left(4y^2 - y^4\right) dy = \pi\left[\tfrac{4}{3}y^3 - \tfrac{1}{5}y^5\right]_0^2 = \pi\left(\tfrac{32}{3} - \tfrac{32}{5}\right) = \tfrac{64\pi}{15}$$

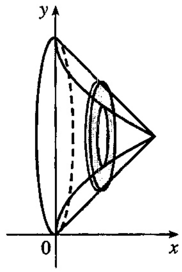




11. A cross-section is a washer with inner radius $1-\sqrt{x}$ and outer radius $1-x$, so its area is
$A(x)=\pi(1-x)^2-\pi(1-\sqrt{x})^2=\pi\left[(1-2x+x^2)-(1-2\sqrt{x}+x)\right]=\pi\left(-3x+x^2+2\sqrt{x}\right)$.

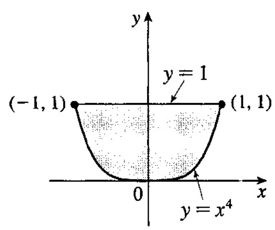
$$\begin{aligned} V &= \int_0^1 A(x)\,dx = \pi\int_0^1\left(-3x+x^2+2\sqrt{x}\right)dx \\ &= \pi\left[-\tfrac{3}{2}x^2+\tfrac{1}{3}x^3+\tfrac{4}{3}x^{3/2}\right]_0^1 = \pi\left(-\tfrac{3}{2}+\tfrac{5}{3}\right)=\tfrac{\pi}{6} \end{aligned}$$

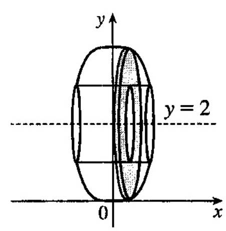




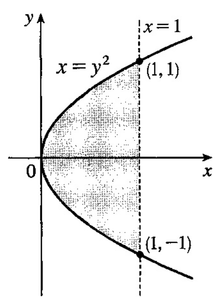
13. A cross-section is an annulus with inner radius $2-1$ and outer radius $2-x^4$, so its area is
$A(x)=\pi\left(2-x^4\right)^2-\pi(2-1)^2=\pi\left(3-4x^4+x^8\right)$.

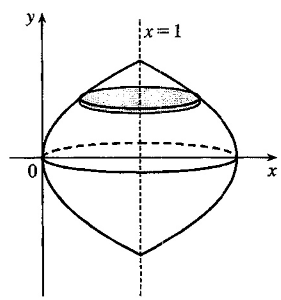
$$\begin{aligned} V &= \int_{-1}^1 A(x)\,dx = 2\int_0^1 A(x)\,dx = 2\pi\int_0^1\left(3-4x^4+x^8\right)dx = 2\pi\left[3x-\tfrac{4}{5}x^5+\tfrac{1}{9}x^9\right]_0^1 \\ &= 2\pi\left(3-\tfrac{4}{5}+\tfrac{1}{9}\right)=\tfrac{208}{45}\pi \end{aligned}$$





15. $V=\int_{-1}^1\pi\left(1-y^2\right)^2dy=2\int_0^1\pi\left(1-y^2\right)^2dy=2\pi\int_0^1\left(1-2y^2+y^4\right)dy$
$=2\pi\left[y-\tfrac{2}{3}y^3+\tfrac{1}{5}y^5\right]_0^1=2\pi\cdot\tfrac{8}{15}=\tfrac{16}{15}\pi$



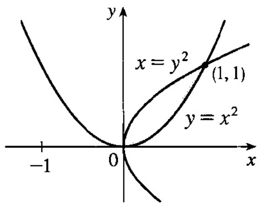


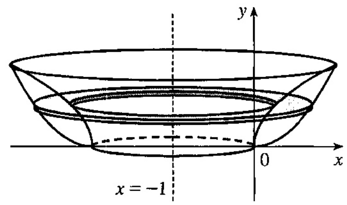
17. $y = x^2 \;\Rightarrow\; x = \sqrt{y}$ for $x \geq 0$. The outer radius is the distance from $x = -1$ to $x = \sqrt{y}$ and the inner radius is the distance from $x = -1$ to $x = y^2$.

$$V = \int_0^1 \pi \left\{ \left[\sqrt{y} - (-1)\right]^2 - \left[y^2 - (-1)\right]^2 \right\} dy = \pi \int_0^1 \left[\left(\sqrt{y} + 1\right)^2 - \left(y^2 + 1\right)^2 \right] dy$$

$$= \pi \int_0^1 \left(y + 2\sqrt{y} + 1 - y^4 - 2y^2 - 1\right) dy = \pi \int_0^1 \left(y + 2\sqrt{y} - y^4 - 2y^2\right) dy$$

$$= \pi \left[\tfrac{1}{2}y^2 + \tfrac{4}{3}y^{3/2} - \tfrac{1}{5}y^5 - \tfrac{2}{3}y^3 \right]_0^1 = \pi\left(\tfrac{1}{2} + \tfrac{4}{3} - \tfrac{1}{5} - \tfrac{2}{3}\right) = \tfrac{29}{30}\pi$$





19. $\mathfrak{R}_1$ about OA (the line $y = 0$): $V = \int_0^1 A(x)\,dx = \int_0^1 \pi(x^3)^2\,dx = \pi \int_0^1 x^6\,dx = \pi\left[\tfrac{1}{7}x^7\right]_0^1 = \tfrac{\pi}{7}$

21. $\mathfrak{R}_1$ about AB (the line $x = 1$):

$$V = \int_0^1 A(y)\,dy = \int_0^1 \pi\left(1 - \sqrt[3]{y}\right)^2 dy = \pi \int_0^1 \left(1 - 2y^{1/3} + y^{2/3}\right) dy$$

$$= \pi\left[y - \tfrac{3}{2}y^{4/3} + \tfrac{3}{5}y^{5/3}\right]_0^1 = \pi\left(1 - \tfrac{3}{2} + \tfrac{3}{5}\right) = \tfrac{\pi}{10}$$

23. $\mathfrak{R}_2$ about OA (the line $y = 0$):

$$V = \int_0^1 A(x)\,dx = \int_0^1 \left[\pi(1)^2 - \pi\left(\sqrt{x}\right)^2\right] dx = \pi \int_0^1 (1 - x)\,dx = \pi\left[x - \tfrac{1}{2}x^2\right]_0^1 = \pi\left(1 - \tfrac{1}{2}\right) = \tfrac{\pi}{2}$$

25. $\mathfrak{R}_2$ about AB (the line $x = 1$):

$$V = \int_0^1 A(y)\,dy = \int_0^1 \left[\pi(1)^2 - \pi(1 - y^2)^2\right] dy = \pi \int_0^1 \left[1 - (1 - 2y^2 + y^4)\right] dy$$

$$= \pi \int_0^1 (2y^2 - y^4)\,dy = \pi\left[\tfrac{2}{3}y^3 - \tfrac{1}{5}y^5\right]_0^1 = \pi\left(\tfrac{2}{3} - \tfrac{1}{5}\right) = \tfrac{7\pi}{15}$$

27. $\mathfrak{R}_3$ about OA (the line $y = 0$):

$$V = \int_0^1 A(x)\,dx = \int_0^1 \left[\pi\left(\sqrt{x}\right)^2 - \pi(x^3)^2\right] dx = \pi \int_0^1 (x - x^6)\,dx = \pi\left[\tfrac{1}{2}x^2 - \tfrac{1}{7}x^7\right]_0^1 = \pi\left(\tfrac{1}{2} - \tfrac{1}{7}\right) = \tfrac{5\pi}{14}.$$

Note: Let $\mathfrak{R} = \mathfrak{R}_1 + \mathfrak{R}_2 + \mathfrak{R}_3$. If we rotate $\mathfrak{R}$ about any of the segments OA, OC, AB, or BC, we obtain a right circular cylinder of height 1 and radius 1. Its volume is $\pi r^2 h = \pi(1)^2 \cdot 1 = \pi$. As a check for Exercises 19, 23, and 27, we can add the answers, and that sum must equal π. Thus, $\frac{\pi}{7} + \frac{\pi}{2} + \frac{5\pi}{14} = \left(\frac{2+7+5}{14}\right)\pi = \pi$.

29. $\mathfrak{R}_3$ about AB (the line $x = 1$):

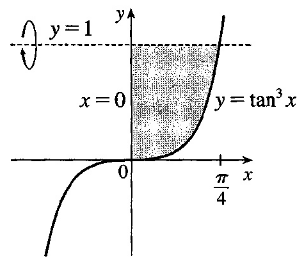
$$V = \int_0^1 A(y)\,dy = \int_0^1 \left[\pi(1 - y^2)^2 - \pi\left(1 - \sqrt[3]{y}\right)^2\right] dy = \pi \int_0^1 \left[(1 - 2y^2 + y^4) - (1 - 2y^{1/3} + y^{2/3})\right] dy$$

$$= \pi \int_0^1 \left(-2y^2 + y^4 + 2y^{1/3} - y^{2/3}\right) dy = \pi\left[-\tfrac{2}{3}y^3 + \tfrac{1}{5}y^5 + \tfrac{3}{2}y^{4/3} - \tfrac{3}{5}y^{5/3}\right]_0^1$$

$$= \pi\left(-\tfrac{2}{3} + \tfrac{1}{5} + \tfrac{3}{2} - \tfrac{3}{5}\right) = \tfrac{13\pi}{30}$$

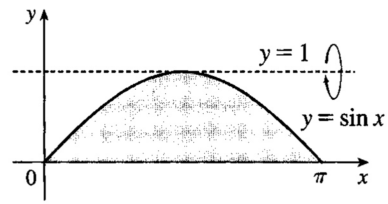
Note: See the note in Exercise 27. For Exercises 21, 25, and 29, we have $\frac{\pi}{10} + \frac{7\pi}{15} + \frac{13\pi}{30} = \left(\frac{3+14+13}{30}\right)\pi = \pi$.

31. $V = \pi \int_0^{\pi/4} (1 - \tan^3 x)^2 \, dx$



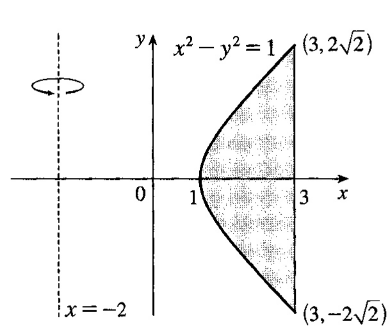
33. $V = \pi \int_0^{\pi} \left[(1-0)^2 - (1-\sin x)^2\right] dx$

$= \pi \int_0^{\pi} \left[1^2 - (1-\sin x)^2\right] dx$

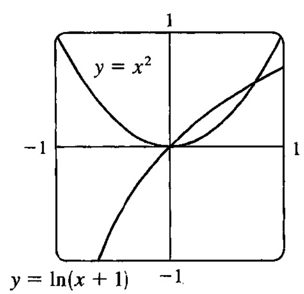


35. $V = \pi \int_{-\sqrt{8}}^{\sqrt{8}} \left\{[3-(-2)]^2 - \left[\sqrt{y^2+1} - (-2)\right]^2\right\} dy$

$= \pi \int_{-2\sqrt{2}}^{2\sqrt{2}} \left[5^2 - \left(\sqrt{1+y^2}+2\right)^2\right] dy$



37.

