7 ☐ TECHNIQUES OF INTEGRATION

7.1 Integration by Parts

1. Let $u = \ln x$, $dv = x\,dx \Rightarrow du = dx/x$, $v = \frac{1}{2}x^2$. Then by Equation 2, $\int u\,dv = uv - \int v\,du$,

$$\begin{aligned}\int x \ln x\,dx &= \tfrac{1}{2}x^2 \ln x - \int \tfrac{1}{2}x^2(dx/x) = \tfrac{1}{2}x^2 \ln x - \tfrac{1}{2}\int x\,dx = \tfrac{1}{2}x^2 \ln x - \tfrac{1}{2} \cdot \tfrac{1}{2}x^2 + C \\ &= \tfrac{1}{2}x^2 \ln x - \tfrac{1}{4}x^2 + C\end{aligned}$$

3. Let $u = x$, $dv = \cos 5x\,dx \Rightarrow du = dx$, $v = \frac{1}{5}\sin 5x$. Then by Equation 2,
$\int x \cos 5x\,dx = \frac{1}{5}x \sin 5x - \int \frac{1}{5}\sin 5x\,dx = \frac{1}{5}x\sin 5x + \frac{1}{25}\cos 5x + C$.

5. Let $u = r$, $dv = e^{r/2}\,dr \Rightarrow du = dr$, $v = 2e^{r/2}$. Then
$\int re^{r/2}\,dr = 2re^{r/2} - \int 2e^{r/2}\,dr = 2re^{r/2} - 4e^{r/2} + C$.

7. Let $u = x^2$, $dv = \sin \pi x\,dx \Rightarrow du = 2x\,dx$ and $v = -\frac{1}{\pi}\cos \pi x$. Then
$I = \int x^2 \sin \pi x\,dx = -\frac{1}{\pi}x^2 \cos \pi x + \frac{2}{\pi}\int x \cos \pi x\,dx$ $(*)$.
Next let $U = x$, $dV = \cos \pi x\,dx \Rightarrow dU = dx$, $V = \frac{1}{\pi}\sin \pi x$, so
$\int x \cos \pi x\,dx = \frac{1}{\pi}x \sin \pi x - \frac{1}{\pi}\int \sin \pi x\,dx = \frac{1}{\pi}x \sin \pi x + \frac{1}{\pi^2}\cos \pi x + C_1$. Substituting for $\int x \cos \pi x\,dx$ in $(*)$,
we get $I = -\frac{1}{\pi}x^2 \cos \pi x + \frac{2}{\pi}\left(\frac{1}{\pi}x \sin \pi x + \frac{1}{\pi^2}\cos \pi x + C_1\right) = -\frac{1}{\pi}x^2 \cos \pi x + \frac{2}{\pi^2}x \sin \pi x + \frac{2}{\pi^3}\cos \pi x + C$,
where $C = \frac{2}{\pi}C_1$.

9. Let $u = \ln(2x+1)$, $dv = dx \Rightarrow du = \dfrac{2}{2x+1}\,dx$, $v = x$. Then

$$\begin{aligned}\int \ln(2x+1)\,dx &= x\ln(2x+1) - \int \frac{2x}{2x+1}\,dx = x\ln(2x+1) - \int \frac{(2x+1)-1}{2x+1}\,dx \\ &= x\ln(2x+1) - \int \left(1 - \frac{1}{2x+1}\right)dx = x\ln(2x+1) - x + \tfrac{1}{2}\ln(2x+1) + C \\ &= \tfrac{1}{2}(2x+1)\,\ln(2x+1) - x + C\end{aligned}$$

11. Let $u = \arctan 4t$, $dv = dt \Rightarrow du = \dfrac{4}{1+(4t)^2}\,dt = \dfrac{4}{1+16t^2}\,dt$, $v = t$. Then

$$\begin{aligned}\int \arctan 4t\,dt &= t\arctan 4t - \int \frac{4t}{1+16t^2}\,dt = t\arctan 4t - \frac{1}{8}\int \frac{32t}{1+16t^2}\,dt \\ &= t\arctan 4t - \tfrac{1}{8}\ln(1+16t^2) + C\end{aligned}$$

13. First let $u = (\ln x)^2$, $dv = dx \Rightarrow du = 2\ln x \cdot \frac{1}{x}\,dx$, $v = x$. Then by Equation 2,
$I = \int (\ln x)^2\,dx = x(\ln x)^2 - 2\int x \ln x \cdot \frac{1}{x}\,dx = x(\ln x)^2 - 2\int \ln x\,dx$. Next let $U = \ln x$, $dV = dx \Rightarrow$
$dU = 1/x\,dx$, $V = x$ to get $\int \ln x\,dx = x\ln x - \int x \cdot (1/x)\,dx = x\ln x - \int dx = x\ln x - x + C_1$. Thus,
$I = x(\ln x)^2 - 2(x\ln x - x + C_1) = x(\ln x)^2 - 2x\ln x + 2x + C$, where $C = -2C_1$.

15. First let $u = \sin 3\theta$, $dv = e^{2\theta}\,d\theta \Rightarrow du = 3\cos 3\theta\,d\theta$, $v = \frac{1}{2}e^{2\theta}$. Then
$I = \int e^{2\theta}\sin 3\theta\,d\theta = \frac{1}{2}e^{2\theta}\sin 3\theta - \frac{3}{2}\int e^{2\theta}\cos 3\theta\,d\theta$. Next let $U = \cos 3\theta$,
$dV = e^{2\theta}\,d\theta \Rightarrow dU = -3\sin 3\theta\,d\theta$, $V = \frac{1}{2}e^{2\theta}$ to get
$\int e^{2\theta}\cos 3\theta\,d\theta = \frac{1}{2}e^{2\theta}\cos 3\theta + \frac{3}{2}\int e^{2\theta}\sin 3\theta\,d\theta$. Substituting in the previous formula gives

$I = \frac{1}{2}e^{2\theta}\sin 3\theta - \frac{3}{4}e^{2\theta}\cos 3\theta - \frac{9}{4}\int e^{2\theta}\sin 3\theta\, d\theta = \frac{1}{2}e^{2\theta}\sin 3\theta - \frac{3}{4}e^{2\theta}\cos 3\theta - \frac{9}{4}I \quad \Rightarrow$
$\frac{13}{4}I = \frac{1}{2}e^{2\theta}\sin 3\theta - \frac{3}{4}e^{2\theta}\cos 3\theta + C_1$. Hence, $I = \frac{1}{13}e^{2\theta}(2\sin 3\theta - 3\cos 3\theta) + C$, where $C = \frac{4}{13}C_1$.

17. Let $u = y$, $dv = \sinh y\, dy \quad \Rightarrow \quad du = dy$, $v = \cosh y$. Then
$\int y\sinh y\, dy = y\cosh y - \int \cosh y\, dy = y\cosh y - \sinh y + C$.

19. Let $u = t$, $dv = \sin 3t\, dt \quad \Rightarrow \quad du = dt$, $v = -\frac{1}{3}\cos 3t$. Then
$\int_0^\pi t\sin 3t\, dt = \left[-\frac{1}{3}t\cos 3t\right]_0^\pi + \frac{1}{3}\int_0^\pi \cos 3t\, dt = \left(\frac{1}{3}\pi - 0\right) + \frac{1}{9}\left[\sin 3t\right]_0^\pi = \frac{\pi}{3}$.

21. Let $u = \ln x$, $dv = x^{-2}\, dx \quad \Rightarrow \quad du = \dfrac{1}{x}\, dx$, $v = -x^{-1}$. By (6),

$$\int_1^2 \frac{\ln x}{x^2}\, dx = \left[-\frac{\ln x}{x}\right]_1^2 + \int_1^2 x^{-2}\, dx = -\tfrac{1}{2}\ln 2 + \ln 1 + \left[-\frac{1}{x}\right]_1^2 = -\tfrac{1}{2}\ln 2 + 0 - \tfrac{1}{2} + 1 = \tfrac{1}{2} - \tfrac{1}{2}\ln 2.$$

23. Let $u = y$, $dv = \dfrac{dy}{e^{2y}} = e^{-2y}dy \quad \Rightarrow \quad du = dy$, $v = -\dfrac{1}{2}e^{-2y}$. Then

$$\int_0^1 \frac{y}{e^{2y}}\, dy = \left[-\tfrac{1}{2}ye^{-2y}\right]_0^1 + \tfrac{1}{2}\int_0^1 e^{-2y}dy = \left(-\tfrac{1}{2}e^{-2} + 0\right) - \tfrac{1}{4}\left[e^{-2y}\right]_0^1 = -\tfrac{1}{2}e^{-2} - \tfrac{1}{4}e^{-2} + \tfrac{1}{4} = \tfrac{1}{4} - \tfrac{3}{4}e^{-2}.$$

25. Let $u = \cos^{-1} x$, $dv = dx \quad \Rightarrow \quad du = -\dfrac{dx}{\sqrt{1-x^2}}$, $v = x$. Then

$$I = \int_0^{1/2} \cos^{-1} x\, dx = \left[x\cos^{-1} x\right]_0^{1/2} + \int_0^{1/2} \frac{x\, dx}{\sqrt{1-x^2}} = \tfrac{1}{2}\cdot\tfrac{\pi}{3} + \int_1^{3/4} t^{-1/2}\left[-\frac{1}{2}\, dt\right], \text{ where } t = 1 - x^2$$

$\Rightarrow \quad dt = -2x\, dx$. Thus, $I = \frac{\pi}{6} + \frac{1}{2}\int_{3/4}^1 t^{-1/2}\, dt = \frac{\pi}{6} + \left[\sqrt{t}\right]_{3/4}^1 = \frac{\pi}{6} + 1 - \frac{\sqrt{3}}{2} = \frac{1}{6}\left(\pi + 6 - 3\sqrt{3}\right)$.

27. Let $u = \ln(\sin x)$, $dv = \cos x\, dx \quad \Rightarrow \quad du = \dfrac{\cos x}{\sin x}\, dx$, $v = \sin x$. Then
$I = \int \cos x \ln(\sin x)\, dx = \sin x\ln(\sin x) - \int \cos x\, dx = \sin x\ln(\sin x) - \sin x + C$.
Another method: Substitute $t = \sin x$, so $dt = \cos x\, dx$. Then $I = \int \ln t\, dt = t\ln t - t + C$ (see Example 2) and so $I = \sin x\,(\ln\sin x - 1) + C$.

29. Let $w = \ln x \quad \Rightarrow \quad dw = dx/x$. Then $x = e^w$ and $dx = e^w\, dw$, so

$$\begin{aligned}\int \cos(\ln x)\, dx &= \int e^w\cos w\, dw = \tfrac{1}{2}e^w(\sin w + \cos w) + C \quad \text{[by the method of Example 4]}\\ &= \tfrac{1}{2}x\left[\sin(\ln x) + \cos(\ln x)\right] + C\end{aligned}$$

31. Let $u = (\ln x)^2$, $dv = x^4\, dx \quad \Rightarrow \quad du = 2\dfrac{\ln x}{x}\, dx$, $v = \dfrac{x^5}{5}$. By (6),

$$\int_1^2 x^4(\ln x)^2\, dx = \left[\frac{x^5}{5}(\ln x)^2\right]_1^2 - 2\int_1^2 \frac{x^4}{5}\ln x\, dx = \tfrac{32}{5}(\ln 2)^2 - 0 - 2\int_1^2 \frac{x^4}{5}\ln x\, dx.$$

Let $U = \ln x$, $dV = \dfrac{x^4}{5}\, dx \quad \Rightarrow \quad dU = \dfrac{1}{x}\, dx$, $V = \dfrac{x^5}{25}$.

Then $\displaystyle\int_1^2 \frac{x^4}{5}\ln x\, dx = \left[\frac{x^5}{25}\ln x\right]_1^2 - \int_1^2 \frac{x^4}{25}\, dx = \tfrac{32}{25}\ln 2 - 0 - \left[\frac{x^5}{125}\right]_1^2 = \tfrac{32}{25}\ln 2 - \left(\tfrac{32}{125} - \tfrac{1}{125}\right)$.

So $\int_1^2 x^4(\ln x)^2\, dx = \frac{32}{5}(\ln 2)^2 - 2\left(\frac{32}{25}\ln 2 - \frac{31}{125}\right) = \frac{32}{5}(\ln 2)^2 - \frac{64}{25}\ln 2 + \frac{62}{125}$.

33. Let $w = \sqrt{x}$, so that $x = w^2$ and $dx = 2w\, dw$. Thus, $\int \sin\sqrt{x}\, dx = \int 2w\sin w\, dw$. Now use parts with $u = 2w$, $dv = \sin w\, dw$, $du = 2\, dw$, $v = -\cos w$ to get

$$\begin{aligned}\int 2w\sin w\, dw &= -2w\cos w + \int 2\cos w\, dw = -2w\cos w + 2\sin w + C\\ &= -2\sqrt{x}\cos\sqrt{x} + 2\sin\sqrt{x} + C = 2\left(\sin\sqrt{x} - \sqrt{x}\cos\sqrt{x}\right) + C\end{aligned}$$

35. Let $x = \theta^2$, so that $dx = 2\theta\, d\theta$. Thus, $\int_{\sqrt{\pi/2}}^{\sqrt{\pi}} \theta^3 \cos(\theta^2)\, d\theta = \int_{\sqrt{\pi/2}}^{\sqrt{\pi}} \theta^2 \cos(\theta^2) \cdot \frac{1}{2}(2\theta\, d\theta) = \frac{1}{2}\int_{\pi/2}^{\pi} x \cos x\, dx$.
Now use parts with $u = x$, $dv = \cos x\, dx$, $du = dx$, $v = \sin x$ to get

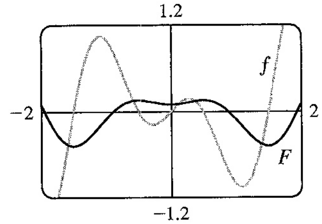
$$\frac{1}{2}\int_{\pi/2}^{\pi} x\cos x\, dx = \frac{1}{2}\left([x \sin x]_{\pi/2}^{\pi} - \int_{\pi/2}^{\pi} \sin x\, dx\right) = \frac{1}{2}\left[x\sin x + \cos x\right]_{\pi/2}^{\pi}$$
$$= \frac{1}{2}(\pi \sin\pi + \cos\pi) - \frac{1}{2}\left(\frac{\pi}{2}\sin\frac{\pi}{2} + \cos\frac{\pi}{2}\right) = \frac{1}{2}(\pi\cdot 0 - 1) - \frac{1}{2}\left(\frac{\pi}{2}\cdot 1 + 0\right) = -\frac{1}{2} - \frac{\pi}{4}$$

In Exercises 37–40, let $f(x)$ denote the integrand and $F(x)$ its antiderivative (with $C = 0$).

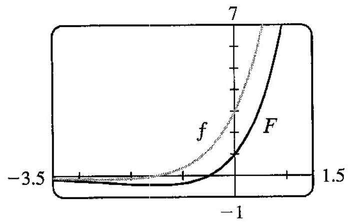
37. Let $u = x$, $dv = \cos \pi x\, dx \Rightarrow du = dx$, $v = (\sin \pi x)/\pi$. Then

$$\int x\cos\pi x\, dx = x\cdot\frac{\sin\pi x}{\pi} - \int\frac{\sin\pi x}{\pi}\,dx = \frac{x\sin\pi x}{\pi} + \frac{\cos\pi x}{\pi^2} + C.$$

We see from the graph that this is reasonable, since F has extreme values where f is 0.



39. Let $u = 2x + 3$, $dv = e^x\, dx \Rightarrow du = 2\,dx$, $v = e^x$. Then $\int(2x+3)e^x\,dx = (2x+3)e^x - 2\int e^x\,dx = (2x+3)e^x - 2e^x + C = (2x+1)\,e^x + C$. We see from the graph that this is reasonable, since F has a minimum where f changes from negative to positive.



41. (a) Take $n = 2$ in Example 6 to get $\int \sin^2 x\, dx = -\frac{1}{2}\cos x \sin x + \frac{1}{2}\int 1\, dx = \frac{x}{2} - \frac{\sin 2x}{4} + C$.

(b) $\int \sin^4 x\,dx = -\frac{1}{4}\cos x\sin^3 x + \frac{3}{4}\int \sin^2 x\,dx = -\frac{1}{4}\cos x \sin^3 x + \frac{3}{8}x - \frac{3}{16}\sin 2x + C$.

43. (a) From Example 6, $\displaystyle\int \sin^n x\,dx = -\frac{1}{n}\cos x\sin^{n-1}x + \frac{n-1}{n}\int\sin^{n-2}x\,dx$. Using (6),

$$\int_0^{\pi/2}\sin^n x\,dx = \left[-\frac{\cos x\sin^{n-1}x}{n}\right]_0^{\pi/2} + \frac{n-1}{n}\int_0^{\pi/2}\sin^{n-2}x\,dx$$
$$= (0-0) + \frac{n-1}{n}\int_0^{\pi/2}\sin^{n-2}x\,dx = \frac{n-1}{n}\int_0^{\pi/2}\sin^{n-2}x\,dx$$

(b) Using $n = 3$ in part (a), we have $\int_0^{\pi/2}\sin^3 x\,dx = \frac{2}{3}\int_0^{\pi/2}\sin x\,dx = \left[-\frac{2}{3}\cos x\right]_0^{\pi/2} = \frac{2}{3}$.
Using $n = 5$ in part (a), we have $\int_0^{\pi/2}\sin^5 x\,dx = \frac{4}{5}\int_0^{\pi/2}\sin^3 x\,dx = \frac{4}{5}\cdot\frac{2}{3} = \frac{8}{15}$.

(c) The formula holds for $n = 1$ (that is, $2n+1 = 3$) by (b). Assume it holds for some $k \geq 1$. Then $\displaystyle\int_0^{\pi/2}\sin^{2k+1}x\,dx = \frac{2\cdot4\cdot6\cdot\cdots\cdot(2k)}{3\cdot5\cdot7\cdot\cdots\cdot(2k+1)}$. By Example 6,

$$\int_0^{\pi/2}\sin^{2k+3}x\,dx = \frac{2k+2}{2k+3}\int_0^{\pi/2}\sin^{2k+1}x\,dx = \frac{2k+2}{2k+3}\cdot\frac{2\cdot4\cdot6\cdot\cdots\cdot(2k)}{3\cdot5\cdot7\cdot\cdots\cdot(2k+1)}$$
$$= \frac{2\cdot4\cdot6\cdot\cdots\cdot(2k)[2(k+1)]}{3\cdot5\cdot7\cdot\cdots\cdot(2k+1)[2(k+1)+1]},$$

so the formula holds for $n = k+1$. By induction, the formula holds for all $n \geq 1$.

45. Let $u = (\ln x)^n$, $dv = dx \Rightarrow du = n(\ln x)^{n-1}(dx/x)$, $v = x$. By Equation 2,
$\int(\ln x)^n\,dx = x(\ln x)^n - \int nx(\ln x)^{n-1}(dx/x) = x(\ln x)^n - n\int(\ln x)^{n-1}\,dx$.

47. Let $u = (x^2+a^2)^n$, $dv = dx \Rightarrow du = n(x^2+a^2)^{n-1}\,2x\,dx$, $v = x$. Then

$$\begin{aligned}\int (x^2+a^2)^n\,dx &= x(x^2+a^2)^n - 2n\int x^2(x^2+a^2)^{n-1}\,dx\\ &= x(x^2+a^2)^n - 2n\left[\int (x^2+a^2)^n\,dx - a^2\int (x^2+a^2)^{n-1}\,dx\right] \quad [\text{since } x^2 = (x^2+a^2)-a^2]\end{aligned}$$

$$\Rightarrow \quad (2n+1)\int (x^2+a^2)^n\,dx = x(x^2+a^2)^n + 2na^2\int (x^2+a^2)^{n-1}\,dx, \text{ and}$$

$$\int (x^2+a^2)^n\,dx = \frac{x(x^2+a^2)^n}{2n+1} + \frac{2na^2}{2n+1}\int (x^2+a^2)^{n-1}\,dx \quad [\text{provided } 2n+1\neq 0].$$

49. Take $n = 3$ in Exercise 45 to get

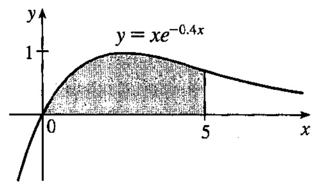
$$\int (\ln x)^3\,dx = x(\ln x)^3 - 3\int(\ln x)^2\,dx = x(\ln x)^3 - 3x(\ln x)^2 + 6x\ln x - 6x + C \quad [\text{by Exercise 13}].$$

Or: Instead of using Exercise 13, apply Exercise 45 again with $n = 2$.

51. Area $= \int_0^5 xe^{-0.4x}dx$. Let $u = x$, $dv = e^{-0.4x}dx \Rightarrow$

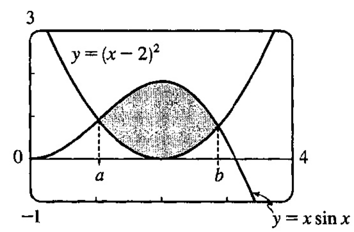
$du = dx$, $v = -2.5e^{-0.4x}$. Then

$$\begin{aligned}\text{area} &= \left[-2.5xe^{-0.4x}\right]_0^5 + 2.5\int_0^5 e^{-0.4x}dx\\ &= -12.5e^{-2} + 0 + 2.5\left[-2.5e^{-0.4x}\right]_0^5\\ &= -12.5e^{-2} - 6.25(e^{-2}-1) = 6.25 - 18.75e^{-2} \quad \text{or} \quad \tfrac{25}{4} - \tfrac{75}{4}e^{-2}\end{aligned}$$



53. The curves $y = x\sin x$ and $y = (x-2)^2$ intersect at $a \approx 1.04748$ and $b \approx 2.87307$, so

$$\begin{aligned}\text{area} &= \int_a^b \left[x\sin x - (x-2)^2\right]dx\\ &= \left[-x\cos x + \sin x - \tfrac{1}{3}(x-2)^3\right]_a^b \quad [\text{by Example 1}]\\ &\approx 2.81358 - 0.63075 = 2.18283\end{aligned}$$



55. $V = \int_0^1 2\pi x\cos(\pi x/2)dx$. Let $u = x$, $dv = \cos(\pi x/2)dx \Rightarrow du = dx$, $v = \frac{2}{\pi}\sin(\pi x/2)$.

$$\begin{aligned}V &= 2\pi\left[\frac{2}{\pi}x\sin\left(\frac{\pi x}{2}\right)\right]_0^1 - 2\pi\cdot\frac{2}{\pi}\int_0^1 \sin\left(\frac{\pi x}{2}\right)dx = 2\pi\left(\frac{2}{\pi}-0\right) - 4\left[-\frac{2}{\pi}\cos\left(\frac{\pi x}{2}\right)\right]_0^1\\ &= 4 + \tfrac{8}{\pi}(0-1) = 4 - \tfrac{8}{\pi}.\end{aligned}$$

57. Volume $= \int_{-1}^0 2\pi(1-x)e^{-x}\,dx$. Let $u = 1-x$, $dv = e^{-x}\,dx \Rightarrow du = -\,dx$, $v = -e^{-x}$.

$$\begin{aligned}V &= 2\pi\left[(1-x)(-e^{-x})\right]_{-1}^0 - 2\pi\int_{-1}^0 e^{-x}\,dx = 2\pi\left[(x-1)(e^{-x}) + e^{-x}\right]_{-1}^0\\ &= 2\pi\left[xe^{-x}\right]_{-1}^0 = 2\pi(0+e) = 2\pi e\end{aligned}$$

59. The average value of $f(x) = x^2\ln x$ on the interval $[1,3]$ is $f_{\text{ave}} = \dfrac{1}{3-1}\displaystyle\int_1^3 x^2\ln x\,dx = \tfrac{1}{2}I$.

Let $u = \ln x$, $dv = x^2\,dx \Rightarrow du = (1/x)\,dx$, $v = \frac{1}{3}x^3$. So

$$I = \left[\tfrac{1}{3}x^3\ln x\right]_1^3 - \int_1^3 \tfrac{1}{3}x^2dx = (9\ln 3 - 0) - \left[\tfrac{1}{9}x^3\right]_1^3 = 9\ln 3 - \left(3-\tfrac{1}{9}\right) = 9\ln 3 - \tfrac{26}{9}.$$

Thus, $f_{\text{ave}} = \frac{1}{2}I = \frac{1}{2}\left(9\ln 3 - \frac{26}{9}\right) = \frac{9}{2}\ln 3 - \frac{13}{9}$.