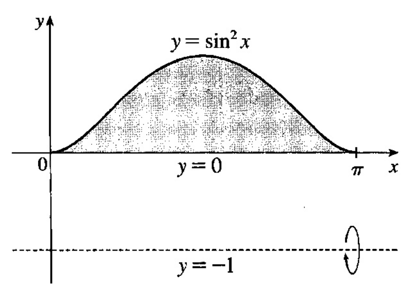


$y = x^2$ and $y = \ln(x+1)$ intersect at $x = 0$ and at $x = a \approx 0.747$.

$$V = \pi \int_0^a \left\{[\ln(x+1)]^2 - \left(x^2\right)^2\right\} dx \approx 0.132$$

39. $V = \pi \int_0^{\pi} \left\{\left[\sin^2 x - (-1)\right]^2 - [0-(-1)]^2\right\} dx$

$\overset{\text{CAS}}{=} \frac{11}{8}\pi^2$



41. $\pi \int_0^{\pi/2} \cos^2 x\, dx$ describes the volume of the solid obtained by rotating the region

$\mathcal{R} = \{(x,y) \mid 0 \le x \le \frac{\pi}{2}, 0 \le y \le \cos x\}$ of the xy-plane about the x-axis.

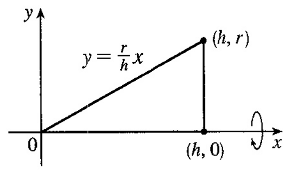
43. $\pi \int_0^1 (y^4 - y^8)\,dy = \pi \int_0^1 \left[(y^2)^2 - (y^4)^2\right] dy$ describes the volume of the solid obtained by rotating the region

$\mathcal{R} = \{(x,y) \mid 0 \le y \le 1, y^4 \le x \le y^2\}$ of the xy-plane about the y-axis.

45. There are 10 subintervals over the 15-cm length, so we'll use $n = 10/2 = 5$ for the Midpoint Rule.

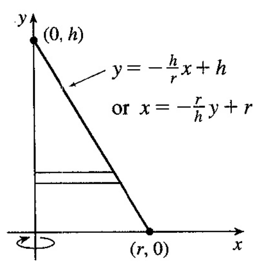
$$V = \int_0^{15} A(x)\,dx \approx M_5 = \frac{15-0}{5}[A(1.5) + A(4.5) + A(7.5) + A(10.5) + A(13.5)]$$
$$= 3(18 + 79 + 106 + 128 + 39) = 3 \cdot 370 = 1110 \text{ cm}^3$$

47. We'll form a right circular cone with height h and base radius r by revolving the line $y = \frac{r}{h}x$ about the x-axis.



$$V = \pi \int_0^h \left(\frac{r}{h}x\right)^2 dx = \pi \int_0^h \frac{r^2}{h^2} x^2\,dx = \pi \frac{r^2}{h^2}\left[\frac{1}{3}x^3\right]_0^h$$
$$= \pi \frac{r^2}{h^2}\left(\frac{1}{3}h^3\right) = \frac{1}{3}\pi r^2 h$$

Another solution: Revolve $x = -\dfrac{r}{h}y + r$ about the y-axis.

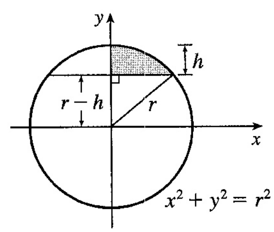


$$V = \pi \int_0^h \left(-\frac{r}{h}y + r\right)^2 dy \stackrel{*}{=} \pi \int_0^h \left[\frac{r^2}{h^2}y^2 - \frac{2r^2}{h}y + r^2\right] dy$$
$$= \pi\left[\frac{r^2}{3h^2}y^3 - \frac{r^2}{h}y^2 + r^2 y\right]_0^h = \pi\left(\tfrac{1}{3}r^2h - r^2h + r^2h\right) = \tfrac{1}{3}\pi r^2 h$$

* Or use substitution with $u = r - \dfrac{r}{h}y$ and $du = -\dfrac{r}{h}dy$ to get

$$\pi \int_r^0 u^2\left(-\frac{h}{r}\,du\right) = -\pi\frac{h}{r}\left[\frac{1}{3}u^3\right]_r^0 = -\pi\frac{h}{r}\left(-\frac{1}{3}r^3\right) = \frac{1}{3}\pi r^2 h.$$

49. $x^2 + y^2 = r^2 \quad \Leftrightarrow \quad x^2 = r^2 - y^2$

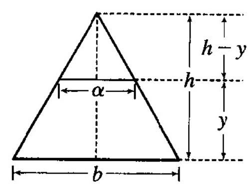


$$V = \pi \int_{r-h}^r (r^2 - y^2)\,dy = \pi\left[r^2 y - \frac{y^3}{3}\right]_{r-h}^r$$
$$= \pi\left\{\left[r^3 - \frac{r^3}{3}\right] - \left[r^2(r-h) - \frac{(r-h)^3}{3}\right]\right\}$$
$$= \pi\left\{\tfrac{2}{3}r^3 - \tfrac{1}{3}(r-h)\left[3r^2 - (r-h)^2\right]\right\}$$
$$= \tfrac{1}{3}\pi\left\{2r^3 - (r-h)\left[3r^2 - (r^2 - 2rh + h^2)\right]\right\}$$
$$= \tfrac{1}{3}\pi\left\{2r^3 - (r-h)\left[2r^2 + 2rh - h^2\right]\right\}$$
$$= \tfrac{1}{3}\pi\left(2r^3 - 2r^3 - 2r^2h + rh^2 + 2r^2h + 2rh^2 - h^3\right)$$
$$= \tfrac{1}{3}\pi\left(3rh^2 - h^3\right) = \tfrac{1}{3}\pi h^2(3r - h), \text{ or, equivalently, } \pi h^2\left(r - \frac{h}{3}\right)$$

51. For a cross-section at height y, we see from similar triangles that $\dfrac{\alpha/2}{b/2} = \dfrac{h-y}{h}$, so $\alpha = b\left(1 - \dfrac{y}{h}\right)$.

Similarly, for cross-sections having $2b$ as their base and β replacing α, $\beta = 2b\left(1 - \dfrac{y}{h}\right)$. So

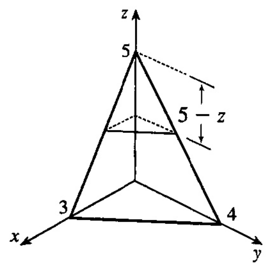
$$\begin{aligned} V &= \int_0^h A(y)\,dy = \int_0^h \left[b\left(1-\frac{y}{h}\right)\right]\left[2b\left(1-\frac{y}{h}\right)\right] dy \\ &= \int_0^h 2b^2\left(1-\frac{y}{h}\right)^2 dy = 2b^2\int_0^h \left(1 - \frac{2y}{h} + \frac{y^2}{h^2}\right) dy \\ &= 2b^2\left[y - \frac{y^2}{h} + \frac{y^3}{3h^2}\right]_0^h = 2b^2\left[h - h + \tfrac{1}{3}h\right] \\ &= \tfrac{2}{3}b^2h \quad \left[= \tfrac{1}{3}Bh \text{ where } B \text{ is the area of the base, as with any pyramid.}\right] \end{aligned}$$



53. A cross-section at height z is a triangle similar to the base, so we'll multiply the legs of the base triangle, 3 and 4, by a proportionality factor of $(5-z)/5$. Thus, the triangle at height z has area

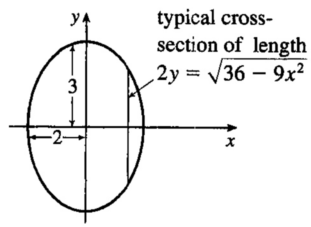
$$A(z) = \frac{1}{2}\cdot 3\left(\frac{5-z}{5}\right)\cdot 4\left(\frac{5-z}{5}\right) = 6\left(1-\frac{z}{5}\right)^2, \text{ so}$$

$$\begin{aligned} V &= \textstyle\int_0^5 A(z)\,dz = 6\int_0^5 (1-z/5)^2\,dz \\ &= 6\textstyle\int_1^0 u^2(-5\,du) \quad \left[u = 1 - z/5,\ du = -\tfrac{1}{5}dz\right] \\ &= -30\left[\tfrac{1}{3}u^3\right]_1^0 = -30\left(-\tfrac{1}{3}\right) = 10 \text{ cm}^3 \end{aligned}$$

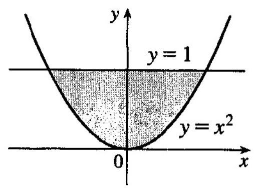


55. If l is a leg of the isosceles right triangle and $2y$ is the hypotenuse, then $l^2 + l^2 = (2y)^2 \Rightarrow 2l^2 = 4y^2 \Rightarrow l^2 = 2y^2$.

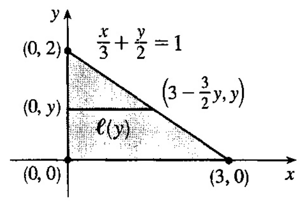
$$\begin{aligned} V &= \textstyle\int_{-2}^2 A(x)\,dx = 2\int_0^2 A(x)\,dx = 2\int_0^2 \tfrac{1}{2}(l)(l)\,dx = 2\int_0^2 y^2\,dx \\ &= 2\textstyle\int_0^2 \tfrac{1}{4}(36 - 9x^2)\,dx = \tfrac{9}{2}\int_0^2 (4 - x^2)\,dx \\ &= \tfrac{9}{2}\left[4x - \tfrac{1}{3}x^3\right]_0^2 = \tfrac{9}{2}\left(8 - \tfrac{8}{3}\right) = 24 \end{aligned}$$



57. The cross-section of the base corresponding to the coordinate y has length $2x = 2\sqrt{y}$. The square has area $A(y) = \left(2\sqrt{y}\right)^2 = 4y$, so $V = \int_0^1 A(y)\,dy = \int_0^1 4y\,dy = \left[2y^2\right]_0^1 = 2$.



59. A typical cross-section perpendicular to the y-axis in the base has length $\ell(y) = 3 - \frac{3}{2}y$. This length is the leg of an isosceles right triangle, so

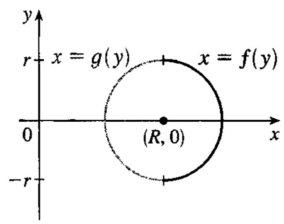


$$A(y) = \tfrac{1}{2}\,[\ell(y)]^2 \qquad \left[\tfrac{1}{2}bh \text{ with base} = \text{height}\right]$$
$$= \tfrac{1}{2}\left[3\left(1 - \tfrac{1}{2}y\right)\right]^2 = \tfrac{9}{2}\left(1 - \tfrac{1}{2}y\right)^2$$

Thus,

$$V = \int_0^2 A(y)\,dy = \tfrac{9}{2}\int_1^0 u^2(-2\,du) \qquad \left[u = 1 - \tfrac{1}{2}y,\ du = -\tfrac{1}{2}\,dy\right]$$
$$= -9\left[\tfrac{1}{3}u^3\right]_1^0 = -9\left(-\tfrac{1}{3}\right) = 3$$

61. (a) The torus is obtained by rotating the circle $(x - R)^2 + y^2 = r^2$ about the y-axis. Solving for x, we see that the right half of the circle is given by $x = R + \sqrt{r^2 - y^2} = f(y)$ and the left half by $x = R - \sqrt{r^2 - y^2} = g(y)$. So



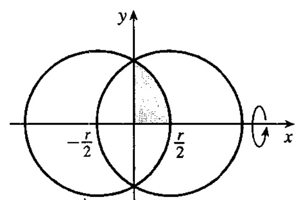
$$V = \pi\int_{-r}^{r}\left\{[f(y)]^2 - [g(y)]^2\right\}dy$$
$$= 2\pi\int_0^r\left[\left(R^2 + 2R\sqrt{r^2 - y^2} + r^2 - y^2\right) - \left(R^2 - 2R\sqrt{r^2 - y^2} + r^2 - y^2\right)\right]dy$$
$$= 2\pi\int_0^r 4R\sqrt{r^2 - y^2}\,dy = 8\pi R\int_0^r \sqrt{r^2 - y^2}\,dy$$

(b) Observe that the integral represents a quarter of the area of a circle with radius r, so $8\pi R\int_0^r \sqrt{r^2 - y^2}\,dy = 8\pi R \cdot \frac{1}{4}\pi r^2 = 2\pi^2 r^2 R$.