61. Since $v(t) > 0$ for all t, the desired distance is $s(t) = \int_0^t v(w)dw = \int_0^t w^2 e^{-w}\,dw$.

First let $u = w^2$, $dv = e^{-w}\,dw \Rightarrow du = 2w\,dw$, $v = -e^{-w}$. Then $s(t) = \left[-w^2e^{-w}\right]_0^t + 2\int_0^t we^{-w}\,dw$.

Next let $U = w$, $dV = e^{-w}\,dw \Rightarrow dU = dw$, $V = -e^{-w}$. Then

$$\begin{aligned} s(t) &= -t^2e^{-t} + 2\left(\left[-we^{-w}\right]_0^t + \int_0^t e^{-w}\,dw\right) = -t^2e^{-t} + 2\left(-te^{-t} + 0 + \left[-e^{-w}\right]_0^t\right) \\ &= -t^2e^{-t} + 2\left(-te^{-t} - e^{-t} + 1\right) = -t^2e^{-t} - 2te^{-t} - 2e^{-t} + 2 \\ &= 2 - e^{-t}\left(t^2 + 2t + 2\right) \text{ meters} \end{aligned}$$

63. For $I = \int_1^4 xf''(x)\,dx$, let $u = x$, $dv = f''(x)\,dx \Rightarrow du = dx$, $v = f'(x)$. Then

$I = \left[xf'(x)\right]_1^4 - \int_1^4 f'(x)\,dx = 4f'(4) - 1\cdot f'(1) - [f(4) - f(1)] = 4\cdot 3 - 1\cdot 5 - (7-2) = 12 - 5 - 5 = 2$.

We used the fact that f'' is continuous to guarantee that I exists.

65. Using the formula for volumes of rotation and the figure, we see that

Volume $= \int_0^d \pi b^2\,dy - \int_0^c \pi a^2\,dy - \int_c^d \pi[g(y)]^2\,dy = \pi b^2 d - \pi a^2 c - \int_c^d \pi[g(y)]^2\,dy$. Let $y = f(x)$, which gives $dy = f'(x)\,dx$ and $g(y) = x$, so that $V = \pi b^2 d - \pi a^2 c - \pi\int_a^b x^2 f'(x)\,dx$. Now integrate by parts with $u = x^2$, and $dv = f'(x)\,dx \Rightarrow du = 2x\,dx$, $v = f(x)$, and

$\int_a^b x^2 f'(x)\,dx = \left[x^2 f(x)\right]_a^b - \int_a^b 2x\,f(x)\,dx = b^2 f(b) - a^2 f(a) - \int_a^b 2x\,f(x)\,dx$, but $f(a) = c$ and $f(b) = d$

$\Rightarrow V = \pi b^2 d - \pi a^2 c - \pi\left[b^2 d - a^2 c - \int_a^b 2xf(x)\,dx\right] = \int_a^b 2\pi x f(x)\,dx$.

7.2 Trigonometric Integrals

The symbols $\overset{s}{=}$ and $\overset{c}{=}$ indicate the use of the substitutions $\{u = \sin x, du = \cos x\,dx\}$ and $\{u = \cos x, du = -\sin x\,dx\}$, respectively.

1. $$\begin{aligned} \int \sin^3 x\cos^2 x\,dx &= \int \sin^2 x\cos^2 x\sin x\,dx = \int (1-\cos^2 x)\cos^2 x\sin x\,dx \overset{c}{=} \int (1-u^2)u^2(-du) \\ &= \int (u^2-1)u^2\,du = \int (u^4-u^2)\,du = \tfrac{1}{5}u^5 - \tfrac{1}{3}u^3 + C = \tfrac{1}{5}\cos^5 x - \tfrac{1}{3}\cos^3 x + C \end{aligned}$$

3. $$\begin{aligned} \int_{\pi/2}^{3\pi/4} \sin^5 x\cos^3 x\,dx &= \int_{\pi/2}^{3\pi/4} \sin^5 x\cos^2 x\cos x\,dx = \int_{\pi/2}^{3\pi/4} \sin^5 x\,(1-\sin^2 x)\cos x\,dx \\ &\overset{s}{=} \int_1^{\sqrt{2}/2} u^5(1-u^2)\,du = \int_1^{\sqrt{2}/2} (u^5-u^7)\,du = \left[\tfrac{1}{6}u^6 - \tfrac{1}{8}u^8\right]_1^{\sqrt{2}/2} \\ &= \left(\tfrac{1/8}{6} - \tfrac{1/16}{8}\right) - \left(\tfrac{1}{6} - \tfrac{1}{8}\right) = -\tfrac{11}{384} \end{aligned}$$

5. $$\begin{aligned} \int \cos^5 x\sin^4 x\,dx &= \int \cos^4 x\sin^4 x\cos x\,dx = \int \left(1-\sin^2 x\right)^2 \sin^4 x\cos x\,dx \overset{s}{=} \int \left(1-u^2\right)^2 u^4\,du \\ &= \int \left(1-2u^2+u^4\right)u^4\,du = \int \left(u^4 - 2u^6 + u^8\right)du = \tfrac{1}{5}u^5 - \tfrac{2}{7}u^7 + \tfrac{1}{9}u^9 + C \\ &= \tfrac{1}{5}\sin^5 x - \tfrac{2}{7}\sin^7 x + \tfrac{1}{9}\sin^9 x + C \end{aligned}$$

7. $$\begin{aligned} \int_0^{\pi/2} \cos^2\theta\,d\theta &= \int_0^{\pi/2} \tfrac{1}{2}(1+\cos 2\theta)\,d\theta \quad \text{[half-angle identity]} \\ &= \tfrac{1}{2}\left[\theta + \tfrac{1}{2}\sin 2\theta\right]_0^{\pi/2} = \tfrac{1}{2}\left[\left(\tfrac{\pi}{2}+0\right) - (0+0)\right] = \tfrac{\pi}{4} \end{aligned}$$

9. $$\begin{aligned} \int_0^\pi \sin^4(3t)\,dt &= \int_0^\pi \left[\sin^2(3t)\right]^2 dt = \int_0^\pi \left[\tfrac{1}{2}(1-\cos 6t)\right]^2 dt = \tfrac{1}{4}\int_0^\pi (1 - 2\cos 6t + \cos^2 6t)\,dt \\ &= \tfrac{1}{4}\int_0^\pi \left[1 - 2\cos 6t + \tfrac{1}{2}(1+\cos 12t)\right] dt = \tfrac{1}{4}\int_0^\pi \left(\tfrac{3}{2} - 2\cos 6t + \tfrac{1}{2}\cos 12t\right) dt \\ &= \tfrac{1}{4}\left[\tfrac{3}{2}t - \tfrac{1}{3}\sin 6t + \tfrac{1}{24}\sin 12t\right]_0^\pi = \tfrac{1}{4}\left[\left(\tfrac{3\pi}{2} - 0 + 0\right) - (0 - 0 + 0)\right] = \tfrac{3\pi}{8} \end{aligned}$$

11. $\int (1+\cos\theta)^2\,d\theta = \int (1+2\cos\theta+\cos^2\theta)\,d\theta = \theta + 2\sin\theta + \frac{1}{2}\int (1+\cos 2\theta)\,d\theta$
$= \theta + 2\sin\theta + \frac{1}{2}\theta + \frac{1}{4}\sin 2\theta + C = \frac{3}{2}\theta + 2\sin\theta + \frac{1}{4}\sin 2\theta + C$

13. $\int_0^{\pi/4} \sin^4 x\cos^2 x\,dx = \int_0^{\pi/4}\sin^2 x\,(\sin x\cos x)^2\,dx = \int_0^{\pi/4}\frac{1}{2}(1-\cos 2x)\left(\frac{1}{2}\sin 2x\right)^2\,dx$
$= \frac{1}{8}\int_0^{\pi/4}(1-\cos 2x)\,\sin^2 2x\,dx = \frac{1}{8}\int_0^{\pi/4}\sin^2 2x\,dx - \frac{1}{8}\int_0^{\pi/4}\sin^2 2x\cos 2x\,dx$
$= \frac{1}{16}\int_0^{\pi/4}(1-\cos 4x)\,dx - \frac{1}{16}\left[\frac{1}{3}\sin^3 2x\right]_0^{\pi/4} = \frac{1}{16}\left[x - \frac{1}{4}\sin 4x - \frac{1}{3}\sin^3 2x\right]_0^{\pi/4}$
$= \frac{1}{16}\left(\frac{\pi}{4} - 0 - \frac{1}{3}\right) = \frac{1}{192}(3\pi - 4)$

15. $\int \sin^3 x\,\sqrt{\cos x}\,dx = \int \left(1-\cos^2 x\right)\sqrt{\cos x}\,\sin x\,dx \overset{c}{=} \int \left(1-u^2\right)u^{1/2}\,(-du) = \int \left(u^{5/2} - u^{1/2}\right)du$
$= \frac{2}{7}u^{7/2} - \frac{2}{3}u^{3/2} + C = \frac{2}{7}(\cos x)^{7/2} - \frac{2}{3}(\cos x)^{3/2} + C$
$= \left(\frac{2}{7}\cos^3 x - \frac{2}{3}\cos x\right)\sqrt{\cos x} + C$

17. $\displaystyle\int \cos^2 x\tan^3 x\,dx = \int \frac{\sin^3 x}{\cos x}dx \overset{c}{=} \int \frac{(1-u^2)(-du)}{u} = \int \left[\frac{-1}{u} + u\right]du$
$= -\ln|u| + \frac{1}{2}u^2 + C = \frac{1}{2}\cos^2 x - \ln|\cos x| + C$

19. $\displaystyle\int \frac{1-\sin x}{\cos x}\,dx = \int (\sec x - \tan x)\,dx = \ln|\sec x + \tan x| - \ln|\sec x| + C$ [by (1) and the boxed formula above it]
$= \ln|(\sec x + \tan x)\cos x| + C = \ln|1+\sin x| + C$
$= \ln(1+\sin x) + C$ since $1 + \sin x \ge 0$

Or: $\displaystyle\int \frac{1-\sin x}{\cos x}\,dx = \int \frac{1-\sin x}{\cos x}\cdot\frac{1+\sin x}{1+\sin x}\,dx = \int \frac{(1-\sin^2 x)\,dx}{\cos x\,(1+\sin x)} = \int \frac{\cos x\,dx}{1+\sin x}$
$\displaystyle= \int \frac{dw}{w}$ [where $w = 1+\sin x$, $dw = \cos x\,dx$]
$= \ln|w| + C = \ln|1+\sin x| + C = \ln(1+\sin x) + C$

21. Let $u = \tan x$, $du = \sec^2 x\,dx$. Then $\int \sec^2 x\tan x\,dx = \int u\,du = \frac{1}{2}u^2 + C = \frac{1}{2}\tan^2 x + C$.
Or: Let $v = \sec x$, $dv = \sec x\tan x\,dx$. Then $\int \sec^2 x\tan x\,dx = \int v\,dv = \frac{1}{2}v^2 + C = \frac{1}{2}\sec^2 x + C$.

23. $\int \tan^2 x\,dx = \int \left(\sec^2 x - 1\right)dx = \tan x - x + C$

25. $\int \sec^6 t\,dt = \int \sec^4 t\cdot\sec^2 t\,dt = \int (\tan^2 t + 1)^2\sec^2 t\,dt = \int (u^2+1)^2\,du$ [$u = \tan t$, $du = \sec^2 t\,dt$]
$= \int (u^4 + 2u^2 + 1)\,du = \frac{1}{5}u^5 + \frac{2}{3}u^3 + u + C = \frac{1}{5}\tan^5 t + \frac{2}{3}\tan^3 t + \tan t + C$

27. $\int_0^{\pi/3}\tan^5 x\,\sec^4 x\,dx = \int_0^{\pi/3}\tan^5 x\,(\tan^2 x + 1)\sec^2 x\,dx$
$= \int_0^{\sqrt{3}} u^5(u^2+1)\,du$ [$u = \tan x$, $du = \sec^2 x\,dx$]
$= \int_0^{\sqrt{3}}(u^7 + u^5)\,du = \left[\frac{1}{8}u^8 + \frac{1}{6}u^6\right]_0^{\sqrt{3}} = \frac{81}{8} + \frac{27}{6} = \frac{81}{8} + \frac{9}{2} = \frac{81}{8} + \frac{36}{8} = \frac{117}{8}$

Alternate solution:
$\int_0^{\pi/3}\tan^5 x\,\sec^4 x\,dx = \int_0^{\pi/3}\tan^4 x\sec^3 x\sec x\tan x\,dx = \int_0^{\pi/3}(\sec^2 x - 1)^2\sec^3 x\sec x\tan x\,dx$
$= \int_1^2 (u^2-1)^2u^3\,du$ [$u = \sec x$, $du = \sec x\tan x\,dx$]
$= \int_1^2 (u^4 - 2u^2 + 1)u^3\,du = \int_1^2 (u^7 - 2u^5 + u^3)\,du$
$= \left[\frac{1}{8}u^8 - \frac{1}{3}u^6 + \frac{1}{4}u^4\right]_1^2 = \left(32 - \frac{64}{3} + 4\right) - \left(\frac{1}{8} - \frac{1}{3} + \frac{1}{4}\right) = \frac{117}{8}$

29. $\int \tan^3 x \sec x \, dx = \int \tan^2 x \sec x \tan x \, dx = \int \left(\sec^2 x - 1\right) \sec x \tan x \, dx$
$= \int (u^2 - 1) \, du \qquad [u = \sec x, \, du = \sec x \tan x \, dx]$
$= \frac{1}{3}u^3 - u + C = \frac{1}{3}\sec^3 x - \sec x + C$

31. $\int \tan^5 x \, dx = \int \left(\sec^2 x - 1\right)^2 \tan x \, dx = \int \sec^4 x \tan x \, dx - 2\int \sec^2 x \tan x \, dx + \int \tan x \, dx$
$= \int \sec^3 x \sec x \tan x \, dx - 2\int \tan x \sec^2 x \, dx + \int \tan x \, dx$
$= \frac{1}{4}\sec^4 x - \tan^2 x + \ln|\sec x| + C \quad [\text{or } \frac{1}{4}\sec^4 x - \sec^2 x + \ln|\sec x| + C\,]$

33. $\displaystyle\int \frac{\tan^3 \theta}{\cos^4 \theta} \, d\theta = \int \tan^3 \theta \sec^4 \theta \, d\theta = \int \tan^3 \theta \cdot (\tan^2 \theta + 1) \cdot \sec^2 \theta \, d\theta$
$= \int u^3(u^2 + 1) \, du \qquad [u = \tan \theta, \, du = \sec^2 \theta \, d\theta]$
$= \int (u^5 + u^3) \, du = \frac{1}{6}u^6 + \frac{1}{4}u^4 + C = \frac{1}{6}\tan^6 \theta + \frac{1}{4}\tan^4 \theta + C$

35. $\int_{\pi/6}^{\pi/2} \cot^2 x \, dx = \int_{\pi/6}^{\pi/2} \left(\csc^2 x - 1\right) dx = [-\cot x - x]_{\pi/6}^{\pi/2} = \left(0 - \frac{\pi}{2}\right) - \left(-\sqrt{3} - \frac{\pi}{6}\right) = \sqrt{3} - \frac{\pi}{3}$

37. $\int \cot^3 \alpha \csc^3 \alpha \, d\alpha = \int \cot^2 \alpha \csc^2 \alpha \cdot \csc \alpha \cot \alpha \, d\alpha = \int (\csc^2 \alpha - 1) \csc^2 \alpha \cdot \csc \alpha \cot \alpha \, d\alpha$
$= \int (u^2 - 1)u^2 \cdot (-du) \qquad [u = \csc \alpha, \, du = -\csc \alpha \cot \alpha \, d\alpha]$
$= \int (u^2 - u^4) \, du = \frac{1}{3}u^3 - \frac{1}{5}u^5 + C = \frac{1}{3}\csc^3 \alpha - \frac{1}{5}\csc^5 \alpha + C$

39. $\displaystyle I = \int \csc x \, dx = \int \frac{\csc x\,(\csc x - \cot x)}{\csc x - \cot x} \, dx = \int \frac{-\csc x \cot x + \csc^2 x}{\csc x - \cot x} \, dx$. Let $u = \csc x - \cot x \quad \Rightarrow$
$du = \left(-\csc x \cot x + \csc^2 x\right) dx$. Then $I = \int du/u = \ln|u| = \ln|\csc x - \cot x| + C$.

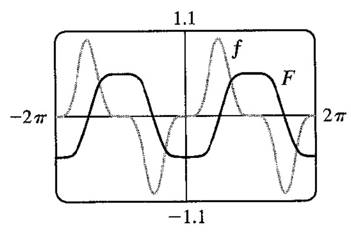
41. Use Equation 2(b):
$\int \sin 5x \sin 2x \, dx = \int \frac{1}{2}[\cos(5x - 2x) - \cos(5x + 2x)] \, dx = \frac{1}{2}\int (\cos 3x - \cos 7x) \, dx$
$= \frac{1}{6}\sin 3x - \frac{1}{14}\sin 7x + C$

43. Use Equation 2(c):
$\int \cos 7\theta \cos 5\theta \, d\theta = \int \frac{1}{2}[\cos(7\theta - 5\theta) + \cos(7\theta + 5\theta)] \, d\theta = \frac{1}{2}\int (\cos 2\theta + \cos 12\theta) \, d\theta$
$= \frac{1}{2}\left(\frac{1}{2}\sin 2\theta + \frac{1}{12}\sin 12\theta\right) + C = \frac{1}{4}\sin 2\theta + \frac{1}{24}\sin 12\theta + C$

45. $\displaystyle\int \frac{1 - \tan^2 x}{\sec^2 x} \, dx = \int \left(\cos^2 x - \sin^2 x\right) dx = \int \cos 2x \, dx = \frac{1}{2}\sin 2x + C$

47. Let $u = \tan(t^2) \quad \Rightarrow \quad du = 2t\sec^2(t^2) \, dt$. Then
$\int t\sec^2(t^2)\tan^4(t^2) \, dt = \int u^4 \left(\frac{1}{2}\,du\right) = \frac{1}{10}u^5 + C = \frac{1}{10}\tan^5(t^2) + C.$

49. Let $u = \cos x \quad \Rightarrow \quad du = -\sin x \, dx$. Then

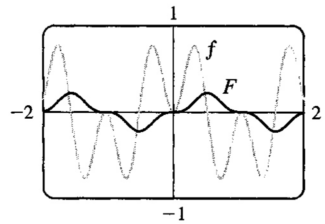


$\int \sin^5 x \, dx = \int \left(1 - \cos^2 x\right)^2 \sin x \, dx = \int \left(1 - u^2\right)^2 (-du)$
$= \int \left(-1 + 2u^2 - u^4\right) du = -\frac{1}{5}u^5 + \frac{2}{3}u^3 - u + C$
$= -\frac{1}{5}\cos^5 x + \frac{2}{3}\cos^3 x - \cos x + C$

Notice that F is increasing when $f(x) > 0$, so the graphs serve as a check on our work.

51. $\int \sin 3x \sin 6x\, dx = \int \frac{1}{2}[\cos(3x - 6x) - \cos(3x + 6x)]\, dx$
$= \frac{1}{2}\int(\cos 3x - \cos 9x)\, dx$
$= \frac{1}{6}\sin 3x - \frac{1}{18}\sin 9x + C$

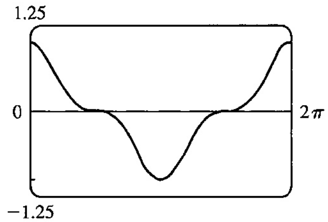
Notice that $f(x) = 0$ whenever F has a horizontal tangent.



53. $f_{\text{ave}} = \frac{1}{2\pi}\int_{-\pi}^{\pi} \sin^2 x \cos^3 x\, dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} \sin^2 x\left(1 - \sin^2 x\right)\cos x\, dx$
$= \frac{1}{2\pi}\int_0^0 u^2\left(1 - u^2\right) du$ [where $u = \sin x$]
$= 0$

55. For $0 < x < \frac{\pi}{2}$, we have $0 < \sin x < 1$, so $\sin^3 x < \sin x$. Hence the area is $\int_0^{\pi/2}\left(\sin x - \sin^3 x\right) dx = \int_0^{\pi/2} \sin x\left(1 - \sin^2 x\right) dx = \int_0^{\pi/2} \cos^2 x \sin x\, dx$. Now let $u = \cos x \Rightarrow du = -\sin x\, dx$. Then area $= \int_1^0 u^2(-du) = \int_0^1 u^2 du = \left[\frac{1}{3}u^3\right]_0^1 = \frac{1}{3}$.

57.



It seems from the graph that $\int_0^{2\pi} \cos^3 x\, dx = 0$, since the area below the x-axis and above the graph looks about equal to the area above the axis and below the graph. By Example 1, the integral is $\left[\sin x - \frac{1}{3}\sin^3 x\right]_0^{2\pi} = 0$. Note that due to symmetry, the integral of any odd power of $\sin x$ or $\cos x$ between limits which differ by $2n\pi$ (n any integer) is 0.

59. $V = \int_{\pi/2}^{\pi} \pi \sin^2 x\, dx = \pi\int_{\pi/2}^{\pi} \frac{1}{2}(1 - \cos 2x)\, dx = \pi\left[\frac{1}{2}x - \frac{1}{4}\sin 2x\right]_{\pi/2}^{\pi} = \pi\left(\frac{\pi}{2} - 0 - \frac{\pi}{4} + 0\right) = \frac{\pi^2}{4}$

61. Volume $= \pi\int_0^{\pi/2}\left[(1 + \cos x)^2 - 1^2\right] dx = \pi\int_0^{\pi/2}\left(2\cos x + \cos^2 x\right) dx$

$= \pi\left[2\sin x + \frac{1}{2}x + \frac{1}{4}\sin 2x\right]_0^{\pi/2} = \pi\left(2 + \frac{\pi}{4}\right) = 2\pi + \frac{\pi^2}{4}$

63. $s = f(t) = \int_0^t \sin \omega u \cos^2 \omega u\, du$. Let $y = \cos \omega u \Rightarrow dy = -\omega \sin \omega u\, du$. Then $s = -\frac{1}{\omega}\int_1^{\cos \omega t} y^2 dy = -\frac{1}{\omega}\left[\frac{1}{3}y^3\right]_1^{\cos \omega t} = \frac{1}{3\omega}\left(1 - \cos^3 \omega t\right)$.

65. Just note that the integrand is odd $[f(-x) = -f(x)]$.

Or: If $m \neq n$, calculate

$$\int_{-\pi}^{\pi} \sin mx \cos nx\, dx = \int_{-\pi}^{\pi} \frac{1}{2}[\sin(m - n)x + \sin(m + n)x]\, dx$$

$$= \frac{1}{2}\left[-\frac{\cos(m - n)x}{m - n} - \frac{\cos(m + n)x}{m + n}\right]_{-\pi}^{\pi} = 0$$

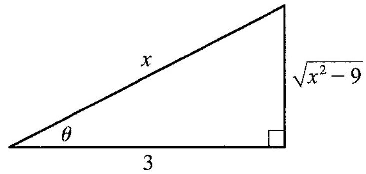
If $m = n$, then the first term in each set of brackets is zero.

67. $\int_{-\pi}^{\pi} \cos mx \cos nx\, dx = \int_{-\pi}^{\pi} \frac{1}{2}[\cos(m - n)x + \cos(m + n)x]\, dx$. If $m \neq n$, this is equal to $\frac{1}{2}\left[\frac{\sin(m - n)x}{m - n} + \frac{\sin(m + n)x}{m + n}\right]_{-\pi}^{\pi} = 0$. If $m = n$, we get

$$\int_{-\pi}^{\pi} \frac{1}{2}[1 + \cos(m + n)x]\, dx = \left[\frac{1}{2}x\right]_{-\pi}^{\pi} + \left[\frac{\sin(m + n)x}{2(m + n)}\right]_{-\pi}^{\pi} = \pi + 0 = \pi.$$

7.3 Trigonometric Substitution

1. Let $x = 3\sec\theta$, where $0 \le \theta < \frac{\pi}{2}$ or $\pi \le \theta < \frac{3\pi}{2}$. Then $dx = 3\sec\theta\tan\theta\,d\theta$ and

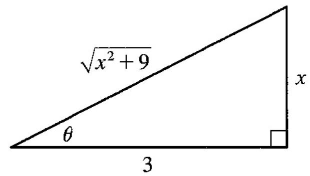


$$\sqrt{x^2-9} = \sqrt{9\sec^2\theta - 9} = \sqrt{9(\sec^2\theta - 1)} = \sqrt{9\tan^2\theta}$$
$$= 3\,|\tan\theta| = 3\tan\theta \text{ for the relevant values of } \theta.$$

$$\int \frac{1}{x^2\sqrt{x^2-9}}\,dx = \int \frac{1}{9\sec^2\theta \cdot 3\tan\theta} 3\sec\theta\tan\theta\,d\theta = \tfrac{1}{9}\int\cos\theta\,d\theta = \tfrac{1}{9}\sin\theta + C = \frac{1}{9}\frac{\sqrt{x^2-9}}{x} + C$$

Note that $-\sec(\theta+\pi) = \sec\theta$, so the figure is sufficient for the case $\pi \le \theta < \frac{3\pi}{2}$.