63. (a) Volume$(S_1) = \int_0^h A(z)\,dz =$ Volume(S_2) since the cross-sectional area $A(z)$ at height z is the same for both solids.

(b) By Cavalieri's Principle, the volume of the cylinder in the figure is the same as that of a right circular cylinder with radius r and height h, that is, $\pi r^2 h$.

65. The volume is obtained by rotating the area common to two circles of radius r, as shown. The volume of the right half is



$$V_{\text{right}} = \pi\int_0^{r/2} y^2\,dx = \pi\int_0^{r/2}\left[r^2 - \left(\tfrac{1}{2}r + x\right)^2\right]dx$$
$$= \pi\left[r^2x - \tfrac{1}{3}\left(\tfrac{1}{2}r + x\right)^3\right]_0^{r/2} = \pi\left[\left(\tfrac{1}{2}r^3 - \tfrac{1}{3}r^3\right) - \left(0 - \tfrac{1}{24}r^3\right)\right] = \tfrac{5}{24}\pi r^3$$

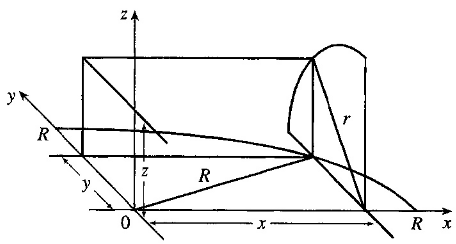
So by symmetry, the total volume is twice this, or $\frac{5}{12}\pi r^3$.

Another solution: We observe that the volume is the twice the volume of a cap of a sphere, so we can use the formula from Exercise 49 with $h = \frac{1}{2}r$: $V = 2 \cdot \frac{1}{3}\pi h^2(3r - h) = \frac{2}{3}\pi\left(\frac{1}{2}r\right)^2\left(3r - \frac{1}{2}r\right) = \frac{5}{12}\pi r^3$.

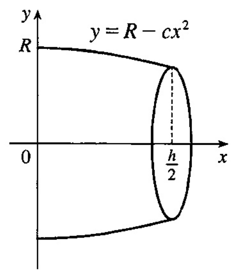
67. Take the x-axis to be the axis of the cylindrical hole of radius r. A quarter of the cross-section through y, perpendicular to the y-axis, is the rectangle shown. Using the Pythagorean Theorem twice, we see that the dimensions of this rectangle are $x = \sqrt{R^2 - y^2}$ and $z = \sqrt{r^2 - y^2}$, so $\frac{1}{4}A(y) = xz = \sqrt{r^2 - y^2}\,\sqrt{R^2 - y^2}$, and

$$V = \int_{-r}^{r} A(y)\,dy = \int_{-r}^{r} 4\sqrt{r^2 - y^2}\,\sqrt{R^2 - y^2}\,dy$$

$$= 8\int_0^r \sqrt{r^2 - y^2}\,\sqrt{R^2 - y^2}\,dy$$



69. (a) The radius of the barrel is the same at each end by symmetry, since the function $y = R - cx^2$ is even. Since the barrel is obtained by rotating the graph of the function y about the x-axis, this radius is equal to the value of y at $x = \frac{1}{2}h$, which is $R - c\left(\frac{1}{2}h\right)^2 = R - d = r$.



(b) The barrel is symmetric about the y-axis, so its volume is twice the volume of that part of the barrel for $x > 0$. Also, the barrel is a volume of rotation, so

$$V = 2\int_0^{h/2} \pi y^2\,dx = 2\pi\int_0^{h/2} \left(R - cx^2\right)^2 dx = 2\pi\left[R^2x - \tfrac{2}{3}Rcx^3 + \tfrac{1}{5}c^2x^5\right]_0^{h/2}$$

$$= 2\pi\left(\tfrac{1}{2}R^2h - \tfrac{1}{12}Rch^3 + \tfrac{1}{160}c^2h^5\right)$$

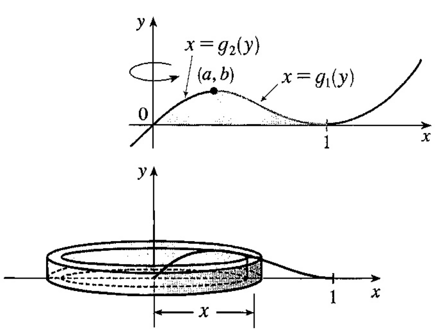
Trying to make this look more like the expression we want, we rewrite it as $V = \frac{1}{3}\pi h\left[2R^2 + \left(R^2 - \frac{1}{2}Rch^2 + \frac{3}{80}c^2h^4\right)\right]$. But

$R^2 - \frac{1}{2}Rch^2 + \frac{3}{80}c^2h^4 = \left(R - \frac{1}{4}ch^2\right)^2 - \frac{1}{40}c^2h^4 = (R - d)^2 - \frac{2}{5}\left(\frac{1}{4}ch^2\right)^2 = r^2 - \frac{2}{5}d^2$.

Substituting this back into V, we see that $V = \frac{1}{3}\pi h\left(2R^2 + r^2 - \frac{2}{5}d^2\right)$, as required.

6.3 Volumes by Cylindrical Shells

1.



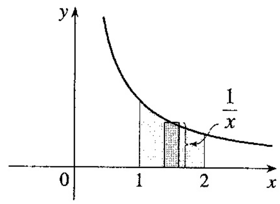
If we were to use the "washer" method, we would first have to locate the local maximum point (a, b) of $y = x(x-1)^2$ using the methods of Chapter 4. Then we would have to solve the equation $y = x(x-1)^2$ for x in terms of y to obtain the functions $x = g_1(y)$ and $x = g_2(y)$ shown in the first figure. This step would be difficult because it involves the cubic formula. Finally we would find the volume using $V = \pi\int_0^b \left\{[g_1(y)]^2 - [g_2(y)]^2\right\} dy$.

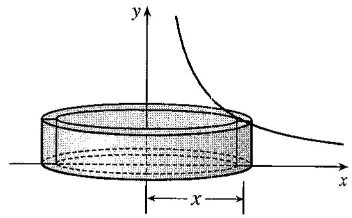
Using shells, we find that a typical approximating shell has radius x, so its circumference is $2\pi x$. Its height is y, that is, $x(x-1)^2$. So the total volume is

$$V = \int_0^1 2\pi x\left[x(x-1)^2\right] dx = 2\pi \int_0^1 \left(x^4 - 2x^3 + x^2\right) dx = 2\pi \left[\frac{x^5}{5} - 2\frac{x^4}{4} + \frac{x^3}{3}\right]_0^1 = \frac{\pi}{15}$$

3. $V = \displaystyle\int_1^2 2\pi x \cdot \frac{1}{x}\, dx = 2\pi \int_1^2 1\, dx$

$= 2\pi\, [x]_1^2 = 2\pi(2-1) = 2\pi$



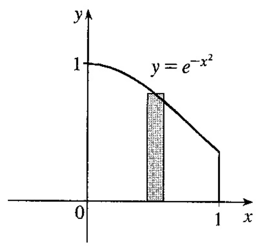


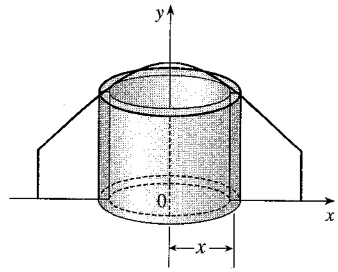
5. $V = \int_0^1 2\pi x e^{-x^2}\, dx$. Let $u = x^2$.

Thus, $du = 2x\, dx$, so

$V = \pi \int_0^1 e^{-u}\, du = \pi\left[-e^{-u}\right]_0^1$

$= \pi(1 - 1/e)$





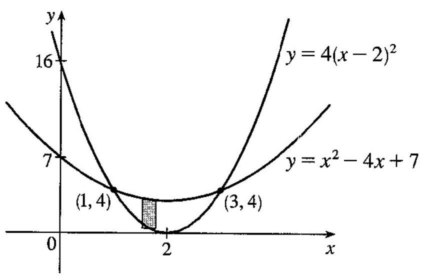
7. The curves intersect when $4(x-2)^2 = x^2 - 4x + 7 \;\Leftrightarrow\; 4x^2 - 16x + 16 = x^2 - 4x + 7 \;\Leftrightarrow$

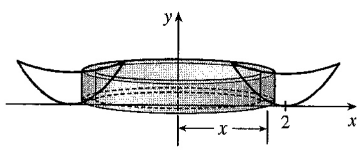
$3x^2 - 12x + 9 = 0 \;\Leftrightarrow\; 3(x^2 - 4x + 3) = 0 \;\Leftrightarrow\; 3(x-1)(x-3) = 0$, so $x = 1$ or 3.

$V = 2\pi \int_1^3 \left\{x\left[(x^2 - 4x + 7) - 4(x-2)^2\right]\right\} dx = 2\pi \int_1^3 \left[x(x^2 - 4x + 7 - 4x^2 + 16x - 16)\right] dx$

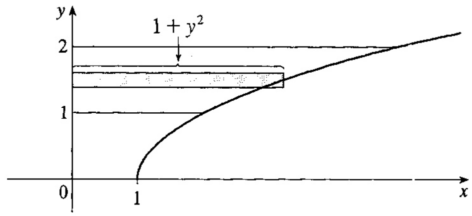
$= 2\pi \int_1^3 \left[x(-3x^2 + 12x - 9)\right] dx = 2\pi(-3)\int_1^3 (x^3 - 4x^2 + 3x)\, dx = -6\pi\left[\frac{1}{4}x^4 - \frac{4}{3}x^3 + \frac{3}{2}x^2\right]_1^3$

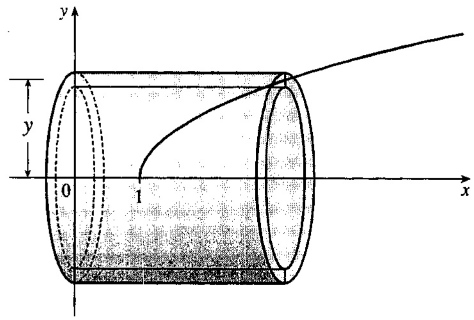
$= -6\pi\left[\left(\frac{81}{4} - 36 + \frac{27}{2}\right) - \left(\frac{1}{4} - \frac{4}{3} + \frac{3}{2}\right)\right] = -6\pi\left(20 - 36 + 12 + \frac{4}{3}\right) = -6\pi\left(-\frac{8}{3}\right) = 16\pi$





9. $V = \int_1^2 2\pi y(1+y^2)\,dy = 2\pi \int_1^2 (y+y^3)\,dy = 2\pi\left[\frac{1}{2}y^2 + \frac{1}{4}y^4\right]_1^2$
$= 2\pi\left[(2+4) - \left(\frac{1}{2}+\frac{1}{4}\right)\right] = 2\pi\left(\frac{21}{4}\right) = \frac{21\pi}{2}$

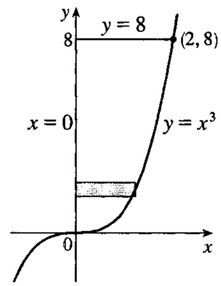


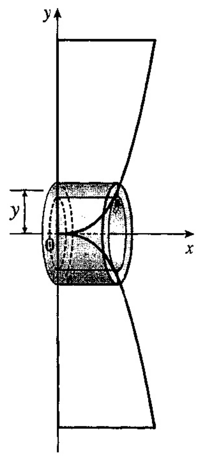


11. $V = 2\pi \int_0^8 [y(\sqrt[3]{y} - 0)]\,dy$

$= 2\pi \int_0^8 y^{4/3}\,dy = 2\pi\left[\frac{3}{7}y^{7/3}\right]_0^8$

$= \frac{6\pi}{7}(8^{7/3}) = \frac{6\pi}{7}(2^7) = \frac{768\pi}{7}$





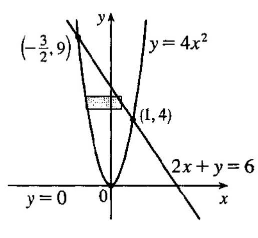
13. The curves intersect when $4x^2 = 6 - 2x \;\Leftrightarrow\; 2x^2 + x - 3 = 0 \;\Leftrightarrow\; (2x+3)(x-1) = 0 \;\Leftrightarrow\; x = -\frac{3}{2}$ or 1.
Solving the equations for x gives us $y = 4x^2 \;\Rightarrow\; x = \pm\frac{1}{2}\sqrt{y}$ and $2x + y = 6 \;\Rightarrow\; x = -\frac{1}{2}y + 3$.

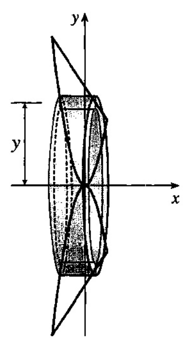
$$V = 2\pi \int_0^4 \left\{y\left[\left(\tfrac{1}{2}\sqrt{y}\right) - \left(-\tfrac{1}{2}\sqrt{y}\right)\right]\right\} dy + 2\pi \int_4^9 \left\{y\left[\left(-\tfrac{1}{2}y + 3\right) - \left(-\tfrac{1}{2}\sqrt{y}\right)\right]\right\} dy$$

$$= 2\pi \int_0^4 (y\sqrt{y})\,dy + 2\pi \int_4^9 \left(-\tfrac{1}{2}y^2 + 3y + \tfrac{1}{2}y^{3/2}\right) dy = 2\pi\left[\tfrac{2}{5}y^{5/2}\right]_0^4 + 2\pi\left[-\tfrac{1}{6}y^3 + \tfrac{3}{2}y^2 + \tfrac{1}{5}y^{5/2}\right]_4^9$$

$$= 2\pi\left(\tfrac{2}{5}\cdot 32\right) + 2\pi\left[\left(-\tfrac{243}{2} + \tfrac{243}{2} + \tfrac{243}{5}\right) - \left(-\tfrac{32}{3} + 24 + \tfrac{32}{5}\right)\right]$$

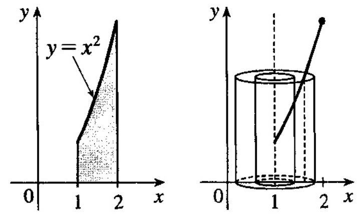
$$= \tfrac{128}{5}\pi + 2\pi\left(\tfrac{433}{15}\right) = \tfrac{1250}{15}\pi = \tfrac{250}{3}\pi$$





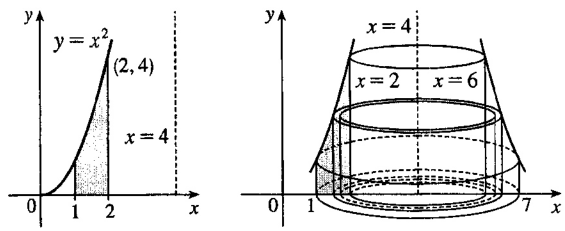
15. $V = \int_1^2 2\pi(x-1)x^2\,dx = 2\pi\left[\frac{1}{4}x^4 - \frac{1}{3}x^3\right]_1^2$

$= 2\pi\left[\left(4 - \frac{8}{3}\right) - \left(\frac{1}{4} - \frac{1}{3}\right)\right] = \frac{17}{6}\pi$



17. $V = \int_1^2 2\pi(4-x)x^2\,dx = 2\pi\left[\frac{4}{3}x^3 - \frac{1}{4}x^4\right]_1^2$

$= 2\pi\left[\left(\frac{32}{3} - 4\right) - \left(\frac{4}{3} - \frac{1}{4}\right)\right] = \frac{67}{6}\pi$



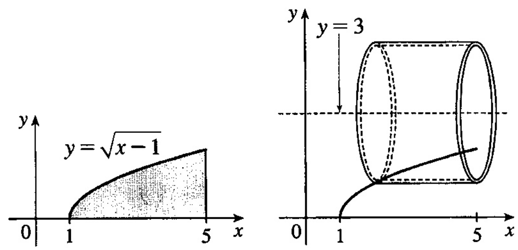
19. $V = \int_0^2 2\pi(3-y)(5-x)dy$

$= \int_0^2 2\pi(3-y)\left(5 - y^2 - 1\right)dy$

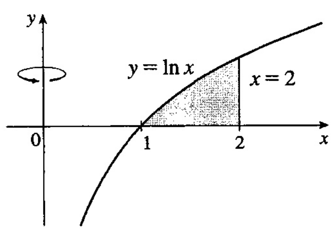
$= \int_0^2 2\pi\left(12 - 4y - 3y^2 + y^3\right)dy$

$= 2\pi\left[12y - 2y^2 - y^3 + \frac{1}{4}y^4\right]_0^2$

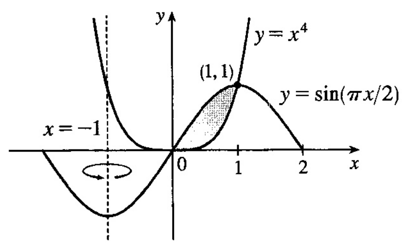
$= 2\pi(24 - 8 - 8 + 4) = 24\pi$



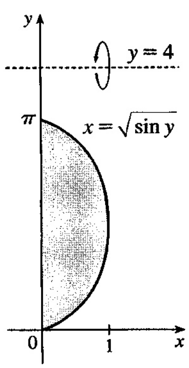
21. $V = \int_1^2 2\pi x \ln x\,dx$



23. $V = \int_0^1 2\pi[x - (-1)]\left(\sin \frac{\pi}{2}x - x^4\right)dx$



25. $V = \int_0^\pi 2\pi(4-y)\sqrt{\sin y}\,dy$



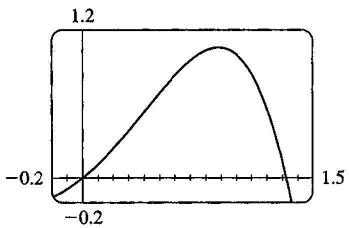
27. $\Delta x = \dfrac{\pi/4 - 0}{4} = \dfrac{\pi}{16}$.

$V = \int_0^{\pi/4} 2\pi x \, \tan x \, dx \approx 2\pi \cdot \frac{\pi}{16} \left(\frac{\pi}{32} \tan \frac{\pi}{32} + \frac{3\pi}{32} \tan \frac{3\pi}{32} + \frac{5\pi}{32} \tan \frac{5\pi}{32} + \frac{7\pi}{32} \tan \frac{7\pi}{32} \right) \approx 1.142$

29. $\int_0^3 2\pi x^5 \, dx = 2\pi \int_0^3 x(x^4) \, dx$. The solid is obtained by rotating the region $0 \le y \le x^4$, $0 \le x \le 3$ about the y-axis using cylindrical shells.

31. $\int_0^1 2\pi(3-y)(1-y^2) \, dy$. The solid is obtained by rotating the region bounded by (i) $x = 1 - y^2$, $x = 0$, and $y = 0$ or (ii) $x = y^2$, $x = 1$, and $y = 0$ about the line $y = 3$ using cylindrical shells.

33.



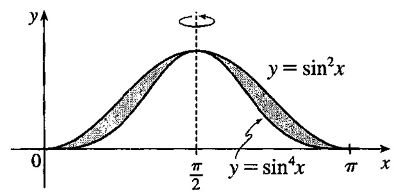
From the graph, the curves intersect at $x = 0$ and at $x = a \approx 1.32$, with $x + x^2 - x^4 > 0$ on the interval $(0, a)$. So the volume of the solid obtained by rotating the region about the y-axis is

$$V = 2\pi \int_0^a \left[x(x + x^2 - x^4) \right] dx = 2\pi \int_0^a (x^2 + x^3 - x^5) \, dx$$

$$= 2\pi \left[\tfrac{1}{3}x^3 + \tfrac{1}{4}x^4 - \tfrac{1}{6}x^6 \right]_0^a \approx 4.05$$

35. $V = 2\pi \displaystyle\int_0^{\pi/2} \left[\left(\tfrac{\pi}{2} - x \right) \left(\sin^2 x - \sin^4 x \right) \right] dx$

$\overset{\text{CAS}}{=} \frac{1}{32}\pi^3$



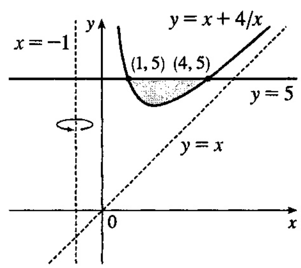
37. Use disks:

$$V = \int_{-2}^{1} \pi \left(x^2 + x - 2 \right)^2 dx = \pi \int_{-2}^{1} \left(x^4 + 2x^3 - 3x^2 - 4x + 4 \right) dx$$

$$= \pi \left[\tfrac{1}{5}x^5 + \tfrac{1}{2}x^4 - x^3 - 2x^2 + 4x \right]_{-2}^{1} = \pi \left[\left(\tfrac{1}{5} + \tfrac{1}{2} - 1 - 2 + 4 \right) - \left(-\tfrac{32}{5} + 8 + 8 - 8 - 8 \right) \right]$$

$$= \pi \left(\tfrac{33}{5} + \tfrac{3}{2} \right) = \tfrac{81}{10}\pi$$

39. Use shells:



$$V = \int_1^4 2\pi[x - (-1)][5 - (x + 4/x)] \, dx$$

$$= 2\pi \int_1^4 (x + 1)(5 - x - 4/x) \, dx$$

$$= 2\pi \int_1^4 \left(5x - x^2 - 4 + 5 - x - 4/x \right) dx$$

$$= 2\pi \int_1^4 \left(-x^2 + 4x + 1 - 4/x \right) dx = 2\pi \left[-\tfrac{1}{3}x^3 + 2x^2 + x - 4 \ln x \right]_1^4$$

$$= 2\pi \left[\left(-\tfrac{64}{3} + 32 + 4 - 4 \ln 4 \right) - \left(-\tfrac{1}{3} + 2 + 1 - 0 \right) \right]$$

$$= 2\pi(12 - 4 \ln 4) = 8\pi(3 - \ln 4)$$

41. Use disks: $V = \pi \int_0^2 \left[\sqrt{1-(y-1)^2}\right]^2 dy = \pi \int_0^2 (2y - y^2)\,dy = \pi\left[y^2 - \frac{1}{3}y^3\right]_0^2 = \pi\left(4 - \frac{8}{3}\right) = \frac{4}{3}\pi$