3. Let $x = 3\tan\theta$, where $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$. Then $dx = 3\sec^2\theta\,d\theta$ and



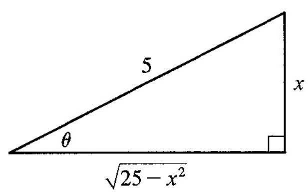
$$\sqrt{x^2+9} = \sqrt{9\tan^2\theta + 9} = \sqrt{9(\tan^2\theta + 1)} = \sqrt{9\sec^2\theta}$$
$$= 3\,|\sec\theta| = 3\sec\theta \text{ for the relevant values of } \theta.$$

$$\int \frac{x^3}{\sqrt{x^2+9}}\,dx = \int \frac{3^3\tan^3\theta}{3\sec\theta} 3\sec^2\theta\,d\theta = 3^3\int\tan^3\theta\sec\theta d\theta = 3^3\int\tan^2\theta\tan\theta\sec\theta\,d\theta$$
$$= 3^3\int(\sec^2\theta - 1)\tan\theta\sec\theta\,d\theta = 3^3\int(u^2-1)\,du \qquad [u = \sec\theta,\ du = \sec\theta\tan\theta\,d\theta]$$
$$= 3^3\left(\tfrac{1}{3}u^3 - u\right) + C = 3^3\left(\tfrac{1}{3}\sec^3\theta - \sec\theta\right) + C = 3^3\left[\frac{1}{3}\frac{(x^2+9)^{3/2}}{3^3} - \frac{\sqrt{x^2+9}}{3}\right] + C$$
$$= \tfrac{1}{3}(x^2+9)^{3/2} - 9\sqrt{x^2+9} + C \quad\text{or}\quad \tfrac{1}{3}(x^2-18)\sqrt{x^2+9} + C$$

5. Let $t = \sec\theta$, so $dt = \sec\theta\tan\theta\,d\theta$, $t = \sqrt{2} \;\Rightarrow\; \theta = \frac{\pi}{4}$, and $t = 2 \;\Rightarrow\; \theta = \frac{\pi}{3}$. Then

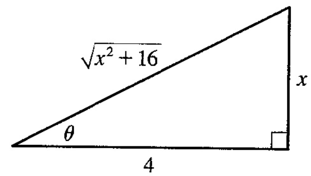
$$\int_{\sqrt{2}}^{2} \frac{1}{t^3\sqrt{t^2-1}}\,dt = \int_{\pi/4}^{\pi/3} \frac{1}{\sec^3\theta\tan\theta}\sec\theta\tan\theta\,d\theta = \int_{\pi/4}^{\pi/3}\frac{1}{\sec^2\theta}\,d\theta = \int_{\pi/4}^{\pi/3}\cos^2\theta\,d\theta$$
$$= \int_{\pi/4}^{\pi/3}\tfrac{1}{2}(1+\cos 2\theta)\,d\theta = \tfrac{1}{2}\left[\theta + \tfrac{1}{2}\sin 2\theta\right]_{\pi/4}^{\pi/3}$$
$$= \tfrac{1}{2}\left[\left(\tfrac{\pi}{3} + \tfrac{1}{2}\tfrac{\sqrt{3}}{2}\right) - \left(\tfrac{\pi}{4} + \tfrac{1}{2}\cdot 1\right)\right] = \tfrac{1}{2}\left(\tfrac{\pi}{12} + \tfrac{\sqrt{3}}{4} - \tfrac{1}{2}\right) = \tfrac{\pi}{24} + \tfrac{\sqrt{3}}{8} - \tfrac{1}{4}$$

7. Let $x = 5\sin\theta$, so $dx = 5\cos\theta\,d\theta$. Then



$$\int \frac{1}{x^2\sqrt{25-x^2}}\,dx = \int \frac{1}{5^2\sin^2\theta\cdot 5\cos\theta}5\cos\theta\,d\theta$$
$$= \tfrac{1}{25}\int\csc^2\theta\,d\theta = -\tfrac{1}{25}\cot\theta + C$$
$$= -\frac{1}{25}\frac{\sqrt{25-x^2}}{x} + C$$

9. Let $x = 4\tan\theta$, where $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$. Then $dx = 4\sec^2\theta\,d\theta$ and



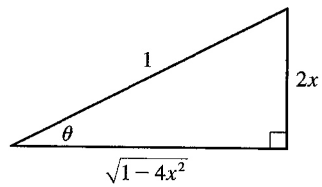
$$\begin{aligned}\sqrt{x^2+16} &= \sqrt{16\tan^2\theta+16} = \sqrt{16(\tan^2\theta+1)}\\ &= \sqrt{16\sec^2\theta} = 4\,|\sec\theta|\\ &= 4\sec\theta \text{ for the relevant values of } \theta.\end{aligned}$$

$$\begin{aligned}\int\frac{dx}{\sqrt{x^2+16}} &= \int\frac{4\sec^2\theta\,d\theta}{4\sec\theta} = \int\sec\theta\,d\theta = \ln|\sec\theta+\tan\theta| + C_1\\ &= \ln\left|\frac{\sqrt{x^2+16}}{4}+\frac{x}{4}\right| + C_1 = \ln\left|\sqrt{x^2+16}+x\right| - \ln|4| + C_1\\ &= \ln\left(\sqrt{x^2+16}+x\right) + C, \text{ where } C = C_1 - \ln 4.\end{aligned}$$

(Since $\sqrt{x^2+16}+x > 0$, we don't need the absolute value.)

11. Let $2x = \sin\theta$, where $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$. Then $x = \frac{1}{2}\sin\theta$,

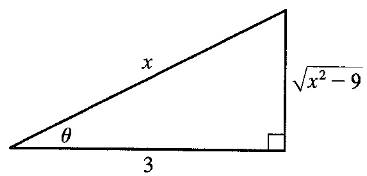
$dx = \frac{1}{2}\cos\theta\,d\theta$, and $\sqrt{1-4x^2} = \sqrt{1-(2x)^2} = \cos\theta$.



$$\begin{aligned}\int\sqrt{1-4x^2}\,dx &= \int\cos\theta\left(\tfrac{1}{2}\cos\theta\right)d\theta = \tfrac{1}{4}\int(1+\cos 2\theta)\,d\theta\\ &= \tfrac{1}{4}\left(\theta+\tfrac{1}{2}\sin 2\theta\right) + C = \tfrac{1}{4}(\theta+\sin\theta\cos\theta) + C\\ &= \tfrac{1}{4}\left[\sin^{-1}(2x) + 2x\sqrt{1-4x^2}\right] + C\end{aligned}$$

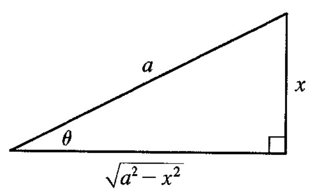
13. Let $x = 3\sec\theta$, where $0 \le \theta < \frac{\pi}{2}$ or $\pi \le \theta < \frac{3\pi}{2}$. Then

$dx = 3\sec\theta\tan\theta\,d\theta$ and $\sqrt{x^2-9} = 3\tan\theta$, so



$$\begin{aligned}\int\frac{\sqrt{x^2-9}}{x^3}\,dx &= \int\frac{3\tan\theta}{27\sec^3\theta}\,3\sec\theta\tan\theta\,d\theta = \frac{1}{3}\int\frac{\tan^2\theta}{\sec^2\theta}\,d\theta\\ &= \tfrac{1}{3}\int\sin^2\theta\,d\theta = \tfrac{1}{3}\int\tfrac{1}{2}(1-\cos 2\theta)\,d\theta = \tfrac{1}{6}\theta - \tfrac{1}{12}\sin 2\theta + C = \tfrac{1}{6}\theta - \tfrac{1}{6}\sin\theta\cos\theta + C\\ &= \frac{1}{6}\sec^{-1}\left(\frac{x}{3}\right) - \frac{1}{6}\frac{\sqrt{x^2-9}}{x}\frac{3}{x} + C = \frac{1}{6}\sec^{-1}\left(\frac{x}{3}\right) - \frac{\sqrt{x^2-9}}{2x^2} + C\end{aligned}$$

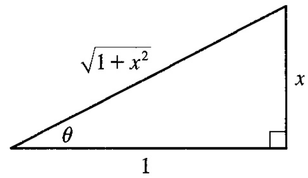
15. Let $x = a\sin\theta$, where $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$. Then $dx = a\cos\theta\,d\theta$ and



$$\begin{aligned}\int\frac{x^2\,dx}{(a^2-x^2)^{3/2}} &= \int\frac{a^2\sin^2\theta\,a\cos\theta\,d\theta}{a^3\cos^3\theta} = \int\tan^2\theta\,d\theta\\ &= \int\left(\sec^2\theta-1\right)d\theta = \tan\theta - \theta + C\\ &= \frac{x}{\sqrt{a^2-x^2}} - \sin^{-1}\frac{x}{a} + C\end{aligned}$$

17. Let $u = x^2 - 7$, so $du = 2x\,dx$. Then $\displaystyle\int\frac{x}{\sqrt{x^2-7}}\,dx = \frac{1}{2}\int\frac{1}{\sqrt{u}}\,du = \tfrac{1}{2}\cdot 2\sqrt{u} + C = \sqrt{x^2-7} + C$.

19. Let $x = \tan\theta$, where $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$. Then $dx = \sec^2\theta\,d\theta$ and $\sqrt{1+x^2} = \sec\theta$, so



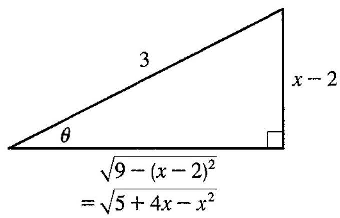
$$\begin{aligned}\int \frac{\sqrt{1+x^2}}{x}\,dx &= \int \frac{\sec\theta}{\tan\theta}\sec^2\theta\,d\theta = \int \frac{\sec\theta}{\tan\theta}(1+\tan^2\theta)\,d\theta \\ &= \int(\csc\theta + \sec\theta\tan\theta)\,d\theta \\ &= \ln|\csc\theta - \cot\theta| + \sec\theta + C \quad \text{[by Exercise 7.2.39]} \\ &= \ln\left|\frac{\sqrt{1+x^2}}{x} - \frac{1}{x}\right| + \frac{\sqrt{1+x^2}}{1} + C = \ln\left|\frac{\sqrt{1+x^2}-1}{x}\right| + \sqrt{1+x^2} + C\end{aligned}$$

21. Let $u = 4 - 9x^2 \quad\Rightarrow\quad du = -18x\,dx$. Then $x^2 = \frac{1}{9}(4-u)$ and

$$\begin{aligned}\int_0^{2/3} x^3\sqrt{4-9x^2}\,dx &= \int_4^0 \tfrac{1}{9}(4-u)u^{1/2}\left(-\tfrac{1}{18}\right)du = \tfrac{1}{162}\int_0^4\left(4u^{1/2} - u^{3/2}\right)du \\ &= \tfrac{1}{162}\left[\tfrac{8}{3}u^{3/2} - \tfrac{2}{5}u^{5/2}\right]_0^4 = \tfrac{1}{162}\left[\tfrac{64}{3} - \tfrac{64}{5}\right] = \tfrac{64}{1215}\end{aligned}$$

Or: Let $3x = 2\sin\theta$, where $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$.

23. $5 + 4x - x^2 = -(x^2 - 4x + 4) + 9 = -(x-2)^2 + 9$. Let $x - 2 = 3\sin\theta$, $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$, so $dx = 3\cos\theta\,d\theta$. Then



$$\begin{aligned}\int\sqrt{5+4x-x^2}\,dx &= \int\sqrt{9-(x-2)^2}\,dx = \int\sqrt{9-9\sin^2\theta}\,3\cos\theta\,d\theta \\ &= \int\sqrt{9\cos^2\theta}\,3\cos\theta\,d\theta = \int 9\cos^2\theta\,d\theta \\ &= \tfrac{9}{2}\int(1+\cos 2\theta)\,d\theta = \tfrac{9}{2}\left(\theta + \tfrac{1}{2}\sin 2\theta\right) + C \\ &= \tfrac{9}{2}\theta + \tfrac{9}{4}\sin 2\theta + C = \tfrac{9}{2}\theta + \tfrac{9}{4}(2\sin\theta\cos\theta) + C \\ &= \frac{9}{2}\sin^{-1}\left(\frac{x-2}{3}\right) + \frac{9}{2}\cdot\frac{x-2}{3}\cdot\frac{\sqrt{5+4x-x^2}}{3} + C \\ &= \frac{9}{2}\sin^{-1}\left(\frac{x-2}{3}\right) + \frac{1}{2}(x-2)\sqrt{5+4x-x^2} + C\end{aligned}$$

25. $9x^2 + 6x - 8 = (3x+1)^2 - 9$, so let $u = 3x + 1$, $du = 3dx$. Then $\displaystyle\int\frac{dx}{\sqrt{9x^2+6x-8}} = \int\frac{\frac{1}{3}\,du}{\sqrt{u^2-9}}$. Now let $u = 3\sec\theta$, where $0 \le \theta < \frac{\pi}{2}$ or $\pi \le \theta < \frac{3\pi}{2}$. Then $du = 3\sec\theta\tan\theta\,d\theta$ and $\sqrt{u^2-9} = 3\tan\theta$, so

$$\begin{aligned}\int\frac{\frac{1}{3}\,du}{\sqrt{u^2-9}} &= \int\frac{\sec\theta\tan\theta\,d\theta}{3\tan\theta} = \tfrac{1}{3}\int\sec\theta d\theta = \tfrac{1}{3}\ln|\sec\theta + \tan\theta| + C_1 = \tfrac{1}{3}\ln\left|\frac{u+\sqrt{u^2-9}}{3}\right| + C_1 \\ &= \tfrac{1}{3}\ln\left|u + \sqrt{u^2-9}\right| + C = \tfrac{1}{3}\ln\left|3x + 1 + \sqrt{9x^2+6x-8}\right| + C\end{aligned}$$

27. $x^2 + 2x + 2 = (x+1)^2 + 1$. Let $u = x + 1$, $du = dx$. Then

$$\begin{aligned}\int\frac{dx}{(x^2+2x+2)^2} &= \int\frac{du}{(u^2+1)^2} = \int\frac{\sec^2\theta d\theta}{\sec^4\theta} \quad \left[\begin{array}{c}\text{where } u = \tan\theta,\ du = \sec^2\theta\,d\theta, \\ \text{and } u^2+1 = \sec^2\theta\end{array}\right] \\ &= \int\cos^2\theta\,d\theta = \tfrac{1}{2}\int(1+\cos 2\theta)\,d\theta = \tfrac{1}{2}(\theta + \sin\theta\cos\theta) + C \\ &= \frac{1}{2}\left[\tan^{-1}u + \frac{u}{1+u^2}\right] + C = \frac{1}{2}\left[\tan^{-1}(x+1) + \frac{x+1}{x^2+2x+2}\right] + C\end{aligned}$$

29. Let $u = x^2$, $du = 2x\,dx$. Then

$$\int x\sqrt{1-x^4}\,dx = \int \sqrt{1-u^2}\left(\tfrac{1}{2}\,du\right) = \tfrac{1}{2}\int \cos\theta \cdot \cos\theta\,d\theta \quad \left[\begin{array}{c}\text{where } u = \sin\theta,\ du = \cos\theta\,d\theta, \\ \text{and } \sqrt{1-u^2} = \cos\theta\end{array}\right]$$

$$= \tfrac{1}{2}\int \tfrac{1}{2}(1+\cos 2\theta)d\theta = \tfrac{1}{4}\theta + \tfrac{1}{8}\sin 2\theta + C = \tfrac{1}{4}\theta + \tfrac{1}{4}\sin\theta\cos\theta + C$$

$$= \tfrac{1}{4}\sin^{-1}u + \tfrac{1}{4}u\sqrt{1-u^2} + C = \tfrac{1}{4}\sin^{-1}(x^2) + \tfrac{1}{4}x^2\sqrt{1-x^4} + C$$

31. (a) Let $x = a\tan\theta$, where $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$. Then $\sqrt{x^2+a^2} = a\sec\theta$ and

$$\int \frac{dx}{\sqrt{x^2+a^2}} = \int \frac{a\sec^2\theta\,d\theta}{a\sec\theta} = \int \sec\theta\,d\theta = \ln|\sec\theta + \tan\theta| + C_1 = \ln\left|\frac{\sqrt{x^2+a^2}}{a} + \frac{x}{a}\right| + C_1$$

$$= \ln\left(x + \sqrt{x^2+a^2}\right) + C \quad \text{where } C = C_1 - \ln|a|$$

(b) Let $x = a\sinh t$, so that $dx = a\cosh t\,dt$ and $\sqrt{x^2+a^2} = a\cosh t$. Then

$$\int \frac{dx}{\sqrt{x^2+a^2}} = \int \frac{a\cosh t\,dt}{a\cosh t} = t + C = \sinh^{-1}\frac{x}{a} + C.$$

33. The average value of $f(x) = \sqrt{x^2-1}/x$ on the interval $[1,7]$ is

$$\frac{1}{7-1}\int_1^7 \frac{\sqrt{x^2-1}}{x}\,dx = \frac{1}{6}\int_0^\alpha \frac{\tan\theta}{\sec\theta}\cdot\sec\theta\tan\theta\,d\theta \quad \left[\begin{array}{c}\text{where } x = \sec\theta,\ dx = \sec\theta\tan\theta\,d\theta, \\ \sqrt{x^2-1} = \tan\theta, \text{ and } \alpha = \sec^{-1}7\end{array}\right]$$

$$= \tfrac{1}{6}\int_0^\alpha \tan^2\theta\,d\theta = \tfrac{1}{6}\int_0^\alpha (\sec^2\theta - 1)\,d\theta$$

$$= \tfrac{1}{6}\Big[\tan\theta - \theta\Big]_0^\alpha = \tfrac{1}{6}(\tan\alpha - \alpha)$$

$$= \tfrac{1}{6}\left(\sqrt{48} - \sec^{-1}7\right)$$

35. Area of $\triangle POQ = \frac{1}{2}(r\cos\theta)(r\sin\theta) = \frac{1}{2}r^2\sin\theta\cos\theta$. Area of region $PQR = \int_{r\cos\theta}^{r}\sqrt{r^2-x^2}\,dx$. Let $x = r\cos u \Rightarrow dx = -r\sin u\,du$ for $\theta \le u \le \frac{\pi}{2}$. Then we obtain

$$\int \sqrt{r^2-x^2}\,dx = \int r\sin u\,(-r\sin u)\,du = -r^2\int \sin^2 u\,du = -\tfrac{1}{2}r^2(u - \sin u\cos u) + C$$

$$= -\tfrac{1}{2}r^2\cos^{-1}(x/r) + \tfrac{1}{2}x\sqrt{r^2-x^2} + C$$

so

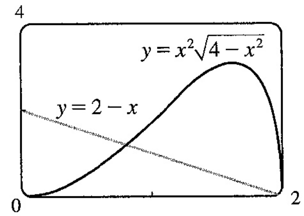
$$\text{area of region } PQR = \tfrac{1}{2}\Big[-r^2\cos^{-1}(x/r) + x\sqrt{r^2-x^2}\Big]_{r\cos\theta}^{r}$$

$$= \tfrac{1}{2}\left[0 - \left(-r^2\theta + r\cos\theta\, r\sin\theta\right)\right]$$

$$= \tfrac{1}{2}r^2\theta - \tfrac{1}{2}r^2\sin\theta\cos\theta$$

and thus, (area of sector POR) = (area of $\triangle POQ$) + (area of region PQR) $= \frac{1}{2}r^2\theta$.

37. From the graph, it appears that the curve $y = x^2\sqrt{4-x^2}$ and the line $y = 2 - x$ intersect at about $x = 0.81$ and $x = 2$, with $x^2\sqrt{4-x^2} > 2 - x$ on $(0.81, 2)$. So the area bounded by the curve and the line is $A \approx \int_{0.81}^{2} \left[x^2\sqrt{4-x^2} - (2-x)\right] dx = \int_{0.81}^{2} x^2\sqrt{4-x^2}\,dx - \left[2x - \frac{1}{2}x^2\right]_{0.81}^{2}$.



To evaluate the integral, we put $x = 2\sin\theta$, where $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$. Then $dx = 2\cos\theta\,d\theta$, $x = 2 \;\Rightarrow\; \theta = \sin^{-1}1 = \frac{\pi}{2}$, and $x = 0.81 \;\Rightarrow\; \theta = \sin^{-1}0.405 \approx 0.417$. So

$$\int_{0.81}^{2} x^2\sqrt{4-x^2}\,dx \approx \int_{0.417}^{\pi/2} 4\sin^2\theta\,(2\cos\theta)(2\cos\theta\,d\theta) = 4\int_{0.417}^{\pi/2}\sin^2 2\theta\,d\theta = 4\int_{0.417}^{\pi/2}\tfrac{1}{2}(1-\cos 4\theta)\,d\theta$$

$$= 2\left[\theta - \tfrac{1}{4}\sin 4\theta\right]_{0.417}^{\pi/2} = 2\left[\left(\tfrac{\pi}{2} - 0\right) - \left(0.417 - \tfrac{1}{4}(0.995)\right)\right] \approx 2.81$$

Thus, $A \approx 2.81 - \left[\left(2\cdot 2 - \frac{1}{2}\cdot 2^2\right) - \left(2\cdot 0.81 - \frac{1}{2}\cdot 0.81^2\right)\right] \approx 2.10$.

39. Let the equation of the large circle be $x^2 + y^2 = R^2$. Then the equation of the small circle is $x^2 + (y-b)^2 = r^2$, where $b = \sqrt{R^2 - r^2}$ is the distance between the centers of the circles. The desired area is

$$A = \int_{-r}^{r}\left[\left(b + \sqrt{r^2-x^2}\right) - \sqrt{R^2-x^2}\right]dx = 2\int_0^r \left(b + \sqrt{r^2-x^2} - \sqrt{R^2-x^2}\right)dx$$

$$= 2\int_0^r b\,dx + 2\int_0^r \sqrt{r^2-x^2}\,dx - 2\int_0^r \sqrt{R^2-x^2}\,dx$$

The first integral is just $2br = 2r\sqrt{R^2-r^2}$. To evaluate the other two integrals, note that

$$\int \sqrt{a^2-x^2}\,dx = \int a^2\cos^2\theta\,d\theta \quad [x = a\sin\theta,\ dx = a\cos\theta\,d\theta] \quad = \tfrac{1}{2}a^2\int(1+\cos 2\theta)\,d\theta$$

$$= \tfrac{1}{2}a^2\left(\theta + \tfrac{1}{2}\sin 2\theta\right) + C = \tfrac{1}{2}a^2(\theta + \sin\theta\cos\theta) + C$$

$$= \frac{a^2}{2}\arcsin\left(\frac{x}{a}\right) + \frac{a^2}{2}\left(\frac{x}{a}\right)\frac{\sqrt{a^2-x^2}}{a} + C = \frac{a^2}{2}\arcsin\left(\frac{x}{a}\right) + \frac{x}{2}\sqrt{a^2-x^2} + C$$

so the desired area is

$$A = 2r\sqrt{R^2-r^2} + \left[r^2\arcsin(x/r) + x\sqrt{r^2-x^2}\right]_0^r - \left[R^2\arcsin(x/R) + x\sqrt{R^2-x^2}\right]_0^r$$

$$= 2r\sqrt{R^2-r^2} + r^2\left(\tfrac{\pi}{2}\right) - \left[R^2\arcsin(r/R) + r\sqrt{R^2-r^2}\right] = r\sqrt{R^2-r^2} + \tfrac{\pi}{2}r^2 - R^2\arcsin(r/R)$$

41. We use cylindrical shells and assume that $R > r$. $x^2 = r^2 - (y-R)^2 \;\Rightarrow\; x = \pm\sqrt{r^2-(y-R)^2}$, so $g(y) = 2\sqrt{r^2-(y-R)^2}$ and

$$V = \int_{R-r}^{R+r} 2\pi y\cdot 2\sqrt{r^2-(y-R)^2}\,dy = \int_{-r}^{r} 4\pi(u+R)\sqrt{r^2-u^2}\,du \quad [\text{where } u = y - R]$$

$$= 4\pi\int_{-r}^{r} u\sqrt{r^2-u^2}\,du + 4\pi R\int_{-r}^{r}\sqrt{r^2-u^2}\,du \quad \left[\begin{array}{c}\text{where } u = r\sin\theta,\ du = r\cos\theta\,d\theta\\ \text{in the second integral}\end{array}\right]$$

$$= 4\pi\left[-\tfrac{1}{3}\left(r^2-u^2\right)^{3/2}\right]_{-r}^{r} + 4\pi R\int_{-\pi/2}^{\pi/2} r^2\cos^2\theta\,d\theta = -\tfrac{4\pi}{3}(0-0) + 4\pi R r^2\int_{-\pi/2}^{\pi/2}\cos^2\theta\,d\theta$$

$$= 2\pi R r^2\int_{-\pi/2}^{\pi/2}(1+\cos 2\theta)\,d\theta = 2\pi R r^2\left[\theta + \tfrac{1}{2}\sin 2\theta\right]_{-\pi/2}^{\pi/2} = 2\pi^2 R r^2$$

Another method: Use washers instead of shells, so $V = 8\pi R\int_0^r \sqrt{r^2-y^2}\,dy$ as in Exercise 6.2.61(a), but evaluate the integral using $y = r\sin\theta$.

7.4 Integration of Rational Functions by Partial Fractions

1. (a) $\dfrac{2x}{(x+3)(3x+1)} = \dfrac{A}{x+3} + \dfrac{B}{3x+1}$

(b) $\dfrac{1}{x^3+2x^2+x} = \dfrac{1}{x(x^2+2x+1)} = \dfrac{1}{x(x+1)^2} = \dfrac{A}{x} + \dfrac{B}{x+1} + \dfrac{C}{(x+1)^2}$

3. (a) $\dfrac{2}{x^2+3x-4} = \dfrac{2}{(x+4)(x-1)} = \dfrac{A}{x+4} + \dfrac{B}{x-1}$

(b) x^2+x+1 is irreducible, so $\dfrac{x^2}{(x-1)(x^2+x+1)} = \dfrac{A}{x-1} + \dfrac{Bx+C}{x^2+x+1}$.

5. (a) $\dfrac{x^4}{x^4-1} = \dfrac{(x^4-1)+1}{x^4-1} = 1 + \dfrac{1}{x^4-1}$ [or use long division] $= 1 + \dfrac{1}{(x^2-1)(x^2+1)}$

$$= 1 + \frac{1}{(x-1)(x+1)(x^2+1)} = 1 + \frac{A}{x-1} + \frac{B}{x+1} + \frac{Cx+D}{x^2+1}$$

(b) $\dfrac{t^4+t^2+1}{(t^2+1)(t^2+4)^2} = \dfrac{At+B}{t^2+1} + \dfrac{Ct+D}{t^2+4} + \dfrac{Et+F}{(t^2+4)^2}$

7. $\displaystyle\int \frac{x}{x-6}\,dx = \int \frac{(x-6)+6}{x-6}\,dx = \int \left(1 + \frac{6}{x-6}\right) dx = x + 6\ln|x-6| + C$

9. $\dfrac{x-9}{(x+5)(x-2)} = \dfrac{A}{x+5} + \dfrac{B}{x-2}$. Multiply both sides by $(x+5)(x-2)$ to get $x-9 = A(x-2) + B(x+5)$. Substituting 2 for x gives $-7 = 7B \Leftrightarrow B = -1$. Substituting -5 for x gives $-14 = -7A \Leftrightarrow A = 2$. Thus,

$$\int \frac{x-9}{(x+5)(x-2)}\,dx = \int \left(\frac{2}{x+5} + \frac{-1}{x-2}\right) dx = 2\ln|x+5| - \ln|x-2| + C$$

11. $\dfrac{1}{x^2-1} = \dfrac{1}{(x+1)(x-1)} = \dfrac{A}{x+1} + \dfrac{B}{x-1}$. Multiply both sides by $(x+1)(x-1)$ to get $1 = A(x-1) + B(x+1)$. Substituting 1 for x gives $1 = 2B \Leftrightarrow B = \frac{1}{2}$. Substituting -1 for x gives $1 = -2A \Leftrightarrow A = -\frac{1}{2}$. Thus,

$$\int_2^3 \frac{1}{x^2-1}\,dx = \int_2^3 \left(\frac{-1/2}{x+1} + \frac{1/2}{x-1}\right) dx = \left[-\tfrac{1}{2}\ln|x+1| + \tfrac{1}{2}\ln|x-1|\right]_2^3$$

$$= \left(-\tfrac{1}{2}\ln 4 + \tfrac{1}{2}\ln 2\right) - \left(-\tfrac{1}{2}\ln 3 + \tfrac{1}{2}\ln 1\right) = \tfrac{1}{2}(\ln 2 + \ln 3 - \ln 4) \quad \left[\text{or } \tfrac{1}{2}\ln\tfrac{3}{2}\right]$$

13. $\displaystyle\int \frac{ax}{x^2-bx}\,dx = \int \frac{ax}{x(x-b)}\,dx = \int \frac{a}{x-b}\,dx = a\ln|x-b| + C$

15. $\dfrac{2x+3}{(x+1)^2} = \dfrac{A}{x+1} + \dfrac{B}{(x+1)^2} \Rightarrow 2x+3 = A(x+1) + B$. Take $x = -1$ to get $B = 1$, and equate coefficients of x to get $A = 2$. Now

$$\int_0^1 \frac{2x+3}{(x+1)^2}\,dx = \int_0^1 \left[\frac{2}{x+1} + \frac{1}{(x+1)^2}\right] dx = \left[2\ln(x+1) - \frac{1}{x+1}\right]_0^1$$

$$= 2\ln 2 - \tfrac{1}{2} - (2\ln 1 - 1) = 2\ln 2 + \tfrac{1}{2}$$

17. $\dfrac{4y^2-7y-12}{y(y+2)(y-3)} = \dfrac{A}{y} + \dfrac{B}{y+2} + \dfrac{C}{y-3} \quad\Rightarrow\quad 4y^2-7y-12 = A(y+2)(y-3) + By(y-3) + Cy(y+2).$

Setting $y=0$ gives $-12=-6A$, so $A=2$. Setting $y=-2$ gives $18=10B$, so $B=\frac{9}{5}$. Setting $y=3$ gives $3=15C$, so $C=\frac{1}{5}$. Now

$$\int_1^2 \frac{4y^2-7y-12}{y(y+2)(y-3)}\,dy = \int_1^2\left(\frac{2}{y}+\frac{9/5}{y+2}+\frac{1/5}{y-3}\right)dy = \left[2\ln|y| + \tfrac{9}{5}\ln|y+2| + \tfrac{1}{5}\ln|y-3|\right]_1^2$$

$$= 2\ln 2 + \tfrac{9}{5}\ln 4 + \tfrac{1}{5}\ln 1 - 2\ln 1 - \tfrac{9}{5}\ln 3 - \tfrac{1}{5}\ln 2$$

$$= 2\ln 2 + \tfrac{18}{5}\ln 2 - \tfrac{1}{5}\ln 2 - \tfrac{9}{5}\ln 3 = \tfrac{27}{5}\ln 2 - \tfrac{9}{5}\ln 3 = \tfrac{9}{5}(3\ln 2 - \ln 3) = \tfrac{9}{5}\ln\tfrac{8}{3}$$

19. $\dfrac{1}{(x+5)^2(x-1)} = \dfrac{A}{x+5} + \dfrac{B}{(x+5)^2} + \dfrac{C}{x-1} \quad\Rightarrow\quad 1 = A(x+5)(x-1) + B(x-1) + C(x+5)^2.$ Setting $x=-5$ gives $1=-6B$, so $B=-\frac{1}{6}$. Setting $x=1$ gives $1=36C$, so $C=\frac{1}{36}$. Setting $x=-2$ gives $1 = A(3)(-3) + B(-3) + C(3^2) = -9A - 3B + 9C = -9A + \frac{1}{2} + \frac{1}{4} = -9A + \frac{3}{4}$, so $9A = -\frac{1}{4}$ and $A = -\frac{1}{36}$. Now

$$\int \frac{1}{(x+5)^2(x-1)}\,dx = \int\left[\frac{-1/36}{x+5} - \frac{1/6}{(x+5)^2} + \frac{1/36}{x-1}\right]dx$$

$$= -\frac{1}{36}\ln|x+5| + \frac{1}{6(x+5)} + \frac{1}{36}\ln|x-1| + C$$

21. $\dfrac{5x^2+3x-2}{x^3+2x^2} = \dfrac{5x^2+3x-2}{x^2(x+2)} = \dfrac{A}{x} + \dfrac{B}{x^2} + \dfrac{C}{x+2}.$ Multiply by $x^2(x+2)$ to get $5x^2+3x-2 = Ax(x+2) + B(x+2) + Cx^2$. Set $x=-2$ to get $C=3$, and take $x=0$ to get $B=-1$. Equating the coefficients of x^2 gives $5 = A + C \;\Rightarrow\; A = 2$. So

$$\int \frac{5x^2+3x-2}{x^3+2x^2}\,dx = \int\left(\frac{2}{x} - \frac{1}{x^2} + \frac{3}{x+2}\right)dx = 2\ln|x| + \frac{1}{x} + 3\ln|x+2| + C.$$

23. $\dfrac{x^2}{(x+1)^3} = \dfrac{A}{x+1} + \dfrac{B}{(x+1)^2} + \dfrac{C}{(x+1)^3}.$ Multiply by $(x+1)^3$ to get $x^2 = A(x+1)^2 + B(x+1) + C$. Setting $x=-1$ gives $C=1$. Equating the coefficients of x^2 gives $A=1$, and setting $x=0$ gives $B=-2$.

Now $\displaystyle\int \frac{x^2\,dx}{(x+1)^3} = \int\left[\frac{1}{x+1} - \frac{2}{(x+1)^2} + \frac{1}{(x+1)^3}\right]dx = \ln|x+1| + \frac{2}{x+1} - \frac{1}{2(x+1)^2} + C.$

25. $\dfrac{10}{(x-1)(x^2+9)} = \dfrac{A}{x-1} + \dfrac{Bx+C}{x^2+9}.$ Multiply both sides by $(x-1)(x^2+9)$ to get $10 = A(x^2+9) + (Bx+C)(x-1)$ $(*)$. Substituting 1 for x gives $10 = 10A \;\Leftrightarrow\; A=1$. Substituting 0 for x gives $10 = 9A - C \;\Rightarrow\; C = 9(1) - 10 = -1$. The coefficients of the x^2-terms in $(*)$ must be equal, so $0 = A + B \;\Rightarrow\; B = -1$. Thus,

$$\int \frac{10}{(x-1)(x^2+9)}\,dx = \int\left(\frac{1}{x-1} + \frac{-x-1}{x^2+9}\right)dx = \int\left(\frac{1}{x-1} - \frac{x}{x^2+9} - \frac{1}{x^2+9}\right)dx$$

$$= \ln|x-1| - \tfrac{1}{2}\ln(x^2+9)\ \text{[let } u = x^2+9\text{]} - \tfrac{1}{3}\tan^{-1}\left(\tfrac{x}{3}\right)\ \text{[Formula 10]} + C$$

27. $\dfrac{x^3+x^2+2x+1}{(x^2+1)(x^2+2)}=\dfrac{Ax+B}{x^2+1}+\dfrac{Cx+D}{x^2+2}$. Multiply both sides by $(x^2+1)(x^2+2)$ to get

$x^3+x^2+2x+1=(Ax+B)(x^2+2)+(Cx+D)(x^2+1)\;\Leftrightarrow$

$x^3+x^2+2x+1=(Ax^3+Bx^2+2Ax+2B)+(Cx^3+Dx^2+Cx+D)\;\Leftrightarrow$

$x^3+x^2+2x+1=(A+C)x^3+(B+D)x^2+(2A+C)x+(2B+D)$. Comparing coefficients gives us the following system of equations:

$$A+C=1\quad\textbf{(1)}\qquad B+D=1\quad\textbf{(2)}$$
$$2A+C=2\quad\textbf{(3)}\qquad 2B+D=1\quad\textbf{(4)}$$

Subtracting equation **(1)** from equation **(3)** gives us $A=1$, so $C=0$. Subtracting equation **(2)** from equation **(4)** gives us $B=0$, so $D=1$. Thus, $I=\displaystyle\int\frac{x^3+x^2+2x+1}{(x^2+1)(x^2+2)}\,dx=\int\left(\frac{x}{x^2+1}+\frac{1}{x^2+2}\right)dx$. For $\displaystyle\int\frac{x}{x^2+1}\,dx$, let $u=x^2+1$ so $du=2x\,dx$ and then $\displaystyle\int\frac{x}{x^2+1}\,dx=\frac{1}{2}\int\frac{1}{u}\,du=\frac{1}{2}\ln|u|+C=\frac{1}{2}\ln(x^2+1)+C$. For $\displaystyle\int\frac{1}{x^2+2}\,dx$, use Formula 10 with $a=\sqrt{2}$. So $\displaystyle\int\frac{1}{x^2+2}\,dx=\int\frac{1}{x^2+(\sqrt{2})^2}\,dx=\frac{1}{\sqrt{2}}\tan^{-1}\frac{x}{\sqrt{2}}+C$.

Thus, $I=\dfrac{1}{2}\ln(x^2+1)+\dfrac{1}{\sqrt{2}}\tan^{-1}\dfrac{x}{\sqrt{2}}+C$.

29.
$$\begin{aligned}\int\frac{x+4}{x^2+2x+5}\,dx&=\int\frac{x+1}{x^2+2x+5}\,dx+\int\frac{3}{x^2+2x+5}\,dx=\frac{1}{2}\int\frac{(2x+2)\,dx}{x^2+2x+5}+\int\frac{3\,dx}{(x+1)^2+4}\\&=\frac{1}{2}\ln|x^2+2x+5|+3\int\frac{2\,du}{4(u^2+1)}\qquad\left[\begin{array}{c}\text{where } x+1=2u,\\ \text{and } dx=2\,du\end{array}\right]\\&=\frac{1}{2}\ln(x^2+2x+5)+\frac{3}{2}\tan^{-1}u+C=\frac{1}{2}\ln(x^2+2x+5)+\frac{3}{2}\tan^{-1}\left(\frac{x+1}{2}\right)+C\end{aligned}$$

31. $\dfrac{1}{x^3-1}=\dfrac{1}{(x-1)(x^2+x+1)}=\dfrac{A}{x-1}+\dfrac{Bx+C}{x^2+x+1}\;\Rightarrow\;1=A(x^2+x+1)+(Bx+C)(x-1)$.

Take $x=1$ to get $A=\frac{1}{3}$. Equating coefficients of x^2 and then comparing the constant terms, we get $0=\frac{1}{3}+B$, $1=\frac{1}{3}-C$, so $B=-\frac{1}{3}$, $C=-\frac{2}{3}\;\Rightarrow$

$$\begin{aligned}\int\frac{1}{x^3-1}dx&=\int\frac{\frac{1}{3}}{x-1}\,dx+\int\frac{-\frac{1}{3}x-\frac{2}{3}}{x^2+x+1}\,dx=\tfrac{1}{3}\ln|x-1|-\frac{1}{3}\int\frac{x+2}{x^2+x+1}dx\\&=\tfrac{1}{3}\ln|x-1|-\frac{1}{3}\int\frac{x+1/2}{x^2+x+1}dx-\frac{1}{3}\int\frac{(3/2)\,dx}{(x+1/2)^2+3/4}\\&=\tfrac{1}{3}\ln|x-1|-\tfrac{1}{6}\ln(x^2+x+1)-\tfrac{1}{2}\left(\tfrac{2}{\sqrt{3}}\right)\tan^{-1}\left(\frac{x+\frac{1}{2}}{\sqrt{3}/2}\right)+K\\&=\tfrac{1}{3}\ln|x-1|-\tfrac{1}{6}\ln(x^2+x+1)-\tfrac{1}{\sqrt{3}}\tan^{-1}\left(\tfrac{1}{\sqrt{3}}(2x+1)\right)+K\end{aligned}$$

33. Let $u=x^3+3x^2+4$. Then $du=3(x^2+2x)\,dx\;\Rightarrow$

$$\int_2^5\frac{x^2+2x}{x^3+3x^2+4}\,dx=\frac{1}{3}\int_{24}^{204}\frac{du}{u}=\tfrac{1}{3}[\ln u]_{24}^{204}=\tfrac{1}{3}(\ln 204-\ln 24)=\tfrac{1}{3}\ln\tfrac{204}{24}=\tfrac{1}{3}\ln\tfrac{17}{2}.$$

35. $\dfrac{1}{x^4-x^2}=\dfrac{1}{x^2(x-1)(x+1)}=\dfrac{A}{x}+\dfrac{B}{x^2}+\dfrac{C}{x-1}+\dfrac{D}{x+1}$. Multiply by $x^2(x-1)(x+1)$ to get $1=Ax(x-1)(x+1)+B(x-1)(x+1)+Cx^2(x+1)+Dx^2(x-1)$. Setting $x=1$ gives $C=\frac{1}{2}$, taking $x=-1$ gives $D=-\frac{1}{2}$. Equating the coefficients of x^3 gives $0=A+C+D=A$. Finally, setting $x=0$ yields $B=-1$. Now $\displaystyle\int\frac{dx}{x^4-x^2}=\int\left[\frac{-1}{x^2}+\frac{1/2}{x-1}-\frac{1/2}{x+1}\right]dx=\frac{1}{x}+\frac{1}{2}\ln\left|\frac{x-1}{x+1}\right|+C.$

37. $\displaystyle\int\frac{x-3}{(x^2+2x+4)^2}\,dx=\int\frac{x-3}{\left[(x+1)^2+3\right]^2}\,dx=\int\frac{u-4}{(u^2+3)^2}\,du$ [with $u=x+1$]

$$\begin{aligned}
&=\int\frac{u\,du}{(u^2+3)^2}-4\int\frac{du}{(u^2+3)^2}=\frac{1}{2}\int\frac{dv}{v^2}-4\int\frac{\sqrt{3}\sec^2\theta\,d\theta}{9\sec^4\theta}\qquad\left[\begin{array}{c}v=u^2+3\text{ in the first integral;}\\ u=\sqrt{3}\tan\theta\text{ in the second}\end{array}\right]\\
&=\frac{-1}{(2v)}-\frac{4\sqrt{3}}{9}\int\cos^2\theta\,d\theta=\frac{-1}{2(u^2+3)}-\frac{2\sqrt{3}}{9}(\theta+\sin\theta\cos\theta)+C\\
&=\frac{-1}{2(x^2+2x+4)}-\frac{2\sqrt{3}}{9}\left[\tan^{-1}\left(\frac{x+1}{\sqrt{3}}\right)+\frac{\sqrt{3}(x+1)}{x^2+2x+4}\right]+C\\
&=\frac{-1}{2(x^2+2x+4)}-\frac{2\sqrt{3}}{9}\tan^{-1}\left(\frac{x+1}{\sqrt{3}}\right)-\frac{2(x+1)}{3(x^2+2x+4)}+C
\end{aligned}$$

39. Let $u=\sqrt{x+1}$. Then $x=u^2-1$, $dx=2u\,du$ $\Rightarrow$

$$\int\frac{dx}{x\sqrt{x+1}}=\int\frac{2u\,du}{(u^2-1)\,u}=2\int\frac{du}{u^2-1}=\ln\left|\frac{u-1}{u+1}\right|+C=\ln\left|\frac{\sqrt{x+1}-1}{\sqrt{x+1}+1}\right|+C.$$

41. Let $u=\sqrt{x}$, so $u^2=x$ and $dx=2u\,du$. Thus,

$$\begin{aligned}
\int_9^{16}\frac{\sqrt{x}}{x-4}\,dx&=\int_3^4\frac{u}{u^2-4}2u\,du=2\int_3^4\frac{u^2}{u^2-4}\,du=2\int_3^4\left(1+\frac{4}{u^2-4}\right)du\qquad\text{[by long division]}\\
&=2+8\int_3^4\frac{du}{(u+2)(u-2)}.\ (*)
\end{aligned}$$

Multiply $\dfrac{1}{(u+2)(u-2)}=\dfrac{A}{u+2}+\dfrac{B}{u-2}$ by $(u+2)(u-2)$ to get $1=A(u-2)+B(u+2)$. Equating coefficients we get $A+B=0$ and $-2A+2B=1$. Solving gives us $B=\frac{1}{4}$ and $A=-\frac{1}{4}$, so $\dfrac{1}{(u+2)(u-2)}=\dfrac{-1/4}{u+2}+\dfrac{1/4}{u-2}$ and $(*)$ is

$$\begin{aligned}
2+8\int_3^4\left(\frac{-1/4}{u+2}+\frac{1/4}{u-2}\right)du&=2+8\left[-\tfrac{1}{4}\ln|u+2|+\tfrac{1}{4}\ln|u-2|\right]_3^4\\
&=2+\Big[2\ln|u-2|-2\ln|u+2|\Big]_3^4=2+2\left[\ln\left|\frac{u-2}{u+2}\right|\right]_3^4\\
&=2+2\left(\ln\tfrac{2}{6}-\ln\tfrac{1}{5}\right)=2+2\ln\tfrac{2/6}{1/5}\\
&=2+2\ln\tfrac{5}{3}\ \text{ or }\ 2+\ln\left(\tfrac{5}{3}\right)^2=2+\ln\tfrac{25}{9}
\end{aligned}$$

43. Let $u=\sqrt[3]{x^2+1}$. Then $x^2=u^3-1$, $2x\,dx=3u^2\,du$ $\Rightarrow$

$$\begin{aligned}
\int\frac{x^3\,dx}{\sqrt[3]{x^2+1}}&=\int\frac{(u^3-1)\frac{3}{2}u^2\,du}{u}=\tfrac{3}{2}\int(u^4-u)\,du=\tfrac{3}{10}u^5-\tfrac{3}{4}u^2+C\\
&=\tfrac{3}{10}(x^2+1)^{5/3}-\tfrac{3}{4}(x^2+1)^{2/3}+C
\end{aligned}$$

45. If we were to substitute $u = \sqrt{x}$, then the square root would disappear but a cube root would remain. On the other hand, the substitution $u = \sqrt[3]{x}$ would eliminate the cube root but leave a square root. We can eliminate both roots by means of the substitution $u = \sqrt[6]{x}$. (Note that 6 is the least common multiple of 2 and 3.)

Let $u = \sqrt[6]{x}$. Then $x = u^6$, so $dx = 6u^5\,du$ and $\sqrt{x} = u^3$, $\sqrt[3]{x} = u^2$. Thus,

$$\int \frac{dx}{\sqrt{x} - \sqrt[3]{x}} = \int \frac{6u^5\,du}{u^3 - u^2} = 6\int \frac{u^5}{u^2(u-1)}\,du = 6\int \frac{u^3}{u-1}\,du$$

$$= 6\int \left(u^2 + u + 1 + \frac{1}{u-1}\right) du \qquad \text{[by long division]}$$

$$= 6\left(\tfrac{1}{3}u^3 + \tfrac{1}{2}u^2 + u + \ln|u-1|\right) + C = 2\sqrt{x} + 3\sqrt[3]{x} + 6\sqrt[6]{x} + 6\ln\left|\sqrt[6]{x} - 1\right| + C$$

47. Let $u = e^x$. Then $x = \ln u$, $dx = \dfrac{du}{u}$ $\Rightarrow$

$$\int \frac{e^{2x}\,dx}{e^{2x} + 3e^x + 2} = \int \frac{u^2\,(du/u)}{u^2 + 3u + 2} = \int \frac{u\,du}{(u+1)(u+2)} = \int \left[\frac{-1}{u+1} + \frac{2}{u+2}\right] du$$

$$= 2\ln|u+2| - \ln|u+1| + C = \ln\left[(e^x+2)^2/(e^x+1)\right] + C$$

49. Let $u = \ln(x^2 - x + 2)$, $dv = dx$. Then $du = \dfrac{2x-1}{x^2 - x + 2}\,dx$, $v = x$, and (by integration by parts)

$$\int \ln(x^2 - x + 2)\,dx = x\ln(x^2 - x + 2) - \int \frac{2x^2 - x}{x^2 - x + 2}\,dx = x\ln(x^2 - x + 2) - \int \left(2 + \frac{x-4}{x^2 - x + 2}\right) dx$$

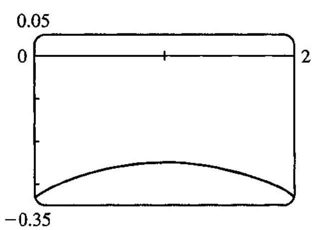
$$= x\ln(x^2 - x + 2) - 2x - \int \frac{\frac{1}{2}(2x-1)}{x^2 - x + 2}\,dx + \frac{7}{2}\int \frac{dx}{(x - \frac{1}{2})^2 + \frac{7}{4}}$$

$$= x\ln(x^2 - x + 2) - 2x - \frac{1}{2}\ln(x^2 - x + 2) + \frac{7}{2}\int \frac{\frac{\sqrt{7}}{2}\,du}{\frac{7}{4}(u^2+1)} \qquad \left[\begin{array}{c}\text{where } x - \frac{1}{2} = \frac{\sqrt{7}}{2}u,\\ dx = \frac{\sqrt{7}}{2}\,du,\\ (x - \frac{1}{2})^2 + \frac{7}{4} = \frac{7}{4}(u^2+1)\end{array}\right]$$

$$= (x - \tfrac{1}{2})\ln(x^2 - x + 2) - 2x + \sqrt{7}\tan^{-1}u + C$$

$$= (x - \tfrac{1}{2})\ln(x^2 - x + 2) - 2x + \sqrt{7}\tan^{-1}\frac{2x-1}{\sqrt{7}} + C$$

51.