43. $V = 2\int_0^r 2\pi x\sqrt{r^2 - x^2}\,dx = -2\pi \int_0^r (r^2 - x^2)^{1/2}(-2x)\,dx = \left[-2\pi \cdot \frac{2}{3}(r^2 - x^2)^{3/2}\right]_0^r$
$= -\frac{4}{3}\pi(0 - r^3) = \frac{4}{3}\pi r^3$

45. $V = 2\pi \int_0^r x\left(-\frac{h}{r}x + h\right)dx = 2\pi h \int_0^r \left(-\frac{x^2}{r} + x\right)dx = 2\pi h\left[-\frac{x^3}{3r} + \frac{x^2}{2}\right]_0^r = 2\pi h\,\frac{r^2}{6} = \frac{\pi r^2 h}{3}$

6.4 Work

1. By Equation 2, $W = Fd = (900)(8) = 7200$ J.

3. $W = \int_a^b f(x)\,dx = \int_0^9 \frac{10}{(1+x)^2}\,dx = 10\int_1^{10} \frac{1}{u^2}\,du \qquad [u = 1 + x,\ du = dx]$
$= 10\left[-\frac{1}{u}\right]_1^{10} = 10\left(-\frac{1}{10} + 1\right) = 9$ ft-lb

5. The force function is given by $F(x)$ (in newtons) and the work (in joules) is the area under the curve, given by $\int_0^8 F(x)\,dx = \int_0^4 F(x)\,dx + \int_4^8 F(x)\,dx = \frac{1}{2}(4)(30) + (4)(30) = 180$ J.

7. $10 = f(x) = kx = \frac{1}{3}k$ [4 inches $= \frac{1}{3}$ foot], so $k = 30$ lb/ft and $f(x) = 30x$. Now 6 inches $= \frac{1}{2}$ foot, so $W = \int_0^{1/2} 30x\,dx = \left[15x^2\right]_0^{1/2} = \frac{15}{4}$ ft-lb.

9. If $\int_0^{0.12} kx\,dx = 2$ J, then $2 = \left[\frac{1}{2}kx^2\right]_0^{0.12} = \frac{1}{2}k(0.0144) = 0.0072k$ and $k = \frac{2}{0.0072} = \frac{2500}{9} \approx 277.78$ N/m. Thus, the work needed to stretch the spring from 35 cm to 40 cm is
$\int_{0.05}^{0.10} \frac{2500}{9}x\,dx = \left[\frac{1250}{9}x^2\right]_{1/20}^{1/10} = \frac{1250}{9}\left(\frac{1}{100} - \frac{1}{400}\right) = \frac{25}{24} \approx 1.04$ J.

11. $f(x) = kx$, so $30 = \frac{2500}{9}x$ and $x = \frac{270}{2500}$ m $= 10.8$ cm

In Exercises 13–20, n is the number of subintervals of length Δx, and x_i^* is a sample point in the ith subinterval $[x_{i-1}, x_i]$.

13. (a) The portion of the rope from x ft to $(x + \Delta x)$ ft below the top of the building weighs $\frac{1}{2}\,\Delta x$ lb and must be lifted x_i^* ft, so its contribution to the total work is $\frac{1}{2}x_i^*\,\Delta x$ ft-lb. The total work is

$$W = \lim_{n\to\infty} \sum_{i=1}^{n} \tfrac{1}{2}x_i^*\,\Delta x = \int_0^{50} \tfrac{1}{2}x\,dx = \left[\tfrac{1}{4}x^2\right]_0^{50} = \tfrac{2500}{4} = 625 \text{ ft-lb}$$

Notice that the exact height of the building does not matter (as long as it is more than 50 ft).

(b) When half the rope is pulled to the top of the building, the work to lift the top half of the rope is $W_1 = \int_0^{25} \frac{1}{2}x\,dx = \left[\frac{1}{4}x^2\right]_0^{25} = \frac{625}{4}$ ft-lb. The bottom half of the rope is lifted 25 ft and the work needed to accomplish that is $W_2 = \int_{25}^{50} \frac{1}{2} \cdot 25\,dx = \frac{25}{2}[x]_{25}^{50} = \frac{625}{2}$ ft-lb. The total work done in pulling half the rope to the top of the building is $W = W_1 + W_2 = \frac{625}{2} + \frac{625}{4} = \frac{3}{4} \cdot 625 = \frac{1875}{4}$ ft-lb.

15. The work needed to lift the cable is $\lim_{n\to\infty} \sum_{i=1}^{n} 2x_i^*\,\Delta x = \int_0^{500} 2x\,dx = \left[x^2\right]_0^{500} = 250{,}000$ ft-lb. The work needed to lift the coal is 800 lb $\cdot$ 500 ft $= 400{,}000$ ft-lb. Thus, the total work required is $250{,}000 + 400{,}000 = 650{,}000$ ft-lb.

17. At a height of x meters ($0 \le x \le 12$), the mass of the rope is $(0.8 \text{ kg/m})(12 - x \text{ m}) = (9.6 - 0.8x)$ kg and the mass of the water is $\left(\frac{36}{12} \text{ kg/m}\right)(12 - x \text{ m}) = (36 - 3x)$ kg. The mass of the bucket is 10 kg, so the total mass is $(9.6 - 0.8x) + (36 - 3x) + 10 = (55.6 - 3.8x)$ kg, and hence, the total force is $9.8(55.6 - 3.8x)$ N.

The work needed to lift the bucket Δx m through the ith subinterval of $[0, 12]$ is $9.8(55.6 - 3.8x_i^*)\Delta x$, so the total work is

$$W = \lim_{n\to\infty} \sum_{i=1}^{n} 9.8(55.6 - 3.8x_i^*)\,\Delta x = \int_0^{12} (9.8)(55.6 - 3.8x)\,dx = 9.8\Big[55.6x - 1.9x^2\Big]_0^{12}$$
$$= 9.8(393.6) \approx 3857 \text{ J}$$

19. A "slice" of water Δx m thick and lying at a depth of x_i^* m (where $0 \le x_i^* \le \frac{1}{2}$) has volume $(2 \times 1 \times \Delta x)$ m^3, a mass of $2000\,\Delta x$ kg, weighs about $(9.8)(2000\,\Delta x) = 19{,}600\,\Delta x$ N, and thus requires about $19{,}600 x_i^*\,\Delta x$ J of work for its removal. So $W = \lim_{n\to\infty} \sum_{i=1}^{n} 19{,}600 x_i^*\,\Delta x = \int_0^{1/2} 19{,}600x\,dx = \left[9800x^2\right]_0^{1/2} = 2450$ J.

21. A rectangular "slice" of water Δx m thick and lying x ft above the bottom has width x ft and volume $8x\,\Delta x$ m^3. It weighs about $(9.8 \times 1000)(8x\,\Delta x)$ N, and must be lifted $(5 - x)$ m by the pump, so the work needed is about $(9.8 \times 10^3)(5 - x)(8x\,\Delta x)$J. The total work required is

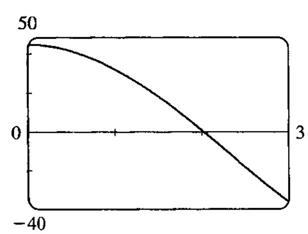
$$W \approx \int_0^3 (9.8 \times 10^3)(5 - x)8x\,dx = (9.8 \times 10^3)\int_0^3 (40x - 8x^2)dx = (9.8 \times 10^3)\left[20x^2 - \tfrac{8}{3}x^3\right]_0^3$$
$$= (9.8 \times 10^3)(180 - 72) = (9.8 \times 10^3)(108) = 1058.4 \times 10^3 \approx 1.06 \times 10^6 \text{ J}$$

23. Measure depth x downward from the flat top of the tank, so that $0 \le x \le 2$ ft. Then $\Delta W = (62.5)\left(2\sqrt{4 - x^2}\right)(8\,\Delta x)(x + 1)$ ft-lb, so

$$W \approx (62.5)(16)\int_0^2 (x + 1)\sqrt{4 - x^2}\,dx = 1000\left(\int_0^2 x\sqrt{4 - x^2}\,dx + \int_0^2 \sqrt{4 - x^2}\,dx\right)$$
$$= 1000\left[\int_0^4 u^{1/2}\left(\tfrac{1}{2}\,du\right) + \tfrac{1}{4}\pi(2^2)\right] \qquad \text{[Put } u = 4 - x^2\text{, so } du = -2x\,dx\text{]}$$
$$= 1000\left(\left[\tfrac{1}{2} \cdot \tfrac{2}{3}u^{3/2}\right]_0^4 + \pi\right) = 1000\left(\tfrac{8}{3} + \pi\right) \approx 5.8 \times 10^3 \text{ ft-lb}$$

Note: The second integral represents the area of a quarter-circle of radius 2.

25. If only 4.7×10^5 J of work is done, then only the water above a certain level (call it h) will be pumped out. So we use the same formula as in Exercise 21, except that the work is fixed, and we are trying to find the lower limit of integration: $4.7 \times 10^5 \approx \int_h^3 (9.8 \times 10^3)(5 - x)8x\,dx = (9.8 \times 10^3)\left[20x^2 - \frac{8}{3}x^3\right]_h^3 \Leftrightarrow$
$\frac{4.7}{9.8} \times 10^2 \approx 48 = \left(20 \cdot 3^2 - \frac{8}{3} \cdot 3^3\right) - \left(20h^2 - \frac{8}{3}h^3\right) \Leftrightarrow$
$2h^3 - 15h^2 + 45 = 0$. To find the solution of this equation, we plot $2h^3 - 15h^2 + 45$ between $h = 0$ and $h = 3$. We see that the equation is satisfied for $h \approx 2.0$. So the depth of water remaining in the tank is about 2.0 m.