From the graph, we see that the integral will be negative, and we guess that the area is about the same as that of a rectangle with width 2 and height 0.3, so we estimate the integral to be $-(2 \cdot 0.3) = -0.6$. Now

$$\frac{1}{x^2 - 2x - 3} = \frac{1}{(x-3)(x+1)} = \frac{A}{x-3} + \frac{B}{x+1} \quad \Leftrightarrow$$

$1 = (A+B)x + A - 3B$, so $A = -B$ and $A - 3B = 1$ $\Leftrightarrow$ $A = \frac{1}{4}$ and $B = -\frac{1}{4}$, so the integral becomes

$$\int_0^2 \frac{dx}{x^2 - 2x - 3} = \frac{1}{4}\int_0^2 \frac{dx}{x-3} - \frac{1}{4}\int_0^2 \frac{dx}{x+1} = \tfrac{1}{4}\Big[\ln|x-3| - \ln|x+1|\Big]_0^2$$

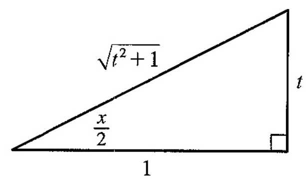
$$= \frac{1}{4}\left[\ln\left|\frac{x-3}{x+1}\right|\right]_0^2 = \tfrac{1}{4}\left(\ln\tfrac{1}{3} - \ln 3\right) = -\tfrac{1}{2}\ln 3 \approx -0.55$$

53. $\displaystyle\int \frac{dx}{x^2-2x} = \int \frac{dx}{(x-1)^2-1} = \int \frac{du}{u^2-1}$ [put $u = x - 1$]

$$= \frac{1}{2}\ln\left|\frac{u-1}{u+1}\right| + C \quad \text{[by Equation 6]} \quad = \frac{1}{2}\ln\left|\frac{x-2}{x}\right| + C$$

55. (a) If $t = \tan\left(\frac{x}{2}\right)$, then $\frac{x}{2} = \tan^{-1} t$. The figure gives

$$\cos\left(\frac{x}{2}\right) = \frac{1}{\sqrt{1+t^2}} \text{ and } \sin\left(\frac{x}{2}\right) = \frac{t}{\sqrt{1+t^2}}.$$



(b) $\cos x = \cos\left(2 \cdot \frac{x}{2}\right) = 2\cos^2\left(\frac{x}{2}\right) - 1$

$$= 2\left(\frac{1}{\sqrt{1+t^2}}\right)^2 - 1 = \frac{2}{1+t^2} - 1 = \frac{1-t^2}{1+t^2}$$

$$\sin x = \sin\left(2 \cdot \frac{x}{2}\right) = 2\sin\left(\frac{x}{2}\right)\cos\left(\frac{x}{2}\right) = 2\,\frac{t}{\sqrt{1+t^2}}\,\frac{1}{\sqrt{1+t^2}} = \frac{2t}{1+t^2}$$

(c) $\frac{x}{2} = \arctan t \;\Rightarrow\; x = 2\arctan t \;\Rightarrow\; dx = \frac{2}{1+t^2}\,dt$

57. Let $t = \tan(x/2)$. Then, using the expressions in Exercise 55, we have

$$\int \frac{1}{3\sin x - 4\cos x}\,dx = \int \frac{1}{3\left(\frac{2t}{1+t^2}\right) - 4\left(\frac{1-t^2}{1+t^2}\right)}\,\frac{2\,dt}{1+t^2} = 2\int \frac{dt}{3(2t) - 4(1-t^2)} = \int \frac{dt}{2t^2+3t-2}$$

$$= \int \frac{dt}{(2t-1)(t+2)} = \int \left[\frac{2}{5}\frac{1}{2t-1} - \frac{1}{5}\frac{1}{t+2}\right] dt \quad \text{[using partial fractions]}$$

$$= \tfrac{1}{5}\Big[\ln|2t-1| - \ln|t+2|\Big] + C = \frac{1}{5}\ln\left|\frac{2t-1}{t+2}\right| + C = \frac{1}{5}\ln\left|\frac{2\tan(x/2)-1}{\tan(x/2)+2}\right| + C$$

59. Let $t = \tan(x/2)$. Then, by Exercise 55,

$$\int \frac{dx}{2\sin x + \sin 2x} = \frac{1}{2}\int \frac{dx}{\sin x + \sin x\cos x} = \frac{1}{2}\int \frac{2\,dt/(1+t^2)}{2t/(1+t^2) + 2t(1-t^2)/(1+t^2)^2}$$

$$= \frac{1}{2}\int \frac{(1+t^2)\,dt}{t(1+t^2) + t(1-t^2)} = \frac{1}{4}\int \frac{(1+t^2)\,dt}{t} = \frac{1}{4}\int\left(\frac{1}{t} + t\right) dt$$

$$= \tfrac{1}{4}\ln|t| + \tfrac{1}{8}t^2 + C = \tfrac{1}{4}\ln\left|\tan\left(\tfrac{1}{2}x\right)\right| + \tfrac{1}{8}\tan^2\left(\tfrac{1}{2}x\right) + C$$

61. $\frac{x+1}{x-1} = 1 + \frac{2}{x-1} > 0$ for $2 \le x \le 3$, so

$$\text{area} = \int_2^3 \left[1 + \frac{2}{x-1}\right] dx = \Big[x + 2\ln|x-1|\Big]_2^3 = (3 + 2\ln 2) - (2 + 2\ln 1) = 1 + 2\ln 2.$$

63. $\dfrac{P+S}{P[(r-1)P-S]} = \dfrac{A}{P} + \dfrac{B}{(r-1)P-S} \Rightarrow P+S = A[(r-1)P-S] + BP = [(r-1)A+B]P - AS$

$\Rightarrow (r-1)A + B = 1, -A = 1 \Rightarrow A = -1, B = r$. Now

$$t = \int \frac{P+S}{P[(r-1)P-S]}\,dP = \int \left[\frac{-1}{P} + \frac{r}{(r-1)P-S}\right] dP = -\int \frac{dP}{P} + \frac{r}{r-1}\int \frac{r-1}{(r-1)P-S}\,dP$$

so $t = -\ln P + \dfrac{r}{r-1}\ln|(r-1)P-S| + C$. Here $r = 0.10$ and $S = 900$, so

$$t = -\ln P + \tfrac{0.1}{-0.9}\ln|-0.9P - 900| + C = -\ln P - \tfrac{1}{9}\ln(|-1|\,|0.9P+900|)$$
$$= -\ln P - \tfrac{1}{9}\ln(0.9P+900) + C$$

When $t = 0$, $P = 10{,}000$, so $0 = -\ln 10{,}000 - \frac{1}{9}\ln(9900) + C$. Thus, $C = \ln 10{,}000 + \frac{1}{9}\ln 9900$ [≈ 10.2326], so our equation becomes

$$t = \ln 10{,}000 - \ln P + \tfrac{1}{9}\ln 9900 - \tfrac{1}{9}\ln(0.9P+900) = \ln\frac{10{,}000}{P} + \frac{1}{9}\ln\frac{9900}{0.9P+900}$$

$$= \ln\frac{10{,}000}{P} + \frac{1}{9}\ln\frac{1100}{0.1P+100} = \ln\frac{10{,}000}{P} + \frac{1}{9}\ln\frac{11{,}000}{P+1000}$$

65. (a) In Maple, we define $f(x)$, and then use `convert(f,parfrac,x);` to obtain

$$f(x) = \frac{24{,}110/4879}{5x+2} - \frac{668/323}{2x+1} - \frac{9438/80{,}155}{3x-7} + \frac{(22{,}098x + 48{,}935)/260{,}015}{x^2+x+5}.$$

In Mathematica, we use the command `Apart`, and in Derive, we use `Expand`.

(b) $$\int f(x)\,dx = \tfrac{24{,}110}{4879}\cdot\tfrac{1}{5}\ln|5x+2| - \tfrac{668}{323}\cdot\tfrac{1}{2}\ln|2x+1| - \tfrac{9438}{80{,}155}\cdot\tfrac{1}{3}\ln|3x-7|$$
$$+ \frac{1}{260{,}015}\int \frac{22{,}098\left(x+\frac{1}{2}\right) + 37{,}886}{\left(x+\frac{1}{2}\right)^2 + \frac{19}{4}}\,dx + C$$
$$= \tfrac{24{,}110}{4879}\cdot\tfrac{1}{5}\ln|5x+2| - \tfrac{668}{323}\cdot\tfrac{1}{2}\ln|2x+1| - \tfrac{9438}{80{,}155}\cdot\tfrac{1}{3}\ln|3x-7|$$
$$+ \tfrac{1}{260{,}015}\left[22{,}098\cdot\tfrac{1}{2}\ln(x^2+x+5) + 37{,}886\cdot\sqrt{\tfrac{4}{19}}\tan^{-1}\left(\tfrac{1}{\sqrt{19/4}}\left(x+\tfrac{1}{2}\right)\right)\right] + C$$
$$= \tfrac{4822}{4879}\ln|5x+2| - \tfrac{334}{323}\ln|2x+1| - \tfrac{3146}{80{,}155}\ln|3x-7| + \tfrac{11{,}049}{260{,}015}\ln(x^2+x+5)$$
$$+ \tfrac{75{,}772}{260{,}015\sqrt{19}}\tan^{-1}\left[\tfrac{1}{\sqrt{19}}(2x+1)\right] + C$$

Using a CAS, we get

$$\frac{4822\ln(5x+2)}{4879} - \frac{334\ln(2x+1)}{323} - \frac{3146\ln(3x-7)}{80{,}155}$$
$$+ \frac{11{,}049\ln(x^2+x+5)}{260{,}015} + \frac{3988\sqrt{19}}{260{,}015}\tan^{-1}\left[\frac{\sqrt{19}}{19}(2x+1)\right]$$

The main difference in this answer is that the absolute value signs and the constant of integration have been omitted. Also, the fractions have been reduced and the denominators rationalized.

67. There are only finitely many values of x where $Q(x) = 0$ (assuming that Q is not the zero polynomial). At all other values of x, $F(x)/Q(x) = G(x)/Q(x)$, so $F(x) = G(x)$. In other words, the values of F and G agree at all except perhaps finitely many values of x. By continuity of F and G, the polynomials F and G must agree at those values of x too.

More explicitly: if a is a value of x such that $Q(a) = 0$, then $Q(x) \neq 0$ for all x sufficiently close to a. Thus,

$$F(a) = \lim_{x\to a} F(x) \quad [\text{by continuity of } F] \quad = \lim_{x\to a} G(x) \quad [\text{whenever } Q(x) \neq 0]$$
$$= G(a) \quad [\text{by continuity of } G]$$

7.5 Strategy for Integration

1. $\displaystyle\int \frac{\sin x + \sec x}{\tan x}\,dx = \int\left(\frac{\sin x}{\tan x} + \frac{\sec x}{\tan x}\right)dx = \int(\cos x + \csc x)\,dx = \sin x + \ln|\csc x - \cot x| + C$

3. $\displaystyle\int_0^2 \frac{2t}{(t-3)^2}\,dt = \int_{-3}^{-1} \frac{2(u+3)}{u^2}\,du \quad [u = t-3,\ du = dt] \quad = \int_{-3}^{-1}\left(\frac{2}{u} + \frac{6}{u^2}\right)du = \left[2\ln|u| - \frac{6}{u}\right]_{-3}^{-1}$

$= (2\ln 1 + 6) - (2\ln 3 + 2) = 4 - 2\ln 3$ or $4 - \ln 9$

5. Let $u = \arctan y$. Then $du = \dfrac{dy}{1+y^2} \Rightarrow \displaystyle\int_{-1}^{1}\frac{e^{\arctan y}}{1+y^2}\,dy = \int_{-\pi/4}^{\pi/4} e^u\,du = \left[e^u\right]_{-\pi/4}^{\pi/4} = e^{\pi/4} - e^{-\pi/4}$.

7. $\displaystyle\int_1^3 r^4 \ln r\,dr \quad \left[\begin{matrix} u = \ln r, & dv = r^4\,dr, \\ du = \frac{dr}{r} & v = \frac{1}{5}r^5 \end{matrix}\right] = \left[\frac{1}{5}r^5 \ln r\right]_1^3 - \int_1^3 \frac{1}{5}r^4\,dr = \frac{243}{5}\ln 3 - 0 - \left[\frac{1}{25}r^5\right]_1^3$

$= \frac{243}{5}\ln 3 - \left(\frac{243}{25} - \frac{1}{25}\right) = \frac{243}{5}\ln 3 - \frac{242}{25}$

9. $\displaystyle\int \frac{x-1}{x^2-4x+5}\,dx = \int \frac{(x-2)+1}{(x-2)^2+1}\,dx = \int\left(\frac{u}{u^2+1} + \frac{1}{u^2+1}\right)du \quad [u = x-2,\ du = dx]$

$= \frac{1}{2}\ln(u^2+1) + \tan^{-1} u + C = \frac{1}{2}\ln(x^2 - 4x + 5) + \tan^{-1}(x-2) + C$

11. $\int \sin^3\theta\,\cos^5\theta\,d\theta = \int \cos^5\theta\,\sin^2\theta\,\sin\theta\,d\theta = -\int \cos^5\theta\,(1-\cos^2\theta)(-\sin\theta)\,d\theta$

$= -\int u^5(1-u^2)\,du \quad \left[\begin{matrix} u = \cos\theta, \\ du = -\sin\theta\,d\theta \end{matrix}\right]$

$= \int (u^7 - u^5)\,du = \frac{1}{8}u^8 - \frac{1}{6}u^6 + C = \frac{1}{8}\cos^8\theta - \frac{1}{6}\cos^6\theta + C$

Another solution:

$\int \sin^3\theta\,\cos^5\theta\,d\theta = \int \sin^3\theta\,(\cos^2\theta)^2\cos\theta\,d\theta = \int \sin^3\theta\,(1-\sin^2\theta)^2\cos\theta\,d\theta$

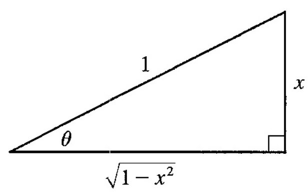
$= \int u^3(1-u^2)^2\,du \quad \left[\begin{matrix} u = \sin\theta, \\ du = \cos\theta\,d\theta \end{matrix}\right] \quad = \int u^3(1 - 2u^2 + u^4)\,du$

$= \int (u^3 - 2u^5 + u^7)\,du = \frac{1}{4}u^4 - \frac{1}{3}u^6 + \frac{1}{8}u^8 + C = \frac{1}{4}\sin^4\theta - \frac{1}{3}\sin^6\theta + \frac{1}{8}\sin^8\theta + C$

13. Let $x = \sin\theta$, where $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$. Then $dx = \cos\theta\,d\theta$ and $(1-x^2)^{1/2} = \cos\theta$, so

$\displaystyle\int \frac{dx}{(1-x^2)^{3/2}} = \int \frac{\cos\theta\,d\theta}{(\cos\theta)^3} = \int \sec^2\theta\,d\theta$

$\displaystyle= \tan\theta + C = \frac{x}{\sqrt{1-x^2}} + C$



15. Let $u = 1 - x^2 \Rightarrow du = -2x\,dx$. Then

$$\int_0^{1/2} \frac{x}{\sqrt{1-x^2}}\,dx = -\frac{1}{2}\int_1^{3/4} \frac{1}{\sqrt{u}}\,du = \frac{1}{2}\int_{3/4}^{1} u^{-1/2}\,du = \tfrac{1}{2}\left[2u^{1/2}\right]_{3/4}^{1} = \left[\sqrt{u}\right]_{3/4}^{1} = 1 - \tfrac{\sqrt{3}}{2}$$

17. $\int x\sin^2 x\,dx$ $\quad \left[\begin{array}{ll} u = x, & dv = \sin^2 x\,dx, \\ du = dx & v = \int \sin^2 x\,dx = \int \frac{1}{2}(1-\cos 2x)\,dx = \frac{1}{2}x - \frac{1}{2}\sin x\cos x \end{array}\right]$

$$= \tfrac{1}{2}x^2 - \tfrac{1}{2}x\sin x\cos x - \int\left(\tfrac{1}{2}x - \tfrac{1}{2}\sin x\cos x\right)dx$$

$$= \tfrac{1}{2}x^2 - \tfrac{1}{2}x\sin x\cos x - \tfrac{1}{4}x^2 + \tfrac{1}{4}\sin^2 x + C = \tfrac{1}{4}x^2 - \tfrac{1}{2}x\sin x\cos x + \tfrac{1}{4}\sin^2 x + C$$

Note: $\int \sin x\cos x\,dx = \int s\,ds = \frac{1}{2}s^2 + C$ [where $s = \sin x$, $ds = \cos x\,dx$].

A slightly different method is to write $\int x\sin^2 x\,dx = \int x\cdot\frac{1}{2}(1-\cos 2x)\,dx = \frac{1}{2}\int x\,dx - \frac{1}{2}\int x\cos 2x\,dx$. If we evaluate the second integral by parts, we arrive at the equivalent answer $\frac{1}{4}x^2 - \frac{1}{4}x\sin 2x - \frac{1}{8}\cos 2x + C$.

19. Let $u = e^x$. Then $\int e^{x+e^x}\,dx = \int e^{e^x}e^x\,dx = \int e^u\,du = e^u + C = e^{e^x} + C$.

21. Integrate by parts three times, first with $u = t^3$, $dv = e^{-2t}\,dt$:

$$\int t^3e^{-2t}\,dt = -\tfrac{1}{2}t^3e^{-2t} + \tfrac{1}{2}\int 3t^2e^{-2t}\,dt = -\tfrac{1}{2}t^3e^{-2t} - \tfrac{3}{4}t^2e^{-2t} + \tfrac{1}{2}\int 3te^{-2t}\,dt$$

$$= -e^{-2t}\left[\tfrac{1}{2}t^3 + \tfrac{3}{4}t^2\right] - \tfrac{3}{4}te^{-2t} + \tfrac{3}{4}\int e^{-2t}\,dt = -e^{-2t}\left[\tfrac{1}{2}t^3 + \tfrac{3}{4}t^2 + \tfrac{3}{4}t + \tfrac{3}{8}\right] + C$$

$$= -\tfrac{1}{8}e^{-2t}\left(4t^3 + 6t^2 + 6t + 3\right) + C$$

23. Let $u = 1 + \sqrt{x}$. Then $x = (u-1)^2$, $dx = 2(u-1)\,du \Rightarrow$

$$\int_0^1 \left(1+\sqrt{x}\right)^8 dx = \int_1^2 u^8\cdot 2(u-1)\,du = 2\int_1^2 \left(u^9 - u^8\right)du = \left[\tfrac{1}{5}u^{10} - 2\cdot\tfrac{1}{9}u^9\right]_1^2$$

$$= \tfrac{1024}{5} - \tfrac{1024}{9} - \tfrac{1}{5} + \tfrac{2}{9} = \tfrac{4097}{45}$$

25. $\dfrac{3x^2-2}{x^2-2x-8} = 3 + \dfrac{6x+22}{(x-4)(x+2)} = 3 + \dfrac{A}{x-4} + \dfrac{B}{x+2} \Rightarrow 6x + 22 = A(x+2) + B(x-4)$. Setting $x = 4$ gives $46 = 6A$, so $A = \frac{23}{3}$. Setting $x = -2$ gives $10 = -6B$, so $B = -\frac{5}{3}$. Now

$$\int \frac{3x^2-2}{x^2-2x-8}\,dx = \int\left(3 + \frac{23/3}{x-4} - \frac{5/3}{x+2}\right)dx = 3x + \tfrac{23}{3}\ln|x-4| - \tfrac{5}{3}\ln|x+2| + C.$$

27. Let $u = \ln(\sin x)$. Then $du = \cot x\,dx \Rightarrow \int \cot x\ln(\sin x)\,dx = \int u\,du = \frac{1}{2}u^2 + C = \frac{1}{2}[\ln(\sin x)]^2 + C$.

29. $\displaystyle\int_0^5 \frac{3w-1}{w+2}\,dw = \int_0^5\left(3 - \frac{7}{w+2}\right)dw = \Big[3w - 7\ln|w+2|\Big]_0^5$

$$= 15 - 7\ln 7 + 7\ln 2 = 15 + 7(\ln 2 - \ln 7) = 15 + 7\ln\tfrac{2}{7}$$

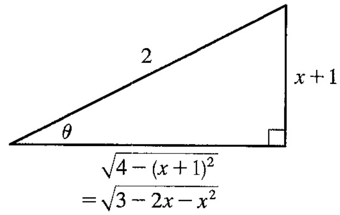
31. As in Example 5,

$$\int\sqrt{\frac{1+x}{1-x}}\,dx = \int\frac{\sqrt{1+x}}{\sqrt{1-x}}\cdot\frac{\sqrt{1+x}}{\sqrt{1+x}}\,dx = \int\frac{1+x}{\sqrt{1-x^2}}\,dx = \int\frac{dx}{\sqrt{1-x^2}} + \int\frac{x\,dx}{\sqrt{1-x^2}}$$

$$= \sin^{-1}x - \sqrt{1-x^2} + C$$

Another method: Substitute $u = \sqrt{(1+x)/(1-x)}$.

33. $3 - 2x - x^2 = -(x^2 + 2x + 1) + 4 = 4 - (x+1)^2$. Let $x + 1 = 2\sin\theta$, where $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$. Then $dx = 2\cos\theta\, d\theta$ and



$$\begin{aligned}\int \sqrt{3 - 2x - x^2}\, dx &= \int \sqrt{4 - (x+1)^2}\, dx = \int \sqrt{4 - 4\sin^2\theta}\, 2\cos\theta\, d\theta \\ &= 4\int \cos^2\theta\, d\theta = 2\int (1 + \cos 2\theta)\, d\theta \\ &= 2\theta + \sin 2\theta + C = 2\theta + 2\sin\theta\cos\theta + C \\ &= 2\sin^{-1}\left(\frac{x+1}{2}\right) + 2\cdot\frac{x+1}{2}\cdot\frac{\sqrt{3 - 2x - x^2}}{2} + C \\ &= 2\sin^{-1}\left(\frac{x+1}{2}\right) + \frac{x+1}{2}\sqrt{3 - 2x - x^2} + C\end{aligned}$$

35. Because $f(x) = x^8 \sin x$ is the product of an even function and an odd function, it is odd. Therefore, $\int_{-1}^{1} x^8 \sin x\, dx = 0$ [by (5.5.7)(b)].