27. $V = \pi r^2 x$, so V is a function of x and P can also be regarded as a function of x. If $V_1 = \pi r^2 x_1$ and $V_2 = \pi r^2 x_2$, then

$$\begin{aligned} W &= \int_{x_1}^{x_2} F(x)\,dx = \int_{x_1}^{x_2} \pi r^2 P(V(x))\,dx \\ &= \int_{x_1}^{x_2} P(V(x))\,dV(x) \qquad [\text{Let } V(x) = \pi r^2 x, \text{ so } dV(x) = \pi r^2\,dx.] \\ &= \int_{V_1}^{V_2} P(V)\,dV \quad \text{by the Substitution Rule.} \end{aligned}$$

29. $W = \int_a^b F(r)\,dr = \int_a^b G\dfrac{m_1 m_2}{r^2}\,dr = Gm_1m_2\left[\dfrac{-1}{r}\right]_a^b = Gm_1m_2\left(\dfrac{1}{a} - \dfrac{1}{b}\right)$

6.5 Average Value of a Function

1. $f_{\text{ave}} = \dfrac{1}{b-a}\int_a^b f(x)\,dx = \dfrac{1}{1-(-1)}\int_{-1}^1 x^2\,dx = \frac{1}{2}\cdot 2\int_0^1 x^2\,dx = \left[\frac{1}{3}x^3\right]_0^1 = \frac{1}{3}$

3. $g_{\text{ave}} = \frac{1}{\frac{\pi}{2}-0}\int_0^{\pi/2} \cos x\,dx = \frac{2}{\pi}[\sin x]_0^{\pi/2} = \frac{2}{\pi}(1-0) = \frac{2}{\pi}$

5. $f_{\text{ave}} = \frac{1}{5-0}\int_0^5 te^{-t^2}\,dt = \frac{1}{5}\int_0^{-25} e^u\left(-\frac{1}{2}\,du\right) \quad [u = -t^2,\ du = -2t\,dt,\ t\,dt = -\frac{1}{2}\,du]$

$= -\frac{1}{10}\left[e^u\right]_0^{-25} = -\frac{1}{10}\left(e^{-25} - 1\right) = \frac{1}{10}\left(1 - e^{-25}\right)$

7. $h_{\text{ave}} = \frac{1}{\pi - 0}\int_0^\pi \cos^4 x\,\sin x\,dx = \frac{1}{\pi}\int_1^{-1} u^4(-du) \quad [u = \cos x,\ du = -\sin x\,dx]$

$= \frac{1}{\pi}\int_{-1}^1 u^4\,du = \frac{1}{\pi}\cdot 2\int_0^1 u^4 du = \frac{2}{\pi}\left[\frac{1}{5}u^5\right]_0^1 = \frac{2}{5\pi}$

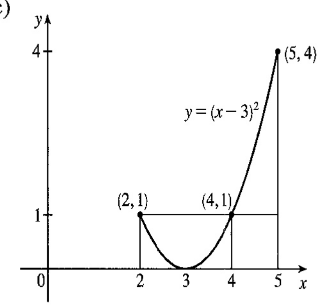
9. (a) $f_{\text{ave}} = \dfrac{1}{5-2}\int_2^5 (x-3)^2\,dx = \frac{1}{3}\left[\frac{1}{3}(x-3)^3\right]_2^5$

$= \frac{1}{9}\left[2^3 - (-1)^3\right] = \frac{1}{9}(8+1) = 1$

(b) $f(c) = f_{\text{ave}} \Leftrightarrow (c-3)^2 = 1 \Leftrightarrow c - 3 = \pm 1$

$\Leftrightarrow c = 2$ or 4

(c)



11. (a) $f_{\text{ave}} = \dfrac{1}{\pi - 0}\int_0^\pi (2\sin x - \sin 2x)\,dx$

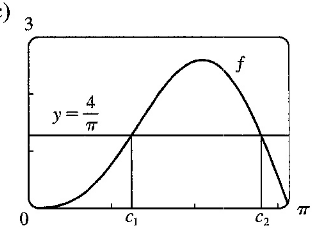
$= \frac{1}{\pi}\left[-2\cos x + \frac{1}{2}\cos 2x\right]_0^\pi$

$= \frac{1}{\pi}\left[\left(2 + \frac{1}{2}\right) - \left(-2 + \frac{1}{2}\right)\right] = \frac{4}{\pi}$

(b) $f(c) = f_{\text{ave}} \Leftrightarrow 2\sin c - \sin 2c = \frac{4}{\pi} \Leftrightarrow$

$c_1 \approx 1.238$ or $c_2 \approx 2.808$

(c)



13. f is continuous on $[1,3]$, so by the Mean Value Theorem for Integrals there exists a number c in $[1,3]$ such that $\int_1^3 f(x)\,dx = f(c)(3-1) \quad\Rightarrow\quad 8 = 2f(c)$; that is, there is a number c such that $f(c) = \frac{8}{2} = 4$.

15. $f_{\text{ave}} = \dfrac{1}{50-20}\displaystyle\int_{20}^{50} f(x)\,dx \approx \frac{1}{30}M_3 = \frac{1}{30}\cdot\frac{50-20}{3}[f(25)+f(35)+f(45)]$

$= \frac{1}{3}(38+29+48) = \frac{115}{3} = 38\frac{1}{3}$

17. Let $t=0$ and $t=12$ correspond to 9 A.M. and 9 P.M., respectively.

$T_{\text{ave}} = \frac{1}{12-0}\int_0^{12}\left[50+14\sin\frac{1}{12}\pi t\right]dt = \frac{1}{12}\left[50t - 14\cdot\frac{12}{\pi}\cos\frac{1}{12}\pi t\right]_0^{12}$

$= \frac{1}{12}\left[50\cdot 12 + 14\cdot\frac{12}{\pi} + 14\cdot\frac{12}{\pi}\right] = \left(50+\frac{28}{\pi}\right)°\text{F} \approx 59°\text{F}$

19. $\rho_{\text{ave}} = \dfrac{1}{8}\displaystyle\int_0^8 \frac{12}{\sqrt{x+1}}\,dx = \frac{3}{2}\int_0^8 (x+1)^{-1/2}\,dx = \left[3\sqrt{x+1}\right]_0^8 = 9-3 = 6\text{ kg/m}$

21. $V_{\text{ave}} = \frac{1}{5}\int_0^5 V(t)\,dt = \frac{1}{5}\int_0^5 \frac{5}{4\pi}\left[1-\cos\left(\frac{2}{5}\pi t\right)\right]dt = \frac{1}{4\pi}\int_0^5\left[1-\cos\left(\frac{2}{5}\pi t\right)\right]dt$

$= \frac{1}{4\pi}\left[t - \frac{5}{2\pi}\sin\left(\frac{2}{5}\pi t\right)\right]_0^5 = \frac{1}{4\pi}[(5-0)-0] = \frac{5}{4\pi} \approx 0.4\text{ L}$

23. Let $F(x) = \int_a^x f(t)\,dt$ for x in $[a,b]$. Then F is continuous on $[a,b]$ and differentiable on (a,b), so by the Mean Value Theorem there is a number c in (a,b) such that $F(b)-F(a) = F'(c)(b-a)$. But $F'(x) = f(x)$ by the Fundamental Theorem of Calculus. Therefore, $\int_a^b f(t)\,dt - 0 = f(c)(b-a)$.

6 Review

CONCEPT CHECK

1. (a) See Section 6.1, Figure 2 and Equations 6.1.1 and 6.1.2.

(b) Instead of using "top minus bottom" and integrating from left to right, we use "right minus left" and integrate from bottom to top. See Figures 11 and 12 in Section 6.1.

2. The numerical value of the area represents the number of meters by which Sue is ahead of Kathy after 1 minute.

3. (a) See the discussion in Section 6.2, near Figures 2 and 3, ending in the Definition of Volume.

(b) See the discussion between Examples 5 and 6 in Section 6.2. If the cross-section is a disk, find the radius in terms of x or y and use $A = \pi(\text{radius})^2$. If the cross-section is a washer, find the inner radius r_{in} and outer radius r_{out} and use $A = \pi\left(r_{\text{out}}^2\right) - \pi\left(r_{\text{in}}^2\right)$.

4. (a) $V = 2\pi r h\,\Delta r = (\text{circumference})\,(\text{height})\,(\text{thickness})$

(b) For a typical shell, find the circumference and height in terms of x or y and calculate $V = \int_a^b (\text{circumference})\,(\text{height})\,(dx \text{ or } dy)$, where a and b are the limits on x or y.

(c) Sometimes slicing produces washers or disks whose radii are difficult (or impossible) to find explicitly. On other occasions, the cylindrical shell method leads to an easier integral than slicing does.

5. $\int_0^6 f(x)\,dx$ represents the amount of work done. Its units are newton-meters, or joules.

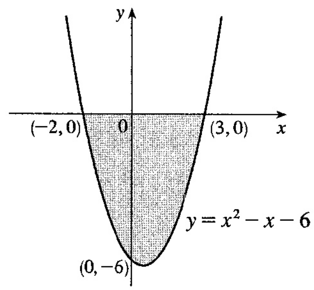
6. (a) The average value of a function f on an interval $[a,b]$ is $f_{\text{ave}} = \dfrac{1}{b-a}\displaystyle\int_a^b f(x)\,dx$.

(b) The Mean Value Theorem for Integrals says that there is a number c at which the value of f is exactly equal to the average value of the function, that is, $f(c) = f_{\text{ave}}$. For a geometric interpretation of the Mean Value Theorem for Integrals, see Figure 2 in Section 6.5 and the discussion that accompanies it.

EXERCISES

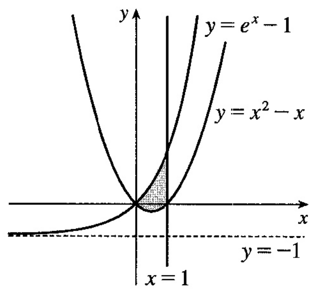
1. $0 = x^2 - x - 6 = (x-3)(x+2) \quad \Leftrightarrow \quad x = 3$ or -2. So

$$A = \int_{-2}^{3} \left[0 - \left(x^2 - x - 6\right)\right] dx = \int_{-2}^{3} \left(-x^2 + x + 6\right) dx$$
$$= \left[-\tfrac{1}{3}x^3 + \tfrac{1}{2}x^2 + 6x\right]_{-2}^{3}$$
$$= \left(-9 + \tfrac{9}{2} + 18\right) - \left(\tfrac{8}{3} + 2 - 12\right)$$
$$= \tfrac{125}{6}$$



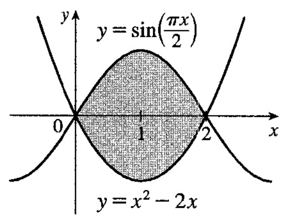
3. $A = \int_0^1 \left[(e^x - 1) - (x^2 - x)\right] dx$

$$= \int_0^1 (e^x - 1 - x^2 + x)\, dx = \left[e^x - x - \tfrac{1}{3}x^3 + \tfrac{1}{2}x^2\right]_0^1$$
$$= \left(e - 1 - \tfrac{1}{3} + \tfrac{1}{2}\right) - (1 - 0 - 0 + 0) = e - \tfrac{11}{6}$$



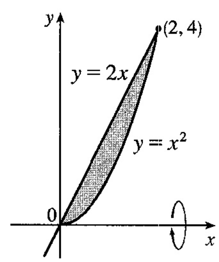
5. $A = \displaystyle\int_0^2 \left[\sin\left(\tfrac{\pi x}{2}\right) - \left(x^2 - 2x\right)\right] dx$

$$= \left[-\tfrac{2}{\pi}\cos\left(\tfrac{\pi x}{2}\right) - \tfrac{1}{3}x^3 + x^2\right]_0^2$$
$$= \left(\tfrac{2}{\pi} - \tfrac{8}{3} + 4\right) - \left(-\tfrac{2}{\pi} - 0 + 0\right) = \tfrac{4}{3} + \tfrac{4}{\pi}$$



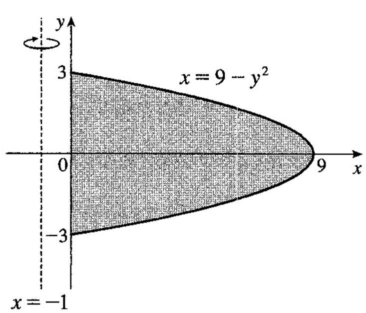
7. Using washers with inner radius x^2 and outer radius $2x$, we have

$$V = \pi \int_0^2 \left[(2x)^2 - \left(x^2\right)^2\right] dx = \pi \int_0^2 \left(4x^2 - x^4\right) dx$$
$$= \pi\left[\tfrac{4}{3}x^3 - \tfrac{1}{5}x^5\right]_0^2 = \pi\left(\tfrac{32}{3} - \tfrac{32}{5}\right) = 32\pi \cdot \tfrac{2}{15}$$
$$= \tfrac{64\pi}{15}$$

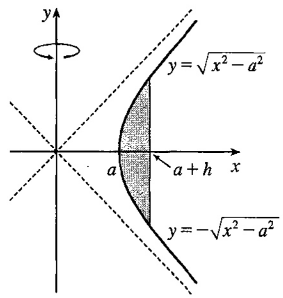


9. $V = \pi \displaystyle\int_{-3}^{3} \left\{\left[(9 - y^2) - (-1)\right]^2 - [0 - (-1)]^2\right\} dy$

$$= 2\pi \int_0^3 \left[(10 - y^2)^2 - 1\right] dy$$
$$= 2\pi \int_0^3 (100 - 20y^2 + y^4 - 1)\, dy$$
$$= 2\pi \int_0^3 (99 - 20y^2 + y^4)\, dy = 2\pi\left[99y - \tfrac{20}{3}y^3 + \tfrac{1}{5}y^5\right]_0^3$$
$$= 2\pi\left(297 - 180 + \tfrac{243}{5}\right) = \tfrac{1656\pi}{5}$$



11. The graph of $x^2 - y^2 = a^2$ is a hyperbola with right and left branches. Solving for y gives us $y^2 = x^2 - a^2 \Rightarrow y = \pm\sqrt{x^2 - a^2}$. We'll use shells and the height of each shell is $\sqrt{x^2 - a^2} - \left(-\sqrt{x^2 - a^2}\right) = 2\sqrt{x^2 - a^2}$.



The volume is $V = \int_a^{a+h} 2\pi x \cdot 2\sqrt{x^2 - a^2}\,dx$. To evaluate, let $u = x^2 - a^2$, so $du = 2x\,dx$ and $x\,dx = \frac{1}{2}\,du$. When $x = a$, $u = 0$, and when $x = a + h$, $u = (a+h)^2 - a^2 = a^2 + 2ah + h^2 - a^2 = 2ah + h^2$.

Thus, $V = 4\pi \int_0^{2ah+h^2} \sqrt{u}\left(\frac{1}{2}\,du\right) = 2\pi\left[\frac{2}{3}u^{3/2}\right]_0^{2ah+h^2} = \frac{4}{3}\pi\left(2ah + h^2\right)^{3/2}$.

13. $V = \int_0^1 \pi\left[\left(1 - x^3\right)^2 - \left(1 - x^2\right)^2\right] dx$

15. (a) A cross-section is a washer with inner radius x^2 and outer radius x.

$V = \int_0^1 \pi\left[(x)^2 - \left(x^2\right)^2\right] dx = \int_0^1 \pi\left(x^2 - x^4\right) dx = \pi\left[\frac{1}{3}x^3 - \frac{1}{5}x^5\right]_0^1 = \pi\left[\frac{1}{3} - \frac{1}{5}\right] = \frac{2\pi}{15}$

(b) A cross-section is a washer with inner radius y and outer radius $\sqrt{y}$.

$V = \int_0^1 \pi\left[\left(\sqrt{y}\right)^2 - y^2\right] dy = \int_0^1 \pi\left(y - y^2\right) dy = \pi\left[\frac{1}{2}y^2 - \frac{1}{3}y^3\right]_0^1 = \pi\left[\frac{1}{2} - \frac{1}{3}\right] = \frac{\pi}{6}$

(c) A cross-section is a washer with inner radius $2 - x$ and outer radius $2 - x^2$.

$V = \int_0^1 \pi\left[\left(2 - x^2\right)^2 - (2 - x)^2\right] dx = \int_0^1 \pi\left(x^4 - 5x^2 + 4x\right) dx = \pi\left[\frac{1}{5}x^5 - \frac{5}{3}x^3 + 2x^2\right]_0^1$

$= \pi\left[\frac{1}{5} - \frac{5}{3} + 2\right] = \frac{8\pi}{15}$

17. (a) Using the Midpoint Rule on $[0, 1]$ with $f(x) = \tan\left(x^2\right)$ and $n = 4$, we estimate

$$A = \int_0^1 \tan\left(x^2\right) dx \approx \frac{1}{4}\left[\tan\left(\left(\frac{1}{8}\right)^2\right) + \tan\left(\left(\frac{3}{8}\right)^2\right) + \tan\left(\left(\frac{5}{8}\right)^2\right) + \tan\left(\left(\frac{7}{8}\right)^2\right)\right] \approx \frac{1}{4}(1.53) \approx 0.38$$

(b) Using the Midpoint Rule on $[0, 1]$ with $f(x) = \pi\tan^2\left(x^2\right)$ (for disks) and $n = 4$, we estimate

$$V = \int_0^1 f(x)\,dx \approx \frac{1}{4}\pi\left[\tan^2\left(\left(\frac{1}{8}\right)^2\right) + \tan^2\left(\left(\frac{3}{8}\right)^2\right) + \tan^2\left(\left(\frac{5}{8}\right)^2\right) + \tan^2\left(\left(\frac{7}{8}\right)^2\right)\right] \approx \frac{\pi}{4}(1.114) \approx 0.87$$

19. The solid is obtained by rotating the region $\mathcal{R} = \left\{(x, y) \mid 0 \le x \le \frac{\pi}{2}, 0 \le y \le \cos x\right\}$ about the y-axis.

21. The solid is obtained by rotating the region $\mathcal{R} = \left\{(x, y) \mid 0 \le y \le 2, 0 \le x \le 4 - y^2\right\}$ about the x-axis.

23. Take the base to be the disk $x^2 + y^2 \le 9$. Then $V = \int_{-3}^3 A(x)\,dx$, where $A(x_0)$ is the area of the isosceles right triangle whose hypotenuse lies along the line $x = x_0$ in the xy-plane. The length of the hypotenuse is $2\sqrt{9 - x^2}$ and the length of each leg is $\sqrt{2}\sqrt{9 - x^2}$. $A(x) = \frac{1}{2}\left(\sqrt{2}\sqrt{9 - x^2}\right)^2 = 9 - x^2$, so

$V = 2\int_0^3 A(x)\,dx = 2\int_0^3 \left(9 - x^2\right) dx = 2\left[9x - \frac{1}{3}x^3\right]_0^3 = 2(27 - 9) = 36.$

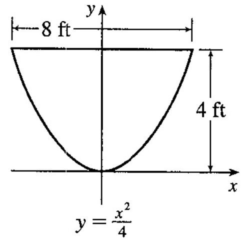
25. Equilateral triangles with sides measuring $\frac{1}{4}x$ meters have height $\frac{1}{4}x\sin 60^\circ = \frac{\sqrt{3}}{8}x$. Therefore,

$A(x) = \frac{1}{2}\cdot\frac{1}{4}x\cdot\frac{\sqrt{3}}{8}x = \frac{\sqrt{3}}{64}x^2$. $V = \int_0^{20} A(x)\,dx = \frac{\sqrt{3}}{64}\int_0^{20} x^2\,dx = \frac{\sqrt{3}}{64}\left[\frac{1}{3}x^3\right]_0^{20} = \frac{8000\sqrt{3}}{64\cdot 3} = \frac{125\sqrt{3}}{3}\text{ m}^3$.

27. $f(x) = kx \Rightarrow 30\text{ N} = k(15-12)\text{ cm} \Rightarrow k = 10\text{ N/cm} = 1000\text{ N/m}$. $20\text{ cm} - 12\text{ cm} = 0.08\text{ m} \Rightarrow$

$W = \int_0^{0.08} kx\,dx = 1000\int_0^{0.08} x\,dx = 500\left[x^2\right]_0^{0.08} = 500(0.08)^2 = 3.2\text{ N-m} = 3.2\text{ J}$.

29. (a) The parabola has equation $y = ax^2$ with vertex at the origin and passing through $(4,4)$. $4 = a\cdot 4^2 \Rightarrow a = \frac{1}{4} \Rightarrow y = \frac{1}{4}x^2 \Rightarrow x^2 = 4y$ $\Rightarrow x = 2\sqrt{y}$. Each circular disk has radius $2\sqrt{y}$ and is moved $4 - y$ ft.

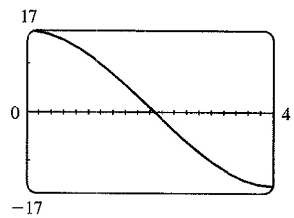


$$W = \int_0^4 \pi\left(2\sqrt{y}\right)^2 62.5\,(4-y)\,dy = 250\pi\int_0^4 y(4-y)\,dy$$

$$= 250\pi\left[2y^2 - \frac{1}{3}y^3\right]_0^4 = 250\pi\left(32 - \frac{64}{3}\right) = \frac{8000\pi}{3} \approx 8378\text{ ft-lb}$$

(b) In part (a) we knew the final water level (0) but not the amount of work done. Here we use the same equation, except with the work fixed, and the lower limit of integration (that is, the final water level — call it h) unknown: $W = 4000 \Leftrightarrow 250\pi\left[2y^2 - \frac{1}{3}y^3\right]_h^4 = 4000 \Leftrightarrow$

$\frac{16}{\pi} = \left[\left(32 - \frac{64}{3}\right) - \left(2h^2 - \frac{1}{3}h^3\right)\right] \Leftrightarrow h^3 - 6h^2 + 32 - \frac{48}{\pi} = 0$.



We graph the function $f(h) = h^3 - 6h^2 + 32 - \frac{48}{\pi}$ on the interval $[0,4]$ to see where it is 0. From the graph, $f(h) = 0$ for $h \approx 2.1$. So the depth of water remaining is about 2.1 ft.

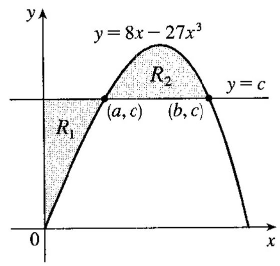
31. $\lim_{h\to 0} f_{\text{ave}} = \lim_{h\to 0} \frac{1}{(x+h)-x}\int_x^{x+h} f(t)\,dt = \lim_{h\to 0}\frac{F(x+h)-F(x)}{h}$, where $F(x) = \int_a^x f(t)\,dt$. But we recognize this limit as being $F'(x)$ by the definition of a derivative. Therefore, $\lim_{h\to 0} f_{\text{ave}} = F'(x) = f(x)$ by FTC1.

☐ PROBLEMS PLUS

1. (a) The area under the graph of f from 0 to t is equal to $\int_0^t f(x)\,dx$, so the requirement is that $\int_0^t f(x)\,dx = t^3$ for all t. We differentiate both sides of this equation with respect to t (with the help of FTC1) to get $f(t) = 3t^2$. This function is positive and continuous, as required.