37. $\int_0^{\pi/4} \cos^2\theta \tan^2\theta\, d\theta = \int_0^{\pi/4} \sin^2\theta\, d\theta = \int_0^{\pi/4} \frac{1}{2}(1 - \cos 2\theta)\, d\theta = \left[\frac{1}{2}\theta - \frac{1}{4}\sin 2\theta\right]_0^{\pi/4}$
$= \left(\frac{\pi}{8} - \frac{1}{4}\right) - (0 - 0) = \frac{\pi}{8} - \frac{1}{4}$

39. Let $u = 1 - x^2$. Then $du = -2x\, dx \Rightarrow$

$$\begin{aligned}\int \frac{x\, dx}{1 - x^2 + \sqrt{1 - x^2}} &= -\frac{1}{2}\int \frac{du}{u + \sqrt{u}} = -\int \frac{v\, dv}{v^2 + v} \quad [v = \sqrt{u},\ u = v^2,\ du = 2v\, dv] \\ &= -\int \frac{dv}{v + 1} = -\ln|v + 1| + C = -\ln\left(\sqrt{1 - x^2} + 1\right) + C\end{aligned}$$

41. Let $u = \theta$, $dv = \tan^2\theta\, d\theta = \left(\sec^2\theta - 1\right) d\theta \Rightarrow du = d\theta$ and $v = \tan\theta - \theta$. So

$$\begin{aligned}\int \theta\tan^2\theta\, d\theta &= \theta(\tan\theta - \theta) - \int (\tan\theta - \theta)\, d\theta = \theta\tan\theta - \theta^2 - \ln|\sec\theta| + \tfrac{1}{2}\theta^2 + C \\ &= \theta\tan\theta - \tfrac{1}{2}\theta^2 - \ln|\sec\theta| + C\end{aligned}$$

43. Let $u = 1 + e^x$, so that $du = e^x\, dx$. Then
$\int e^x\sqrt{1 + e^x}\, dx = \int u^{1/2}\, du = \frac{2}{3}u^{3/2} + C = \frac{2}{3}(1 + e^x)^{3/2} + C$.
Or: Let $u = \sqrt{1 + e^x}$, so that $u^2 = 1 + e^x$ and $2u\, du = e^x\, dx$. Then
$\int e^x\sqrt{1 + e^x}\, dx = \int u \cdot 2u\, du = \int 2u^2 du = \frac{2}{3}u^3 + C = \frac{2}{3}(1 + e^x)^{3/2} + C$.

45. Let $t = x^3$. Then $dt = 3x^2\, dx \Rightarrow I = \int x^5 e^{-x^3}\, dx = \frac{1}{3}\int te^{-t}\, dt$. Now integrate by parts with $u = t$, $dv = e^{-t}\, dt$: $I = -\frac{1}{3}te^{-t} + \frac{1}{3}\int e^{-t}\, dt = -\frac{1}{3}te^{-t} - \frac{1}{3}e^{-t} + C = -\frac{1}{3}e^{-x^3}(x^3 + 1) + C$.

47. $$\begin{aligned}\int \frac{x + a}{x^2 + a^2}\, dx &= \frac{1}{2}\int \frac{2x\, dx}{x^2 + a^2} + a\int \frac{dx}{x^2 + a^2} = \tfrac{1}{2}\ln(x^2 + a^2) + a\cdot\frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C \\ &= \ln\sqrt{x^2 + a^2} + \tan^{-1}(x/a) + C\end{aligned}$$

49. Let $u = \sqrt{4x + 1} \Rightarrow u^2 = 4x + 1 \Rightarrow 2u\, du = 4\, dx \Rightarrow dx = \frac{1}{2}u\, du$. So

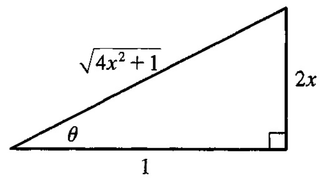
$$\begin{aligned}\int \frac{1}{x\sqrt{4x + 1}}\, dx &= \int \frac{\frac{1}{2}u\, du}{\frac{1}{4}(u^2 - 1)\, u} = 2\int \frac{du}{u^2 - 1} = 2\left(\tfrac{1}{2}\right)\ln\left|\frac{u - 1}{u + 1}\right| + C \quad \text{[by Formula 19]} \\ &= \ln\left|\frac{\sqrt{4x + 1} - 1}{\sqrt{4x + 1} + 1}\right| + C\end{aligned}$$

51. Let $2x = \tan\theta \quad\Rightarrow\quad x = \frac{1}{2}\tan\theta,\ dx = \frac{1}{2}\sec^2\theta\,d\theta,\ \sqrt{4x^2+1} = \sec\theta$, so

$$\int \frac{dx}{x\sqrt{4x^2+1}} = \int \frac{\frac{1}{2}\sec^2\theta\,d\theta}{\frac{1}{2}\tan\theta\sec\theta} = \int \frac{\sec\theta}{\tan\theta}\,d\theta = \int \csc\theta\,d\theta$$

$$= -\ln|\csc\theta + \cot\theta| + C \qquad [\text{or } \ln|\csc\theta - \cot\theta| + C]$$

$$= -\ln\left|\frac{\sqrt{4x^2+1}}{2x} + \frac{1}{2x}\right| + C \qquad \left[\text{or } \ln\left|\frac{\sqrt{4x^2+1}}{2x} - \frac{1}{2x}\right| + C\right]$$



53. $$\int x^2\sinh(mx)dx = \frac{1}{m}x^2\cosh(mx) - \frac{2}{m}\int x\cosh(mx)\,dx \qquad \begin{bmatrix} u = x^2, & dv = \sinh(mx)\,dx, \\ du = 2x\,dx & v = \frac{1}{m}\cosh(mx) \end{bmatrix}$$

$$= \frac{1}{m}x^2\cosh(mx) - \frac{2}{m}\left(\frac{1}{m}x\sinh(mx) - \frac{1}{m}\int\sinh(mx)\,dx\right) \qquad \begin{bmatrix} U = x, & dV = \cosh(mx)\,dx, \\ dU = dx & V = \frac{1}{m}\sinh(mx) \end{bmatrix}$$

$$= \frac{1}{m}x^2\cosh(mx) - \frac{2}{m^2}x\sinh(mx) + \frac{2}{m^3}\cosh(mx) + C$$

55. Let $u = \sqrt{x+1}$. Then $x = u^2 - 1 \quad\Rightarrow$

$$\int \frac{dx}{x + 4 + 4\sqrt{x+1}} = \int \frac{2u\,du}{u^2 + 3 + 4u} = \int \left[\frac{-1}{u+1} + \frac{3}{u+3}\right] du$$

$$= 3\ln|u+3| - \ln|u+1| + C = 3\ln\left(\sqrt{x+1}+3\right) - \ln\left(\sqrt{x+1}+1\right) + C$$

57. Let $u = \sqrt[3]{x+c}$. Then $x = u^3 - c \quad\Rightarrow$

$$\int x\sqrt[3]{x+c}\,dx = \int (u^3 - c)u\cdot 3u^2\,du = 3\int (u^6 - cu^3)\,du = \tfrac{3}{7}u^7 - \tfrac{3}{4}cu^4 + C$$

$$= \tfrac{3}{7}(x+c)^{7/3} - \tfrac{3}{4}c(x+c)^{4/3} + C$$

59. Let $u = e^x$. Then $x = \ln u,\ dx = du/u \quad\Rightarrow$

$$\int \frac{dx}{e^{3x} - e^x} = \int \frac{du/u}{u^3 - u} = \int \frac{du}{(u-1)u^2(u+1)} = \int \left[\frac{1/2}{u-1} - \frac{1}{u^2} - \frac{1/2}{u+1}\right] du$$

$$= \frac{1}{u} + \frac{1}{2}\ln\left|\frac{u-1}{u+1}\right| + C = e^{-x} + \frac{1}{2}\ln\left|\frac{e^x - 1}{e^x + 1}\right| + C$$

61. Let $u = x^5$. Then $du = 5x^4\,dx \quad\Rightarrow$

$$\int \frac{x^4\,dx}{x^{10} + 16} = \int \frac{\frac{1}{5}\,du}{u^2 + 16} = \tfrac{1}{5}\cdot\tfrac{1}{4}\tan^{-1}\left(\tfrac{1}{4}u\right) + C = \tfrac{1}{20}\tan^{-1}\left(\tfrac{1}{4}x^5\right) + C.$$

63. Let $y = \sqrt{x}$ so that $dy = \dfrac{1}{2\sqrt{x}}\,dx \quad\Rightarrow\quad dx = 2\sqrt{x}\,dy = 2y\,dy$. Then

$$\int \sqrt{x}\,e^{\sqrt{x}}\,dx = \int ye^y(2y\,dy) = \int 2y^2e^y\,dy \qquad \begin{bmatrix} u = 2y^2, & dv = e^y\,dy, \\ du = 4y\,dy & v = e^y \end{bmatrix}$$

$$= 2y^2e^y - \int 4ye^y\,dy \qquad \begin{bmatrix} U = 4y, & dV = e^y\,dy, \\ dU = 4\,dy & V = e^y \end{bmatrix}$$

$$= 2y^2e^y - \left(4ye^y - \int 4e^y\,dy\right) = 2y^2e^y - 4ye^y + 4e^y + C$$

$$= 2(y^2 - 2y + 2)e^y + C = 2(x - 2\sqrt{x} + 2)\,e^{\sqrt{x}} + C$$

65. $\displaystyle\int \frac{dx}{\sqrt{x+1}+\sqrt{x}} = \int \left(\frac{1}{\sqrt{x+1}+\sqrt{x}} \cdot \frac{\sqrt{x+1}-\sqrt{x}}{\sqrt{x+1}-\sqrt{x}}\right) dx = \int \left(\sqrt{x+1}-\sqrt{x}\right) dx$

$$= \tfrac{2}{3}\left[(x+1)^{3/2} - x^{3/2}\right] + C$$

67. Let $u = \sqrt{t}$. Then $du = dt/\left(2\sqrt{t}\right) \Rightarrow$

$$\int_1^3 \frac{\arctan\sqrt{t}}{\sqrt{t}}\, dt = \int_1^{\sqrt{3}} \tan^{-1}u\,(2\,du) = 2\left[u\tan^{-1}u - \tfrac{1}{2}\ln(1+u^2)\right]_1^{\sqrt{3}} \quad \text{[Example 5 in Section 7.1]}$$

$$= 2\left[\left(\sqrt{3}\tan^{-1}\sqrt{3} - \tfrac{1}{2}\ln 4\right) - \left(\tan^{-1}1 - \tfrac{1}{2}\ln 2\right)\right]$$

$$= 2\left[\left(\sqrt{3}\cdot\tfrac{\pi}{3} - \ln 2\right) - \left(\tfrac{\pi}{4} - \tfrac{1}{2}\ln 2\right)\right] = \tfrac{2}{3}\sqrt{3}\,\pi - \tfrac{1}{2}\pi - \ln 2$$

69. Let $u = e^x$. Then $x = \ln u$, $dx = du/u \Rightarrow$

$$\int \frac{e^{2x}}{1+e^x}\, dx = \int \frac{u^2}{1+u}\frac{du}{u} = \int \frac{u}{1+u}\, du = \int \left(1 - \frac{1}{1+u}\right) du$$

$$= u - \ln|1+u| + C = e^x - \ln(1+e^x) + C$$

71. $\displaystyle\frac{x}{x^4+4x^2+3} = \frac{x}{(x^2+3)(x^2+1)} = \frac{Ax+B}{x^2+3} + \frac{Cx+D}{x^2+1} \Rightarrow$

$$x = (Ax+B)(x^2+1) + (Cx+D)(x^2+3) = \left(Ax^3+Bx^2+Ax+B\right) + \left(Cx^3+Dx^2+3Cx+3D\right)$$

$$= (A+C)x^3 + (B+D)x^2 + (A+3C)x + (B+3D) \Rightarrow$$

$A+C=0$, $B+D=0$, $A+3C=1$, $B+3D=0 \Rightarrow A=-\tfrac{1}{2}$, $C=\tfrac{1}{2}$, $B=0$, $D=0$. Thus,

$$\int \frac{x}{x^4+4x^2+3}\, dx = \int \left(\frac{-\frac{1}{2}x}{x^2+3} + \frac{\frac{1}{2}x}{x^2+1}\right) dx$$

$$= -\tfrac{1}{4}\ln(x^2+3) + \tfrac{1}{4}\ln(x^2+1) + C \quad \text{or} \quad \frac{1}{4}\ln\left(\frac{x^2+1}{x^2+3}\right) + C$$

73. $\displaystyle\frac{1}{(x-2)(x^2+4)} = \frac{A}{x-2} + \frac{Bx+C}{x^2+4} \Rightarrow$

$1 = A(x^2+4) + (Bx+C)(x-2) = (A+B)x^2 + (C-2B)x + (4A-2C)$. So $0 = A+B = C-2B$, $1 = 4A-2C$. Setting $x=2$ gives $A = \tfrac{1}{8} \Rightarrow B = -\tfrac{1}{8}$ and $C = -\tfrac{1}{4}$. So

$$\int \frac{1}{(x-2)(x^2+4)}\, dx = \int \left(\frac{\frac{1}{8}}{x-2} + \frac{-\frac{1}{8}x - \frac{1}{4}}{x^2+4}\right) dx = \frac{1}{8}\int \frac{dx}{x-2} - \frac{1}{16}\int \frac{2x\,dx}{x^2+4} - \frac{1}{4}\int \frac{dx}{x^2+4}$$

$$= \tfrac{1}{8}\ln|x-2| - \tfrac{1}{16}\ln(x^2+4) - \tfrac{1}{8}\tan^{-1}(x/2) + C$$

75. $\int \sin x \sin 2x \sin 3x\, dx = \int \sin x \cdot \tfrac{1}{2}[\cos(2x-3x) - \cos(2x+3x)]\, dx = \tfrac{1}{2}\int (\sin x\cos x - \sin x\cos 5x)\, dx$

$$= \tfrac{1}{4}\int \sin 2x\, dx - \tfrac{1}{2}\int \tfrac{1}{2}[\sin(x+5x) + \sin(x-5x)]\, dx$$

$$= -\tfrac{1}{8}\cos 2x - \tfrac{1}{4}\int (\sin 6x - \sin 4x)\, dx = -\tfrac{1}{8}\cos 2x + \tfrac{1}{24}\cos 6x - \tfrac{1}{16}\cos 4x + C$$

77. Let $u = x^{3/2}$ so that $u^2 = x^3$ and $du = \tfrac{3}{2}x^{1/2}\, dx \Rightarrow \sqrt{x}\, dx = \tfrac{2}{3}\, du$. Then

$$\int \frac{\sqrt{x}}{1+x^3}\, dx = \int \frac{\frac{2}{3}}{1+u^2} du = \frac{2}{3}\tan^{-1}u + C = \frac{2}{3}\tan^{-1}(x^{3/2}) + C.$$

79. Let $u = x$, $dv = \sin^2 x \cos x\, dx \quad \Rightarrow \quad du = dx$, $v = \frac{1}{3}\sin^3 x$. Then

$$\begin{aligned}\int x\sin^2 x\, \cos x\, dx &= \tfrac{1}{3}x\sin^3 x - \int \tfrac{1}{3}\sin^3 x\, dx = \tfrac{1}{3}x\sin^3 x - \tfrac{1}{3}\int (1-\cos^2 x)\sin x\, dx \\ &= \frac{1}{3}x\sin^3 x + \frac{1}{3}\int (1-y^2)\, dy \qquad \begin{bmatrix} y = \cos x, \\ dy = -\sin x\, dx \end{bmatrix} \\ &= \tfrac{1}{3}x\sin^3 x + \tfrac{1}{3}y - \tfrac{1}{9}y^3 + C = \tfrac{1}{3}x\sin^3 x + \tfrac{1}{3}\cos x - \tfrac{1}{9}\cos^3 x + C\end{aligned}$$

81. The function $y = 2xe^{x^2}$ *does* have an elementary antiderivative, so we'll use this fact to help evaluate the integral.

$$\begin{aligned}\int (2x^2+1)e^{x^2}\, dx &= \int 2x^2 e^{x^2}\, dx + \int e^{x^2}\, dx = \int x(2xe^{x^2})\, dx + \int e^{x^2}\, dx \\ &= xe^{x^2} - \int e^{x^2}\, dx + \int e^{x^2}\, dx \quad \begin{bmatrix} u = x, & dv = 2xe^{x^2}\, dx, \\ du = dx & v = e^{x^2} \end{bmatrix} \quad = xe^{x^2} + C\end{aligned}$$

7.6 Integration Using Tables and Computer Algebra Systems

Keep in mind that there are several ways to approach many of these exercises, and different methods can lead to different forms of the answer.

1. We could make the substitution $u = \sqrt{2}\, x$ to obtain the radical $\sqrt{7-u^2}$ and then use Formula 33 with $a = \sqrt{7}$. Alternatively, we will factor $\sqrt{2}$ out of the radical and use $a = \sqrt{\frac{7}{2}}$.

$$\begin{aligned}\int \frac{\sqrt{7-2x^2}}{x^2}\, dx &= \sqrt{2}\int \frac{\sqrt{\frac{7}{2}-x^2}}{x^2}\, dx \overset{33}{=} \sqrt{2}\left[-\frac{1}{x}\sqrt{\tfrac{7}{2}-x^2} - \sin^{-1}\frac{x}{\sqrt{\frac{7}{2}}}\right] + C \\ &= -\frac{1}{x}\sqrt{7-2x^2} - \sqrt{2}\sin^{-1}\left(\sqrt{\tfrac{2}{7}}\, x\right) + C\end{aligned}$$

3. Let $u = \pi x \quad \Rightarrow \quad du = \pi\, dx$, so

$$\begin{aligned}\int \sec^3(\pi x)\, dx &= \tfrac{1}{\pi}\int \sec^3 u\, du \overset{71}{=} \tfrac{1}{\pi}\left(\tfrac{1}{2}\sec u\tan u + \tfrac{1}{2}\ln|\sec u + \tan u|\right) + C \\ &= \tfrac{1}{2\pi}\sec \pi x\tan \pi x + \tfrac{1}{2\pi}\ln|\sec \pi x + \tan \pi x| + C\end{aligned}$$

5. $\displaystyle\int_0^1 2x\cos^{-1}x\, dx \overset{91}{=} 2\left[\frac{2x^2-1}{4}\cos^{-1}x - \frac{x\sqrt{1-x^2}}{4}\right]_0^1 = 2\left[\left(\tfrac{1}{4}\cdot 0 - 0\right) - \left(-\tfrac{1}{4}\cdot\tfrac{\pi}{2} - 0\right)\right] = 2\left(\tfrac{\pi}{8}\right) = \tfrac{\pi}{4}$

7. By Formula 99 with $a = -3$ and $b = 4$,

$$\int e^{-3x}\cos 4x\, dx = \frac{e^{-3x}}{(-3)^2+4^2}(-3\cos 4x + 4\sin 4x) + C = \frac{e^{-3x}}{25}(-3\cos 4x + 4\sin 4x) + C.$$

9. Let $u = 2x$ and $a = 3$. Then $du = 2\, dx$ and

$$\begin{aligned}\int \frac{dx}{x^2\sqrt{4x^2+9}} &= \int \frac{\frac{1}{2}\, du}{\frac{u^2}{4}\sqrt{u^2+a^2}} = 2\int \frac{du}{u^2\sqrt{a^2+u^2}} \overset{28}{=} -2\,\frac{\sqrt{a^2+u^2}}{a^2 u} + C \\ &= -2\,\frac{\sqrt{4x^2+9}}{9\cdot 2x} + C = -\frac{\sqrt{4x^2+9}}{9x} + C\end{aligned}$$

11. $\displaystyle\int_{-1}^0 t^2 e^{-t}\, dt \overset{97}{=} \left[\frac{1}{-1}t^2 e^{-t}\right]_{-1}^0 - \frac{2}{-1}\int_{-1}^0 te^{-t}\, dt = e + 2\int_{-1}^0 te^{-t}\, dt \overset{96}{=} e + 2\left[\frac{1}{(-1)^2}(-t-1)\, e^{-t}\right]_{-1}^0$

$$= e + 2\left[-e^0 + 0\right] = e - 2$$

13. $\displaystyle\int \frac{\tan^3(1/z)}{z^2}\,dz \quad \begin{bmatrix} u = 1/z, \\ du = -dz/z^2 \end{bmatrix} \quad = -\int \tan^3 u\,du \overset{69}{=} -\frac{1}{2}\tan^2 u - \ln|\cos u| + C$

$$= -\frac{1}{2}\tan^2\left(\frac{1}{z}\right) - \ln\left|\cos\left(\frac{1}{z}\right)\right| + C$$

15. Let $u = e^x$. Then $du = e^x\,dx$, so $\int e^x \operatorname{sech}(e^x)\,dx = \int \operatorname{sech} u\,du \overset{107}{=} \tan^{-1}|\sinh u| + C = \tan^{-1}[\sinh(e^x)] + C$

17. Let $z = 6 + 4y - 4y^2 = 6 - (4y^2 - 4y + 1) + 1 = 7 - (2y-1)^2$, $u = 2y - 1$, and $a = \sqrt{7}$. Then $z = a^2 - u^2$, $du = 2\,dy$, and

$$\begin{aligned}
\int y\sqrt{6+4y-4y^2}\,dy &= \int y\sqrt{z}\,dy = \int \tfrac{1}{2}(u+1)\sqrt{a^2-u^2}\,\tfrac{1}{2}\,du \\
&= \tfrac{1}{4}\int u\sqrt{a^2-u^2}\,du + \tfrac{1}{4}\int \sqrt{a^2-u^2}\,du \\
&= \tfrac{1}{4}\int \sqrt{a^2-u^2}\,du - \tfrac{1}{8}\int (-2u)\sqrt{a^2-u^2}\,du \\
&\overset{30}{=} \frac{u}{8}\sqrt{a^2-u^2} + \frac{a^2}{8}\sin^{-1}\left(\frac{u}{a}\right) - \frac{1}{8}\int \sqrt{w}\,dw \qquad \begin{bmatrix} w = a^2-u^2, \\ dw = -2u\,du \end{bmatrix} \\
&= \frac{2y-1}{8}\sqrt{6+4y-4y^2} + \frac{7}{8}\sin^{-1}\frac{2y-1}{\sqrt{7}} - \frac{1}{8}\cdot\frac{2}{3}w^{3/2} + C \\
&= \frac{2y-1}{8}\sqrt{6+4y-4y^2} + \frac{7}{8}\sin^{-1}\frac{2y-1}{\sqrt{7}} - \frac{1}{12}(6+4y-4y^2)^{3/2} + C.
\end{aligned}$$

This can be rewritten as

$$\begin{aligned}
&\sqrt{6+4y-4y^2}\left[\frac{1}{8}(2y-1) - \frac{1}{12}(6+4y-4y^2)\right] + \frac{7}{8}\sin^{-1}\frac{2y-1}{\sqrt{7}} + C \\
&\qquad = \left(\frac{1}{3}y^2 - \frac{1}{12}y - \frac{5}{8}\right)\sqrt{6+4y-4y^2} + \frac{7}{8}\sin^{-1}\left(\frac{2y-1}{\sqrt{7}}\right) + C \\
&\qquad = \frac{1}{24}(8y^2 - 2y - 15)\sqrt{6+4y-4y^2} + \frac{7}{8}\sin^{-1}\left(\frac{2y-1}{\sqrt{7}}\right) + C
\end{aligned}$$

19. Let $u = \sin x$. Then $du = \cos x\,dx$, so

$$\begin{aligned}
\int \sin^2 x\,\cos x\,\ln(\sin x)\,dx &= \int u^2 \ln u\,du \overset{101}{=} \frac{u^{2+1}}{(2+1)^2}[(2+1)\ln u - 1] + C = \tfrac{1}{9}u^3(3\ln u - 1) + C \\
&= \tfrac{1}{9}\sin^3 x\,[3\ln(\sin x) - 1] + C
\end{aligned}$$

21. Let $u = e^x$ and $a = \sqrt{3}$. Then $du = e^x\,dx$ and

$$\int \frac{e^x}{3 - e^{2x}}\,dx = \int \frac{du}{a^2 - u^2} \overset{19}{=} \frac{1}{2a}\ln\left|\frac{u+a}{u-a}\right| + C = \frac{1}{2\sqrt{3}}\ln\left|\frac{e^x+\sqrt{3}}{e^x-\sqrt{3}}\right| + C.$$

23. $\int \sec^5 x\,dx \overset{77}{=} \frac{1}{4}\tan x\sec^3 x + \frac{3}{4}\int \sec^3 x\,dx \overset{77}{=} \frac{1}{4}\tan x\sec^3 x + \frac{3}{4}\left(\frac{1}{2}\tan x\sec x + \frac{1}{2}\int \sec x\,dx\right)$

$$\overset{14}{=} \tfrac{1}{4}\tan x\sec^3 x + \tfrac{3}{8}\tan x\sec x + \tfrac{3}{8}\ln|\sec x + \tan x| + C$$

25. Let $u = \ln x$ and $a = 2$. Then $du = \dfrac{dx}{x}$ and

$$\begin{aligned}
\int \frac{\sqrt{4+(\ln x)^2}}{x}\,dx &= \int \sqrt{a^2+u^2}\,du \overset{21}{=} \frac{u}{2}\sqrt{a^2+u^2} + \frac{a^2}{2}\ln\left(u + \sqrt{a^2+u^2}\right) + C \\
&= \tfrac{1}{2}(\ln x)\sqrt{4+(\ln x)^2} + 2\ln\left[\ln x + \sqrt{4+(\ln x)^2}\right] + C
\end{aligned}$$

27. Let $u = e^x$. Then $x = \ln u$, $dx = du/u$, so

$$\int \sqrt{e^{2x}-1}\,dx = \int \frac{\sqrt{u^2-1}}{u}\,du \overset{41}{=} \sqrt{u^2-1} - \cos^{-1}(1/u) + C = \sqrt{e^{2x}-1} - \cos^{-1}(e^{-x}) + C.$$

29. $\displaystyle\int \frac{x^4\,dx}{\sqrt{x^{10}-2}} = \int \frac{x^4\,dx}{\sqrt{(x^5)^2-2}} = \frac{1}{5}\int \frac{du}{\sqrt{u^2-2}} \qquad [u = x^5,\, du = 5x^4\,dx]$

$\overset{43}{=} \frac{1}{5}\ln\left|u+\sqrt{u^2-2}\right| + C = \frac{1}{5}\ln\left|x^5+\sqrt{x^{10}-2}\right| + C$

31. Using cylindrical shells, we get

$$V = 2\pi \int_0^2 x \cdot x\sqrt{4-x^2}\,dx = 2\pi \int_0^2 x^2\sqrt{4-x^2}\,dx \overset{31}{=} 2\pi\left[\frac{x}{8}(2x^2-4)\sqrt{4-x^2} + \frac{16}{8}\sin^{-1}\frac{x}{2}\right]_0^2$$

$$= 2\pi\left[(0+2\sin^{-1}1) - (0+2\sin^{-1}0\right] = 2\pi\left(2\cdot\frac{\pi}{2}\right) = 2\pi^2$$

33. (a) $\displaystyle\frac{d}{du}\left[\frac{1}{b^3}\left(a+bu-\frac{a^2}{a+bu} - 2a\ln|a+bu|\right)+C\right] = \frac{1}{b^3}\left[b + \frac{ba^2}{(a+bu)^2} - \frac{2ab}{(a+bu)}\right]$

$$= \frac{1}{b^3}\left[\frac{b(a+bu)^2+ba^2-(a+bu)2ab}{(a+bu)^2}\right] = \frac{1}{b^3}\left[\frac{b^3u^2}{(a+bu)^2}\right] = \frac{u^2}{(a+bu)^2}$$

(b) Let $t = a+bu \;\Rightarrow\; dt = b\,du$. Note that $u = \dfrac{t-a}{b}$ and $du = \dfrac{1}{b}\,dt$.

$$\int \frac{u^2\,du}{(a+bu)^2} = \frac{1}{b^3}\int \frac{(t-a)^2}{t^2}\,dt = \frac{1}{b^3}\int \frac{t^2-2at+a^2}{t^2}\,dt$$

$$= \frac{1}{b^3}\int\left(1-\frac{2a}{t}+\frac{a^2}{t^2}\right)dt = \frac{1}{b^3}\left(t-2a\ln|t|-\frac{a^2}{t}\right)+C$$

$$= \frac{1}{b^3}\left(a+bu-\frac{a^2}{a+bu}-2a\ln|a+bu|\right)+C$$

35. Maple, Mathematica and Derive all give $\int x^2\sqrt{5-x^2}\,dx = -\frac{1}{4}x(5-x^2)^{3/2} + \frac{5}{8}x\sqrt{5-x^2} + \frac{25}{8}\sin^{-1}\left(\frac{1}{\sqrt{5}}x\right)$.