(b) The volume generated from $x = 0$ to $x = b$ is $\int_0^b \pi[f(x)]^2\,dx$. Hence, we are given that $b^2 = \int_0^b \pi[f(x)]^2\,dx$ for all $b > 0$. Differentiating both sides of this equation with respect to b using the Fundamental Theorem of Calculus gives $2b = \pi[f(b)]^2 \Rightarrow f(b) = \sqrt{2b/\pi}$, since f is positive. Therefore, $f(x) = \sqrt{2x/\pi}$.

3. Let a and b be the x-coordinates of the points where the line intersects the curve. From the figure, $R_1 = R_2 \Rightarrow$



$$\int_0^a \left[c - \left(8x - 27x^3\right)\right] dx = \int_a^b \left[\left(8x - 27x^3\right) - c\right] dx$$

$$\left[cx - 4x^2 + \tfrac{27}{4}x^4\right]_0^a = \left[4x^2 - \tfrac{27}{4}x^4 - cx\right]_a^b$$

$$ac - 4a^2 + \tfrac{27}{4}a^4 = \left(4b^2 - \tfrac{27}{4}b^4 - bc\right) - \left(4a^2 - \tfrac{27}{4}a^4 - ac\right)$$

$$0 = 4b^2 - \tfrac{27}{4}b^4 - bc = 4b^2 - \tfrac{27}{4}b^4 - b\left(8b - 27b^3\right)$$

$$= 4b^2 - \tfrac{27}{4}b^4 - 8b^2 + 27b^4 = \tfrac{81}{4}b^4 - 4b^2$$

$$= b^2\left(\tfrac{81}{4}b^2 - 4\right)$$

So for $b > 0$, $b^2 = \frac{16}{81} \Rightarrow b = \frac{4}{9}$. Thus, $c = 8b - 27b^3 = 8\left(\frac{4}{9}\right) - 27\left(\frac{64}{729}\right) = \frac{32}{9} - \frac{64}{27} = \frac{32}{27}$.

5. (a) $V = \pi h^2(r - h/3) = \frac{1}{3}\pi h^2(3r - h)$. See the solution to Exercise 6.2.49.

(b) The smaller segment has height $h = 1 - x$ and so by part (a) its volume is $V = \frac{1}{3}\pi(1 - x)^2[3(1) - (1 - x)] = \frac{1}{3}\pi(x - 1)^2(x + 2)$. This volume must be $\frac{1}{3}$ of the total volume of the sphere, which is $\frac{4}{3}\pi(1)^3$. So $\frac{1}{3}\pi(x - 1)^2(x + 2) = \frac{1}{3}\left(\frac{4}{3}\pi\right) \Rightarrow \left(x^2 - 2x + 1\right)(x + 2) = \frac{4}{3} \Rightarrow$ $x^3 - 3x + 2 = \frac{4}{3} \Rightarrow 3x^3 - 9x + 2 = 0$. Using Newton's method with $f(x) = 3x^3 - 9x + 2$, $f'(x) = 9x^2 - 9$, we get $x_{n+1} = x_n - \dfrac{3x_n^3 - 9x_n + 2}{9x_n^2 - 9}$. Taking $x_1 = 0$, we get $x_2 \approx 0.2222$, and $x_3 \approx 0.2261 \approx x_4$, so, correct to four decimal places, $x \approx 0.2261$.

(c) With $r = 0.5$ and $s = 0.75$, the equation $x^3 - 3rx^2 + 4r^3s = 0$ becomes $x^3 - 3(0.5)x^2 + 4(0.5)^3(0.75) = 0$ $\Rightarrow x^3 - \frac{3}{2}x^2 + 4\left(\frac{1}{8}\right)\frac{3}{4} = 0 \Rightarrow 8x^3 - 12x^2 + 3 = 0$. We use Newton's method with

$f(x) = 8x^3 - 12x^2 + 3$, $f'(x) = 24x^2 - 24x$, so $x_{n+1} = x_n - \dfrac{8x_n^3 - 12x_n^2 + 3}{24x_n^2 - 24x_n}$. Take $x_1 = 0.5$. Then $x_2 \approx 0.6667$, and $x_3 \approx 0.6736 \approx x_4$. So to four decimal places the depth is 0.6736 m.

(d) (i) From part (a) with $r = 5$ in., the volume of water in the bowl is

$V = \frac{1}{3}\pi h^2(3r - h) = \frac{1}{3}\pi h^2(15 - h) = 5\pi h^2 - \frac{1}{3}\pi h^3$. We are given that $\dfrac{dV}{dt} = 0.2\ \text{m}^3/\text{s}$ and we want to find $\dfrac{dh}{dt}$ when $h = 3$. Now $\dfrac{dV}{dt} = 10\pi h\dfrac{dh}{dt} - \pi h^2\dfrac{dh}{dt}$, so $\dfrac{dh}{dt} = \dfrac{0.2}{\pi(10h - h^2)}$. When $h = 3$, we have

$\dfrac{dh}{dt} = \dfrac{0.2}{\pi(10 \cdot 3 - 3^2)} = \dfrac{1}{105\pi} \approx 0.003$ in/s.

(ii) From part (a), the volume of water required to fill the bowl from the instant that the water is 4 in. deep is

$V = \frac{1}{2} \cdot \frac{4}{3}\pi(5)^3 - \frac{1}{3}\pi(4)^2(15 - 4) = \frac{2}{3} \cdot 125\pi - \frac{16}{3} \cdot 11\pi = \frac{74}{3}\pi$. To find the time required to fill the bowl we divide this volume by the rate: Time $= \frac{74\pi/3}{0.2} = \frac{370\pi}{3} \approx 387\ \text{s} \approx 6.5$ min

7. We are given that the rate of change of the volume of water is $\dfrac{dV}{dt} = -kA(x)$, where k is some positive constant and $A(x)$ is the area of the surface when the water has depth x. Now we are concerned with the rate of change of the depth of the water with respect to time, that is, $\dfrac{dx}{dt}$. But by the Chain Rule, $\dfrac{dV}{dt} = \dfrac{dV}{dx}\dfrac{dx}{dt}$, so the first equation can be written $\dfrac{dV}{dx}\dfrac{dx}{dt} = -kA(x)$ $(\star)$. Also, we know that the total volume of water up to a depth x is $V(x) = \int_0^x A(s)\,ds$, where $A(s)$ is the area of a cross-section of the water at a depth s. Differentiating this equation with respect to x, we get $dV/dx = A(x)$. Substituting this into equation $\star$, we get $A(x)(dx/dt) = -kA(x) \;\Rightarrow\; dx/dt = -k$, a constant.

9. We must find expressions for the areas A and B, and then set them equal and see what this says about the curve C. If $P = (a, 2a^2)$, then area A is just $\int_0^a (2x^2 - x^2)\,dx = \int_0^a x^2\,dx = \frac{1}{3}a^3$. To find area B, we use y as the variable of integration. So we find the equation of the middle curve as a function of y: $y = 2x^2 \;\Leftrightarrow\; x = \sqrt{y/2}$, since we are concerned with the first quadrant only. We can express area B as

$\int_0^{2a^2} \left[\sqrt{y/2} - C(y)\right] dy = \left[\frac{4}{3}(y/2)^{3/2}\right]_0^{2a^2} - \int_0^{2a^2} C(y)\,dy = \frac{4}{3}a^3 - \int_0^{2a^2} C(y)\,dy$, where $C(y)$ is the function with graph C. Setting $A = B$, we get $\frac{1}{3}a^3 = \frac{4}{3}a^3 - \int_0^{2a^2} C(y)\,dy \;\Leftrightarrow\; \int_0^{2a^2} C(y)\,dy = a^3$. Now we differentiate this equation with respect to a using the Chain Rule and the Fundamental Theorem:

$C(2a^2)(4a) = 3a^2 \;\Rightarrow\; C(y) = \frac{3}{4}\sqrt{y/2}$, where $y = 2a^2$. Now we can solve for y: $x = \frac{3}{4}\sqrt{y/2} \;\Rightarrow$ $x^2 = \frac{9}{16}(y/2) \;\Rightarrow\; y = \frac{32}{9}x^2$.

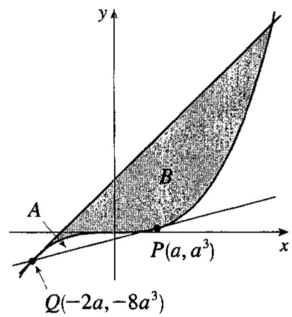
11. (a) Stacking disks along the y-axis gives us $V = \int_0^h \pi\,[f(y)]^2\,dy$.

(b) Using the Chain Rule, $\dfrac{dV}{dt} = \dfrac{dV}{dh}\cdot\dfrac{dh}{dt} = \pi\,[f(h)]^2\,\dfrac{dh}{dt}$.

(c) $kA\sqrt{h} = \pi[f(h)]^2\,\dfrac{dh}{dt}$. Set $\dfrac{dh}{dt} = C$: $\pi[f(h)]^2\,C = kA\sqrt{h} \;\Rightarrow\; [f(h)]^2 = \dfrac{kA}{\pi C}\sqrt{h} \;\Rightarrow$

$f(h) = \sqrt{\dfrac{kA}{\pi C}}\,h^{1/4}$; that is, $f(y) = \sqrt{\dfrac{kA}{\pi C}}\,y^{1/4}$. The advantage of having $\dfrac{dh}{dt} = C$ is that the markings on the container are equally spaced.

13. We assume that P lies in the region of positive x. Since $y = x^3$ is an odd function, this assumption will not affect the result of the calculation. Let $P = (a, a^3)$. The slope of the tangent to the curve $y = x^3$ at P is $3a^2$, and so the equation of the tangent is $y - a^3 = 3a^2(x - a) \;\Leftrightarrow\; y = 3a^2x - 2a^3$. We solve this simultaneously with $y = x^3$ to find the other point of intersection: $x^3 = 3a^2x - 2a^3 \;\Leftrightarrow\; (x - a)^2(x + 2a) = 0$. So $Q = (-2a, -8a^3)$ is the other point of intersection. The equation of the tangent at Q is $y - (-8a^3) = 12a^2[x - (-2a)] \;\Leftrightarrow\; y = 12a^2x + 16a^3$. By symmetry,



this tangent will intersect the curve again at $x = -2(-2a) = 4a$. The curve lies above the first tangent, and below the second, so we are looking for a relationship between $A = \int_{-2a}^{a}\left[x^3 - (3a^2x - 2a^3)\right]dx$ and $B = \int_{-2a}^{4a}\left[(12a^2x + 16a^3) - x^3\right]dx$. We calculate $A = \left[\frac{1}{4}x^4 - \frac{3}{2}a^2x^2 + 2a^3x\right]_{-2a}^{a} = \frac{3}{4}a^4 - (-6a^4) = \frac{27}{4}a^4$, and $B = \left[6a^2x^2 + 16a^3x - \frac{1}{4}x^4\right]_{-2a}^{4a} = 96a^4 - (-12a^4) = 108a^4$. We see that $B = 16A = 2^4A$. This is because our calculation of area B was essentially the same as that of area A, with a replaced by $-2a$, so if we replace a with $-2a$ in our expression for A, we get $\frac{27}{4}(-2a)^4 = 108a^4 = B$.