Using Formula 31, we get $\int x^2\sqrt{5-x^2}\,dx = \frac{1}{8}x(2x^2-5)\sqrt{5-x^2} + \frac{1}{8}(5^2)\sin^{-1}\left(\frac{1}{\sqrt{5}}x\right) + C$. But

$-\frac{1}{4}x(5-x^2)^{3/2} + \frac{5}{8}x\sqrt{5-x^2} = \frac{1}{8}x\sqrt{5-x^2}\left[5-2(5-x^2)\right] = \frac{1}{8}x(2x^2-5)\sqrt{5-x^2}$, and the $\sin^{-1}$ terms are the same in each expression, so the answers are equivalent.

37. Maple and Derive both give $\int \sin^3 x\,\cos^2 x\,dx = -\frac{1}{5}\sin^2 x\,\cos^3 x - \frac{2}{15}\cos^3 x$ (although Derive factors the expression), and Mathematica gives $\int \sin^3 x\,\cos^2 x\,dx = -\frac{1}{8}\cos x - \frac{1}{48}\cos 3x + \frac{1}{80}\cos 5x$. We can use a CAS to show that both of these expressions are equal to $-\frac{1}{3}\cos^3 x + \frac{1}{5}\cos^5 x$. Using Formula 86, we write

$$\int \sin^3 x\,\cos^2 x\,dx = -\tfrac{1}{5}\sin^2 x\,\cos^3 x + \tfrac{2}{5}\int \sin x\,\cos^2 x\,dx = -\tfrac{1}{5}\sin^2 x\,\cos^3 x + \tfrac{2}{5}\left(-\tfrac{1}{3}\cos^3 x\right) + C$$

$$= -\tfrac{1}{5}\sin^2 x\,\cos^3 x - \tfrac{2}{15}\cos^3 x + C$$

39. Maple gives $\int x\sqrt{1+2x}\,dx = \frac{1}{10}(1+2x)^{5/2} - \frac{1}{6}(1+2x)^{3/2}$, Mathematica gives $\sqrt{1+2x}\left(\frac{2}{5}x^2+\frac{1}{15}x-\frac{1}{15}\right)$, and Derive gives $\frac{1}{15}(1+2x)^{3/2}(3x-1)$. The first two expressions can be simplified to Derive's result. If we use Formula 54, we get

$$\int x\sqrt{1+2x}\,dx = \frac{2}{15(2)^2}(3\cdot 2x - 2\cdot 1)(1+2x)^{3/2} + C = \tfrac{1}{30}(6x-2)(1+2x)^{3/2} + C$$

$$= \tfrac{1}{15}(3x-1)(1+2x)^{3/2}$$

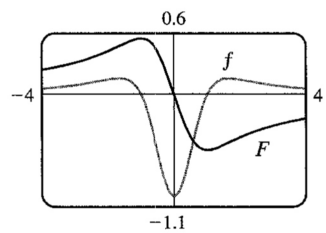
41. Maple gives $\int \tan^5 x\,dx = \frac{1}{4}\tan^4 x - \frac{1}{2}\tan^2 x + \frac{1}{2}\ln(1+\tan^2 x)$, Mathematica gives $\int \tan^5 x\,dx = \frac{1}{4}[-1 - 2\cos(2x)]\sec^4 x - \ln(\cos x)$, and Derive gives $\int \tan^5 x\,dx = \frac{1}{4}\tan^4 x - \frac{1}{2}\tan^2 x - \ln(\cos x)$. These expressions are equivalent, and none includes absolute value bars or a constant of integration. Note that Mathematica's and Derive's expressions suggest that the integral is undefined where $\cos x < 0$, which is not the case.

Using Formula 75, $\int \tan^5 x\,dx = \frac{1}{5-1}\tan^{5-1} x - \int \tan^{5-2} x\,dx = \frac{1}{4}\tan^4 x - \int \tan^3 x\,dx$. Using Formula 69, $\int \tan^3 x\,dx = \frac{1}{2}\tan^2 x + \ln|\cos x| + C$, so $\int \tan^5 x\,dx = \frac{1}{4}\tan^4 x - \frac{1}{2}\tan^2 x - \ln|\cos x| + C$.

43. Derive gives $I = \int 2^x\sqrt{4^x - 1}\,dx = \dfrac{2^{x-1}\sqrt{2^{2x}-1}}{\ln 2} - \dfrac{\ln\left(\sqrt{2^{2x}-1} + 2^x\right)}{2\ln 2}$ immediately. Neither Maple nor Mathematica is able to evaluate I in its given form. However, if we instead write I as $\int 2^x\sqrt{(2^x)^2 - 1}\,dx$, both systems give the same answer as Derive (after minor simplification). Our trick works because the CAS now recognizes 2^x as a promising substitution.

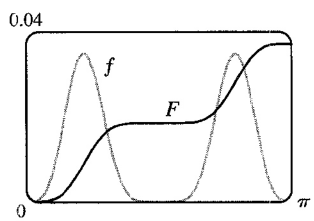
45. Maple gives the antiderivative

$$F(x) = \int \frac{x^2-1}{x^4+x^2+1}\,dx = -\tfrac{1}{2}\ln(x^2+x+1) + \tfrac{1}{2}\ln(x^2-x+1).$$



We can see that at 0, this antiderivative is 0. From the graphs, it appears that F has a maximum at $x = -1$ and a minimum at $x = 1$ [since $F'(x) = f(x)$ changes sign at these x-values], and that F has inflection points at $x \approx -1.7$, $x = 0$, and $x \approx 1.7$ [since $f(x)$ has extrema at these x-values].

47. Since $f(x) = \sin^4 x\cos^6 x$ is everywhere positive, we know that its antiderivative F is increasing. Maple gives $\int f(x)\,dx = -\frac{1}{10}\sin^3 x\cos^7 x - \frac{3}{80}\sin x\cos^7 x + \frac{1}{160}\cos^5 x\sin x + \frac{1}{128}\cos^3 x\sin x + \frac{3}{256}\cos x\sin x + \frac{3}{256}x$ and this expression is 0 at $x = 0$.



F has a minimum at $x = 0$ and a maximum at $x = \pi$. F has inflection points where f' changes sign, that is, at $x \approx 0.7$, $x = \pi/2$, and $x \approx 2.5$.

7.7 Approximate Integration

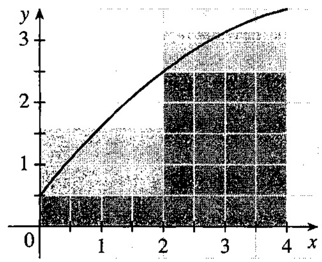
1. (a) $\Delta x = (b-a)/n = (4-0)/2 = 2$

$L_2 = \sum_{i=1}^{2} f(x_{i-1})\,\Delta x = f(x_0)\cdot 2 + f(x_1)\cdot 2 = 2\,[f(0)+f(2)] = 2(0.5+2.5) = 6$

$R_2 = \sum_{i=1}^{2} f(x_i)\,\Delta x = f(x_1)\cdot 2 + f(x_2)\cdot 2 = 2\,[f(2)+f(4)] = 2(2.5+3.5) = 12$

$M_2 = \sum_{i=1}^{2} f(\overline{x}_i)\Delta x = f(\overline{x}_1)\cdot 2 + f(\overline{x}_2)\cdot 2 = 2\,[f(1)+f(3)] \approx 2(1.6+3.2) = 9.6$

(b)



L_2 is an underestimate, since the area under the small rectangles is less than the area under the curve, and R_2 is an overestimate, since the area under the large rectangles is greater than the area under the curve. It appears that M_2 is an overestimate, though it is fairly close to I. See the solution to Exercise 45 for a proof of the fact that if f is concave down on $[a,b]$, then the Midpoint Rule is an overestimate of $\int_a^b f(x)\,dx$.

(c) $T_2 = \left(\frac{1}{2}\,\Delta x\right)[f(x_0) + 2f(x_1) + f(x_2)] = \frac{2}{2}[f(0) + 2f(2) + f(4)] = 0.5 + 2(2.5) + 3.5 = 9.$

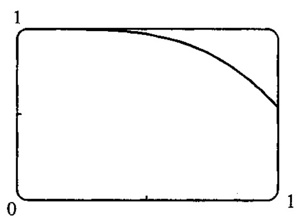
This approximation is an underestimate, since the graph is concave down. Thus, $T_2 = 9 < I$. See the solution to Exercise 45 for a general proof of this conclusion.

(d) For any n, we will have $L_n < T_n < I < M_n < R_n$.

3. $f(x) = \cos(x^2)$, $\Delta x = \frac{1-0}{4} = \frac{1}{4}$

(a) $T_4 = \frac{1}{4\cdot 2}\left[f(0) + 2f\left(\frac{1}{4}\right) + 2f\left(\frac{2}{4}\right) + 2f\left(\frac{3}{4}\right) + f(1)\right] \approx 0.895759$

(b) $M_4 = \frac{1}{4}\left[f\left(\frac{1}{8}\right) + f\left(\frac{3}{8}\right) + f\left(\frac{5}{8}\right) + f\left(\frac{7}{8}\right)\right] \approx 0.908907$



The graph shows that f is concave down on $[0,1]$. So T_4 is an underestimate and M_4 is an overestimate. We can conclude that $0.895759 < \int_0^1 \cos(x^2)\,dx < 0.908907$.

5. $f(x) = x^2 \sin x$, $\Delta x = \dfrac{b-a}{n} = \dfrac{\pi - 0}{8} = \dfrac{\pi}{8}$

(a) $M_8 = \frac{\pi}{8}\left[f\left(\frac{\pi}{16}\right) + f\left(\frac{3\pi}{16}\right) + f\left(\frac{5\pi}{16}\right) + \cdots + f\left(\frac{15\pi}{16}\right)\right] \approx 5.932957$

(b) $S_8 = \frac{\pi}{8\cdot 3}\left[f(0) + 4f\left(\frac{\pi}{8}\right) + 2f\left(\frac{2\pi}{8}\right) + 4f\left(\frac{3\pi}{8}\right) + 2f\left(\frac{4\pi}{8}\right) + 4f\left(\frac{5\pi}{8}\right) + 2f\left(\frac{6\pi}{8}\right) + 4f\left(\frac{7\pi}{8}\right) + f(\pi)\right]$

≈ 5.869247

Actual: $\int_0^\pi x^2 \sin x\,dx \overset{84}{=} \left[-x^2\cos x\right]_0^\pi + 2\int_0^\pi x\cos x\,dx \overset{83}{=} \left[-\pi^2(-1) - 0\right] + 2[\cos x + x\sin x]_0^\pi$

$= \pi^2 + 2[(-1+0) - (1+0)] = \pi^2 - 4 \approx 5.869604$

Errors: $E_M = \text{actual} - M_8 = \int_0^\pi x^2\sin x\,dx - M_8 \approx -0.063353$

$E_S = \text{actual} - S_8 = \int_0^\pi x^2\sin x\,dx - S_8 \approx 0.000357$

7. $f(x) = \sqrt[4]{1+x^2}$, $\Delta x = \dfrac{2-0}{8} = \dfrac{1}{4}$

(a) $T_8 = \frac{1}{4 \cdot 2}\left[f(0) + 2f\left(\frac{1}{4}\right) + 2f\left(\frac{1}{2}\right) + \cdots + 2f\left(\frac{3}{2}\right) + 2f\left(\frac{7}{4}\right) + f(2)\right] \approx 2.413790$

(b) $M_8 = \frac{1}{4}\left[f\left(\frac{1}{8}\right) + f\left(\frac{3}{8}\right) + \cdots + f\left(\frac{13}{8}\right) + f\left(\frac{15}{8}\right)\right] \approx 2.411453$

(c) $S_8 = \frac{1}{4 \cdot 3}\left[f(0) + 4f\left(\frac{1}{4}\right) + 2f\left(\frac{1}{2}\right) + 4f\left(\frac{3}{4}\right) + 2f(1) + 4f\left(\frac{5}{4}\right) + 2f\left(\frac{3}{2}\right) + 4f\left(\frac{7}{4}\right) + f(2)\right] \approx 2.412232$

9. $f(x) = \dfrac{\ln x}{1+x}$, $\Delta x = \dfrac{2-1}{10} = \dfrac{1}{10}$

(a) $T_{10} = \frac{1}{10 \cdot 2}[f(1) + 2f(1.1) + 2f(1.2) + \cdots + 2f(1.8) + 2f(1.9) + f(2)] \approx 0.146879$

(b) $M_{10} = \frac{1}{10}[f(1.05) + f(1.15) + \cdots + f(1.85) + f(1.95)] \approx 0.147391$

(c) $S_{10} = \frac{1}{10 \cdot 3}[f(1) + 4f(1.1) + 2f(1.2) + 4f(1.3) + 2f(1.4) + 4f(1.5) + 2f(1.6) + 4f(1.7)$
$+ 2f(1.8) + 4f(1.9) + f(2)]$
≈ 0.147219

11. $f(t) = \sin(e^{t/2})$, $\Delta t = \dfrac{\frac{1}{2} - 0}{8} = \dfrac{1}{16}$

(a) $T_8 = \frac{1}{16 \cdot 2}\left[f(0) + 2f\left(\frac{1}{16}\right) + 2f\left(\frac{2}{16}\right) + \cdots + 2f\left(\frac{7}{16}\right) + f\left(\frac{1}{2}\right)\right] \approx 0.451948$

(b) $M_8 = \frac{1}{16}\left[f\left(\frac{1}{32}\right) + f\left(\frac{3}{32}\right) + f\left(\frac{5}{32}\right) + \cdots + f\left(\frac{13}{32}\right) + f\left(\frac{15}{32}\right)\right] \approx 0.451991$

(c) $S_8 = \frac{1}{16 \cdot 3}\left[f(0) + 4f\left(\frac{1}{16}\right) + 2f\left(\frac{2}{16}\right) + \cdots + 4f\left(\frac{7}{16}\right) + f\left(\frac{1}{2}\right)\right] \approx 0.451976$

13. $f(x) = e^{1/x}$, $\Delta x = \dfrac{2-1}{4} = \dfrac{1}{4}$

(a) $T_4 = \frac{1}{4 \cdot 2}[f(1) + 2f(1.25) + 2f(1.5) + 2f(1.75) + f(2)] \approx 2.031893$

(b) $M_4 = \frac{1}{4}[f(1.125) + f(1.375) + f(1.625) + f(1.875)] \approx 2.014207$

(c) $S_4 = \frac{1}{4 \cdot 3}[f(1) + 4f(1.25) + 2f(1.5) + 4f(1.75) + f(2)] \approx 2.020651$

15. $f(x) = \dfrac{\cos x}{x}$, $\Delta x = \dfrac{5-1}{8} = \dfrac{1}{2}$

(a) $T_8 = \frac{1}{2 \cdot 2}\left[f(1) + 2f\left(\frac{3}{2}\right) + 2f(2) + \cdots + 2f(4) + 2f\left(\frac{9}{2}\right) + f(5)\right] \approx -0.495333$

(b) $M_8 = \frac{1}{2}\left[f\left(\frac{5}{4}\right) + f\left(\frac{7}{4}\right) + f\left(\frac{9}{4}\right) + f\left(\frac{11}{4}\right) + f\left(\frac{13}{4}\right) + f\left(\frac{15}{4}\right) + f\left(\frac{17}{4}\right) + f\left(\frac{19}{4}\right)\right] \approx -0.543321$

(c) $S_8 = \frac{1}{2 \cdot 3}\left[f(1) + 4f\left(\frac{3}{2}\right) + 2f(2) + 4f\left(\frac{5}{2}\right) + 2f(3) + 4f\left(\frac{7}{2}\right) + 2f(4) + 4f\left(\frac{9}{2}\right) + f(5)\right]$
≈ -0.526123

17. $f(y) = \dfrac{1}{1+y^5}$, $\Delta y = \dfrac{3-0}{6} = \dfrac{1}{2}$

(a) $T_6 = \frac{1}{2 \cdot 2}\left[f(0) + 2f\left(\frac{1}{2}\right) + 2f\left(\frac{2}{2}\right) + 2f\left(\frac{3}{2}\right) + 2f\left(\frac{4}{2}\right) + 2f\left(\frac{5}{2}\right) + f(3)\right] \approx 1.064275$

(b) $M_6 = \frac{1}{2}\left[f\left(\frac{1}{4}\right) + f\left(\frac{3}{4}\right) + f\left(\frac{5}{4}\right) + f\left(\frac{7}{4}\right) + f\left(\frac{9}{4}\right) + f\left(\frac{11}{4}\right)\right] \approx 1.067416$

(c) $S_6 = \frac{1}{2 \cdot 3}\left[f(0) + 4f\left(\frac{1}{2}\right) + 2f\left(\frac{2}{2}\right) + 4f\left(\frac{3}{2}\right) + 2f\left(\frac{4}{2}\right) + 4f\left(\frac{5}{2}\right) + f(3)\right] \approx 1.074915$

19. $f(x) = e^{-x^2}$, $\Delta x = \dfrac{2-0}{10} = \dfrac{1}{5}$

(a) $T_{10} = \frac{1}{5 \cdot 2}\{f(0) + 2[f(0.2) + f(0.4) + \cdots + f(1.8)] + f(2)\} \approx 0.881839$

$M_{10} = \frac{1}{5}[f(0.1) + f(0.3) + f(0.5) + \cdots + f(1.7) + f(1.9)] \approx 0.882202$

(b) $f(x) = e^{-x^2}$, $f'(x) = -2xe^{-x^2}$, $f''(x) = (4x^2-2)e^{-x^2}$, $f'''(x) = 4x(3-2x^2)e^{-x^2}$.

$f'''(x) = 0 \quad \Leftrightarrow \quad x = 0$ or $x = \pm\sqrt{\frac{3}{2}}$. So to find the maximum value of $|f''(x)|$ on $[0, 2]$, we need only consider its values at $x = 0$, $x = 2$, and $x = \sqrt{\frac{3}{2}}$. $|f''(0)| = 2$, $|f''(2)| \approx 0.2564$ and $\left|f''\left(\sqrt{\frac{3}{2}}\right)\right| \approx 0.8925$. Thus, taking $K = 2$, $a = 0$, $b = 2$, and $n = 10$ in Theorem 3, we get $|E_T| \le 2 \cdot 2^3/(12 \cdot 10^2) = \frac{1}{75} = 0.01\overline{3}$, and $|E_M| \le |E_T|/2 \le 0.00\overline{6}$.

(c) Take $K = 2$ [as in part (b)] in Theorem 3. $|E_T| \le \dfrac{K(b-a)^3}{12n^2} \le 10^{-5} \quad \Leftrightarrow \quad \dfrac{2(2-0)^3}{12n^2} \le 10^{-5} \quad \Leftrightarrow$ $\frac{3}{4}n^2 \ge 10^5 \quad \Leftrightarrow \quad n \ge 365.1\ldots \quad \Leftrightarrow \quad n \ge 366$. Take $n = 366$ for T_n. For E_M, again take $K = 2$ in Theorem 3 to get $|E_M| \le 10^{-5} \quad \Leftrightarrow \quad \frac{3}{2}n^2 \ge 10^5 \quad \Leftrightarrow \quad n \ge 258.2 \quad \Rightarrow \quad n \ge 259$. Take $n = 259$ for M_n.

21. (a) $T_{10} = \frac{1}{10 \cdot 2}\{f(0) + 2[f(0.1) + f(0.2) + \cdots + f(0.9)] + f(1)\} \approx 1.71971349$

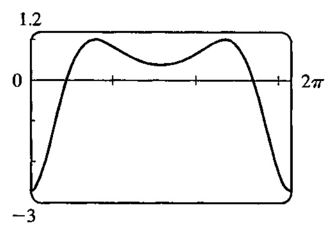
$S_{10} = \frac{1}{10 \cdot 3}[f(0) + 4f(0.1) + 2f(0.2) + 4f(0.3) + \cdots + 4f(0.9) + f(1)] \approx 1.71828278$

Since $I = \int_0^1 e^x\,dx = [e^x]_0^1 = e - 1 \approx 1.71828183$, $E_T = I - T_{10} \approx -0.00143166$ and $E_S = I - S_{10} \approx -0.00000095$.

(b) $f(x) = e^x \quad \Rightarrow \quad f''(x) = e^x \le e$ for $0 \le x \le 1$. Taking $K = e$, $a = 0$, $b = 1$, and $n = 10$ in Theorem 3, we get $|E_T| \le e(1)^3/(12 \cdot 10^2) \approx 0.002265 > 0.00143166$ [actual $|E_T|$ from (a)]. $f^{(4)}(x) = e^x < e$ for $0 \le x \le 1$. Using Theorem 4, we have $|E_S| \le e(1)^5/(180 \cdot 10^4) \approx 0.0000015 > 0.00000095$ [actual $|E_S|$ from (a)]. We see that the actual errors are about two-thirds the size of the error estimates.

(c) From part (b), we take $K = e$ to get $|E_T| \le \dfrac{K(b-a)^3}{12n^2} \le 0.00001 \quad \Rightarrow \quad n^2 \ge \dfrac{e(1^3)}{12(0.00001)} \quad \Rightarrow$ $n \ge 150.5$. Take $n = 151$ for T_n. Now $|E_M| \le \dfrac{K(b-a)^3}{24n^2} \le 0.00001 \quad \Rightarrow \quad n \ge 106.4$. Take $n = 107$ for M_n. Finally, $|E_S| \le \dfrac{K(b-a)^5}{180n^4} \le 0.00001 \quad \Rightarrow \quad n^4 \ge \dfrac{e(1^5)}{180(0.00001)} \quad \Rightarrow \quad n \ge 6.23$. Take $n = 8$ for S_n (since n has to be even for Simpson's Rule).

23. (a) Using a CAS, we differentiate $f(x) = e^{\cos x}$ twice, and find that $f''(x) = e^{\cos x}(\sin^2 x - \cos x)$. From the graph, we see that the maximum value of $|f''(x)|$ occurs at the endpoints of the interval $[0, 2\pi]$. Since $f''(0) = -e$, we can use $K = e$ or $K = 2.8$.



(b) A CAS gives $M_{10} \approx 7.954926518$. (In Maple, use `student[middlesum]`.)

(c) Using Theorem 3 for the Midpoint Rule, with $K = e$, we get $|E_M| \le \dfrac{e(2\pi - 0)^3}{24 \cdot 10^2} \approx 0.280945995$. With $K = 2.8$, we get $|E_M| \le \dfrac{2.8(2\pi - 0)^3}{24 \cdot 10^2} = 0.289391916$.

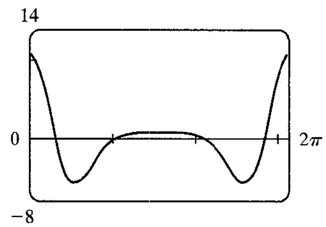
(d) A CAS gives $I \approx 7.954926521$.

(e) The actual error is only about 3×10^{-9}, much less than the estimate in part (c).

(f) We use the CAS to differentiate twice more, and then graph

$f^{(4)}(x) = e^{\cos x}(\sin^4 x - 6\sin^2 x \cos x + 3 - 7\sin^2 x + \cos x)$.

From the graph, we see that the maximum value of $\left|f^{(4)}(x)\right|$ occurs at the endpoints of the interval $[0, 2\pi]$. Since $f^{(4)}(0) = 4e$, we can use $K = 4e$ or $K = 10.9$.



(g) A CAS gives $S_{10} \approx 7.953789422$. (In Maple, use `student[simpson]`.)

(h) Using Theorem 4 with $K = 4e$, we get $|E_S| \le \dfrac{4e(2\pi - 0)^5}{180 \cdot 10^4} \approx 0.059153618$. With $K = 10.9$, we get $|E_S| \le \dfrac{10.9(2\pi - 0)^5}{180 \cdot 10^4} \approx 0.059299814$.

(i) The actual error is about $7.954926521 - 7.953789422 \approx 0.00114$. This is quite a bit smaller than the estimate in part (h), though the difference is not nearly as great as it was in the case of the Midpoint Rule.

(j) To ensure that $|E_S| \le 0.0001$, we use Theorem 4: $|E_S| \le \dfrac{4e(2\pi)^5}{180 \cdot n^4} \le 0.0001 \Rightarrow \dfrac{4e(2\pi)^5}{180 \cdot 0.0001} \le n^4 \Rightarrow$ $n^4 \ge 5{,}915{,}362 \Leftrightarrow n \ge 49.3$. So we must take $n \ge 50$ to ensure that $|I - S_n| \le 0.0001$. ($K = 10.9$ leads to the same value of n.)

25. $I = \int_0^1 x^3 dx = \left[\frac{1}{4}x^4\right]_0^1 = 0.25$. $f(x) = x^3$.

$n = 4$: $L_4 = \frac{1}{4}\left[0^3 + \left(\frac{1}{4}\right)^3 + \left(\frac{2}{4}\right)^3 + \left(\frac{3}{4}\right)^3\right] = 0.140625$

$R_4 = \frac{1}{4}\left[\left(\frac{1}{4}\right)^3 + \left(\frac{2}{4}\right)^3 + \left(\frac{3}{4}\right)^3 + 1^3\right] = 0.390625$

$T_4 = \frac{1}{4 \cdot 2}\left[0^3 + 2\left(\frac{1}{4}\right)^3 + 2\left(\frac{2}{4}\right)^3 + 2\left(\frac{3}{4}\right)^3 + 1^3\right] = 0.265625$,

$M_4 = \frac{1}{4}\left[\left(\frac{1}{8}\right)^3 + \left(\frac{3}{8}\right)^3 + \left(\frac{5}{8}\right)^3 + \left(\frac{7}{8}\right)^3\right] = 0.2421875$,

$E_L = I - L_4 = \frac{1}{4} - 0.140625 = 0.109375$, $E_R = \frac{1}{4} - 0.390625 = -0.140625$,

$E_T = \frac{1}{4} - 0.265625 = -0.015625$, $E_M = \frac{1}{4} - 0.2421875 = 0.0078125$

$n = 8$: $L_8 = \frac{1}{8}\left[f(0) + f\left(\frac{1}{8}\right) + f\left(\frac{2}{8}\right) + \cdots + f\left(\frac{7}{8}\right)\right] \approx 0.191406$

$R_8 = \frac{1}{8}\left[f\left(\frac{1}{8}\right) + f\left(\frac{2}{8}\right) + \cdots + f\left(\frac{7}{8}\right) + f(1)\right] \approx 0.316406$

$T_8 = \frac{1}{8 \cdot 2}\left\{f(0) + 2\left[f\left(\frac{1}{8}\right) + f\left(\frac{2}{8}\right) + \cdots + f\left(\frac{7}{8}\right)\right] + f(1)\right\} \approx 0.253906$

$M_8 = \frac{1}{8}\left[f\left(\frac{1}{16}\right) + f\left(\frac{3}{16}\right) + \cdots + f\left(\frac{13}{16}\right) + f\left(\frac{15}{16}\right)\right] = 0.248047$

$E_L \approx \frac{1}{4} - 0.191406 \approx 0.058594$, $E_R \approx \frac{1}{4} - 0.316406 \approx -0.066406$,

$E_T \approx \frac{1}{4} - 0.253906 \approx -0.003906$, $E_M \approx \frac{1}{4} - 0.248047 \approx 0.001953$.

[continued]

$n = 16$: $L_{16} = \frac{1}{16}\left[f(0) + f\left(\frac{1}{16}\right) + f\left(\frac{2}{16}\right) + \cdots + f\left(\frac{15}{16}\right)\right] \approx 0.219727$

$R_{16} = \frac{1}{16}\left[f\left(\frac{1}{16}\right) + f\left(\frac{2}{16}\right) + \cdots + f\left(\frac{15}{16}\right) + f(1)\right] \approx 0.282227$

$T_{16} = \frac{1}{16 \cdot 2}\left\{f(0) + 2\left[f\left(\frac{1}{16}\right) + f\left(\frac{2}{16}\right) + \cdots + f\left(\frac{15}{16}\right)\right] + f(1)\right\} \approx 0.250977$

$M_{16} = \frac{1}{16}\left[f\left(\frac{1}{32}\right) + f\left(\frac{3}{32}\right) + \cdots + f\left(\frac{31}{32}\right)\right] \approx 0.249512$

$E_L \approx \frac{1}{4} - 0.219727 \approx 0.030273$, $E_R \approx \frac{1}{4} - 0.282227 \approx -0.032227$,

$E_T \approx \frac{1}{4} - 0.250977 \approx -0.000977$, $E_M \approx \frac{1}{4} - 0.249512 \approx 0.000488$.

n	L_n	R_n	T_n	M_n
4	0.140625	0.390625	0.265625	0.242188
8	0.191406	0.316406	0.253906	0.248047
16	0.219727	0.282227	0.250977	0.249512

n	E_L	E_R	E_T	E_M
4	0.109375	−0.140625	−0.015625	0.007813
8	0.058594	−0.066406	−0.003906	0.001953
16	0.030273	−0.032227	−0.000977	0.000488

Observations:

1. E_L and E_R are always opposite in sign, as are E_T and E_M.
2. As n is doubled, E_L and E_R are decreased by about a factor of 2, and E_T and E_M are decreased by a factor of about 4.
3. The Midpoint approximation is about twice as accurate as the Trapezoidal approximation.
4. All the approximations become more accurate as the value of n increases.
5. The Midpoint and Trapezoidal approximations are much more accurate than the endpoint approximations.

27. $\int_1^4 \sqrt{x}\,dx = \left[\frac{2}{3}x^{3/2}\right]_1^4 = \frac{2}{3}(8-1) = \frac{14}{3} \approx 4.666667$

$n = 6$: $\Delta x = (4-1)/6 = \frac{1}{2}$

$T_6 = \frac{1}{2 \cdot 2}\left[\sqrt{1} + 2\sqrt{1.5} + 2\sqrt{2} + 2\sqrt{2.5} + 2\sqrt{3} + 2\sqrt{3.5} + \sqrt{4}\right] \approx 4.661488$

$M_6 = \frac{1}{2}\left[\sqrt{1.25} + \sqrt{1.75} + \sqrt{2.25} + \sqrt{2.75} + \sqrt{3.25} + \sqrt{3.75}\right] \approx 4.669245$

$S_6 = \frac{1}{2 \cdot 3}\left[\sqrt{1} + 4\sqrt{1.5} + 2\sqrt{2} + 4\sqrt{2.5} + 2\sqrt{3} + 4\sqrt{3.5} + \sqrt{4}\right] \approx 4.666563$

$E_T \approx \frac{14}{3} - 4.661488 \approx 0.005178$, $E_M \approx \frac{14}{3} - 4.669245 \approx -0.002578$,

$E_S \approx \frac{14}{3} - 4.666563 \approx 0.000104$.

$n = 12$: $\Delta x = (4-1)/12 = \frac{1}{4}$

$T_{12} = \frac{1}{4 \cdot 2}(f(1) + 2\left[f(1.25) + f(1.5) + \cdots + f(3.5) + f(3.75)\right] + f(4)) \approx 4.665367$

$M_{12} = \frac{1}{4}[f(1.125) + f(1.375) + f(1.625) + \cdots + f(3.875)] \approx 4.667316$

$S_{12} = \frac{1}{4 \cdot 3}[f(1) + 4\,f(1.25) + 2\,f(1.5) + 4\,f(1.75) + \cdots + 4\,f(3.75) + f(4)] \approx 4.666659$

$E_T \approx \frac{14}{3} - 4.665367 \approx 0.001300$, $E_M \approx \frac{14}{3} - 4.667316 \approx -0.000649$,

$E_S \approx \frac{14}{3} - 4.666659 \approx 0.000007$.

Note: These errors were computed more precisely and then rounded to six places. That is, they were not computed by comparing the rounded values of T_n, M_n, and S_n with the rounded value of the actual integral.

n	T_n	M_n	S_n
6	4.661488	4.669245	4.666563
12	4.665367	4.667316	4.666659

n	E_T	E_M	E_S
6	0.005178	−0.002578	0.000104
12	0.001300	−0.000649	0.000007

Observations:

1. E_T and E_M are opposite in sign and decrease by a factor of about 4 as n is doubled.
2. The Simpson's approximation is much more accurate than the Midpoint and Trapezoidal approximations, and seems to decrease by a factor of about 16 as n is doubled.

29. $\Delta x = (4-0)/4 = 1$

(a) $T_4 = \frac{1}{2}[f(0)+2f(1)+2f(2)+2f(3)+f(4)] \approx \frac{1}{2}[0+2(3)+2(5)+2(3)+1] = 11.5$

(b) $M_4 = 1\cdot[f(0.5)+f(1.5)+f(2.5)+f(3.5)] \approx 1+4.5+4.5+2 = 12$

(c) $S_4 = \frac{1}{3}[f(0)+4f(1)+2f(2)+4f(3)+f(4)] \approx \frac{1}{3}[0+4(3)+2(5)+4(3)+1] = 11.\overline{6}$

31. (a) We are given the function values at the endpoints of 8 intervals of length 0.4, so we'll use the Midpoint Rule with $n = 8/2 = 4$ and $\Delta x = (3.2-0)/4 = 0.8$.

$$\begin{aligned}\int_0^{3.2} f(x)\,dx \approx M_4 &= 0.8[f(0.4)+f(1.2)+f(2.0)+f(2.8)]\\ &= 0.8[6.5+6.4+7.6+8.8]\\ &= 0.8(29.3) = 23.44\end{aligned}$$

(b) $-4 \le f''(x) \le 1 \;\Rightarrow\; |f''(x)| \le 4$, so use $K=4$, $a=0$, $b=3.2$, and $n=4$ in Theorem 3. So $|E_M| \le \dfrac{4(3.2-0)^3}{24(4)^2} = \dfrac{128}{375} = 0.341\overline{3}$.

33. By the Net Change Theorem, the increase in velocity is equal to $\int_0^6 a(t)\,dt$. We use Simpson's Rule with $n=6$ and $\Delta t = (6-0)/6 = 1$ to estimate this integral:

$$\begin{aligned}\int_0^6 a(t)\,dt \approx S_6 &= \tfrac{1}{3}[a(0)+4a(1)+2a(2)+4a(3)+2a(4)+4a(5)+a(6)]\\ &\approx \tfrac{1}{3}[0+4(0.5)+2(4.1)+4(9.8)+2(12.9)+4(9.5)+0] = \tfrac{1}{3}(113.2) = 37.7\overline{3}\text{ ft/s}\end{aligned}$$

35. By the Net Change Theorem, the energy used is equal to $\int_0^6 P(t)\,dt$. We use Simpson's Rule with $n=12$ and $\Delta t = (6-0)/12 = \frac{1}{2}$ to estimate this integral:

$$\begin{aligned}\int_0^6 P(t)\,dt \approx S_{12} &= \tfrac{1/2}{3}[P(0)+4P(0.5)+2P(1)+4P(1.5)+2P(2)+4P(2.5)\\ &\qquad +2P(3)+4P(3.5)+2P(4)+4P(4.5)+2P(5)+4P(5.5)+P(6)]\\ &= \tfrac{1}{6}[1814+4(1735)+2(1686)+4(1646)+2(1637)+4(1609)+2(1604)\\ &\qquad +4(1611)+2(1621)+4(1666)+2(1745)+4(1886)+2052]\\ &= \tfrac{1}{6}(61{,}064) = 10{,}177.\overline{3}\text{ megawatt-hours.}\end{aligned}$$

37. Let $y = f(x)$ denote the curve. Using cylindrical shells, $V = \int_2^{10} 2\pi x f(x)\,dx = 2\pi \int_2^{10} x f(x)\,dx = 2\pi I$. Now use Simpson's Rule to approximate I:

$$\begin{aligned} I \approx S_8 &= \tfrac{10-2}{3(8)}\,[2f(2) + 4\cdot 3f(3) + 2\cdot 4f(4) + 4\cdot 5f(5) + 2\cdot 6f(6) \\ &\qquad + 4\cdot 7f(7) + 2\cdot 8f(8) + 4\cdot 9f(9) + 10f(10)] \\ &\approx \tfrac{1}{3}[2(0) + 12(1.5) + 8(1.9) + 20(2.2) + 12(3.0) + 28(3.8) + 16(4.0) + 36(3.1) + 10(0)] \\ &= \tfrac{1}{3}(395.2) \end{aligned}$$

Thus, $V \approx 2\pi \cdot \frac{1}{3}(395.2) \approx 827.7$ or 828 cubic units.

39. Volume $= \pi \int_0^2 \left(\sqrt[3]{1+x^3}\right)^2 dx = \pi \int_0^2 \left(1+x^3\right)^{2/3} dx$. $V \approx \pi \cdot S_{10}$ where $f(x) = \left(1+x^3\right)^{2/3}$ and $\Delta x = (2-0)/10 = \frac{1}{5}$. Therefore,

$$\begin{aligned} V \approx \pi \cdot S_{10} = \pi \tfrac{1}{5\cdot 3}\,[f(0) + 4f(0.2) + 2f(0.4) + 4f(0.6) + 2f(0.8) + 4f(1) \\ + 2f(1.2) + 4f(1.4) + 2f(1.6) + 4f(1.8) + f(2)] \approx 12.325078 \end{aligned}$$

41. $I(\theta) = \dfrac{N^2 \sin^2 k}{k^2}$, where $k = \dfrac{\pi N d \sin\theta}{\lambda}$, $N = 10{,}000$, $d = 10^{-4}$, and $\lambda = 632.8 \times 10^{-9}$. So

$I(\theta) = \dfrac{(10^4)^2 \sin^2 k}{k^2}$, where $k = \dfrac{\pi(10^4)(10^{-4})\sin\theta}{632.8 \times 10^{-9}}$. Now $n = 10$ and $\Delta\theta = \dfrac{10^{-6} - (-10^{-6})}{10} = 2 \times 10^{-7}$,

so $M_{10} = 2 \times 10^{-7}[I(-0.0000009) + I(-0.0000007) + \cdots + I(0.0000009)] \approx 59.4$.

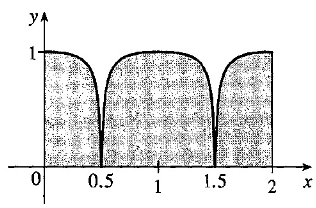
43. Consider the function f whose graph is shown. The area $\int_0^2 f(x)\,dx$ is close to 2. The Trapezoidal Rule gives

$T_2 = \frac{2-0}{2\cdot 2}\,[f(0) + 2f(1) + f(2)] = \frac{1}{2}\,[1 + 2\cdot 1 + 1] = 2.$

The Midpoint Rule gives

$M_2 = \frac{2-0}{2}\,[f(0.5) + f(1.5)] = 1[0 + 0] = 0,$

so the Trapezoidal Rule is more accurate.



45. Since the Trapezoidal and Midpoint approximations on the interval $[a, b]$ are the sums of the Trapezoidal and Midpoint approximations on the subintervals $[x_{i-1}, x_i]$, $i = 1, 2, \ldots, n$, we can focus our attention on one such interval. The condition $f''(x) < 0$ for $a \le x \le b$ means that the graph of f is concave down as in Figure 5. In that figure, T_n is the area of the trapezoid $AQRD$, $\int_a^b f(x)\,dx$ is the area of the region $AQPRD$, and M_n is the area of the trapezoid $ABCD$, so $T_n < \int_a^b f(x)\,dx < M_n$. In general, the condition $f'' < 0$ implies that the graph of f on $[a, b]$ lies above the chord joining the points $(a, f(a))$ and $(b, f(b))$. Thus, $\int_a^b f(x)\,dx > T_n$. Since M_n is the area under a tangent to the graph, and since $f'' < 0$ implies that the tangent lies above the graph, we also have $M_n > \int_a^b f(x)\,dx$. Thus, $T_n < \int_a^b f(x)\,dx < M_n$.

47. $T_n = \frac{1}{2}\,\Delta x\,[f(x_0) + 2f(x_1) + \cdots + 2f(x_{n-1}) + f(x_n)]$ and

$M_n = \Delta x\,[f(\overline{x}_1) + f(\overline{x}_2) + \cdots + f(\overline{x}_{n-1}) + f(\overline{x}_n)]$, where $\overline{x}_i = \frac{1}{2}(x_{i-1} + x_i)$. Now

$$T_{2n} = \tfrac{1}{2}\left(\tfrac{1}{2}\Delta x\right)[f(x_0) + 2f(\overline{x}_1) + 2f(x_1) + 2f(\overline{x}_2) + 2f(x_2) + \cdots$$
$$+ 2f(\overline{x}_{n-1}) + 2f(x_{n-1}) + 2f(\overline{x}_n) + f(x_n)]$$

so
$$\begin{aligned}\tfrac{1}{2}(T_n + M_n) &= \tfrac{1}{2}T_n + \tfrac{1}{2}M_n \\ &= \tfrac{1}{4}\,\Delta x\,[f(x_0) + 2f(x_1) + \cdots + 2f(x_{n-1}) + f(x_n)] \\ &\qquad + \tfrac{1}{4}\Delta x\,[2f(\overline{x}_1) + 2f(\overline{x}_2) + \cdots + 2f(\overline{x}_{n-1}) + 2f(\overline{x}_n)] \\ &= T_{2n}\end{aligned}$$

7.8 Improper Integrals

1. (a) Since $\int_1^\infty x^4 e^{-x^4}\,dx$ has an infinite interval of integration, it is an improper integral of Type I.

(b) Since $y = \sec x$ has an infinite discontinuity at $x = \frac{\pi}{2}$, $\int_0^{\pi/2} \sec x\,dx$ is a Type II improper integral.

(c) Since $y = \dfrac{x}{(x-2)(x-3)}$ has an infinite discontinuity at $x = 2$, $\displaystyle\int_0^2 \frac{x}{x^2 - 5x + 6}\,dx$ is a Type II improper integral.

(d) Since $\displaystyle\int_{-\infty}^0 \frac{1}{x^2+5}\,dx$ has an infinite interval of integration, it is an improper integral of Type I.

3. The area under the graph of $y = 1/x^3 = x^{-3}$ between $x = 1$ and $x = t$ is $A(t) = \int_1^t x^{-3}\,dx = \left[-\frac{1}{2}x^{-2}\right]_1^t = -\frac{1}{2}t^{-2} - \left(-\frac{1}{2}\right) = \frac{1}{2} - 1/(2t^2)$. So the area for $1 \le x \le 10$ is $A(10) = 0.5 - 0.005 = 0.495$, the area for $1 \le x \le 100$ is $A(100) = 0.5 - 0.00005 = 0.49995$, and the area for $1 \le x \le 1000$ is $A(1000) = 0.5 - 0.0000005 = 0.4999995$. The total area under the curve for $x \ge 1$ is $\lim\limits_{t\to\infty} A(t) = \lim\limits_{t\to\infty}\left[\frac{1}{2} - 1/(2t^2)\right] = \frac{1}{2}$.

5. $I = \displaystyle\int_1^\infty \frac{1}{(3x+1)^2}\,dx = \lim_{t\to\infty}\int_1^t \frac{1}{(3x+1)^2}\,dx$. Now

$$\begin{aligned}\int \frac{1}{(3x+1)^2}\,dx &= \frac{1}{3}\int \frac{1}{u^2}\,du \quad [u = 3x+1,\ du = 3\,dx] \\ &= -\frac{1}{3u} + C = -\frac{1}{3(3x+1)} + C,\end{aligned}$$

so $I = \displaystyle\lim_{t\to\infty}\left[-\frac{1}{3(3x+1)}\right]_1^t = \lim_{t\to\infty}\left[-\frac{1}{3(3t+1)} + \frac{1}{12}\right] = 0 + \frac{1}{12} = \frac{1}{12}$. Convergent

7. $\displaystyle\int_{-\infty}^{-1} \frac{1}{\sqrt{2-w}}\,dw = \lim_{t\to-\infty}\int_t^{-1} \frac{1}{\sqrt{2-w}}\,dw = \lim_{t\to-\infty}\left[-2\sqrt{2-w}\right]_t^{-1}$ $[u = 2-w,\ du = -dw]$

$= \displaystyle\lim_{t\to-\infty}\left[-2\sqrt{3} + 2\sqrt{2-t}\right] = \infty$. Divergent

9. $\int_4^\infty e^{-y/2}\,dy = \lim_{t\to\infty}\int_4^t e^{-y/2}\,dy = \lim_{t\to\infty}\left[-2e^{-y/2}\right]_4^t = \lim_{t\to\infty}(-2e^{-t/2}+2e^{-2}) = 0+2e^{-2} = 2e^{-2}.$

Convergent

11. $$\int_{-\infty}^{\infty}\frac{x\,dx}{1+x^2} = \int_{-\infty}^{0}\frac{x\,dx}{1+x^2} + \int_{0}^{\infty}\frac{x\,dx}{1+x^2} \text{ and}$$

$$\int_{-\infty}^{0}\frac{x\,dx}{1+x^2} = \lim_{t\to-\infty}\left[\tfrac{1}{2}\ln(1+x^2)\right]_t^0 = \lim_{t\to-\infty}\left[0-\tfrac{1}{2}\ln(1+t^2)\right] = -\infty. \quad \text{Divergent}$$

13. $\int_{-\infty}^\infty xe^{-x^2}\,dx = \int_{-\infty}^0 xe^{-x^2}\,dx + \int_0^\infty xe^{-x^2}\,dx.$

$\int_{-\infty}^0 xe^{-x^2}\,dx = \lim_{t\to-\infty}\left(-\tfrac{1}{2}\right)\left[e^{-x^2}\right]_t^0 = \lim_{t\to-\infty}\left(-\tfrac{1}{2}\right)\left(1-e^{-t^2}\right) = -\tfrac{1}{2}\cdot 1 = -\tfrac{1}{2}$, and

$\int_0^\infty xe^{-x^2}\,dx = \lim_{t\to\infty}\left(-\tfrac{1}{2}\right)\left[e^{-x^2}\right]_0^t = \lim_{t\to\infty}\left(-\tfrac{1}{2}\right)\left(e^{-t^2}-1\right) = -\tfrac{1}{2}\cdot(-1) = \tfrac{1}{2}.$

Therefore, $\int_{-\infty}^\infty xe^{-x^2}\,dx = -\tfrac{1}{2}+\tfrac{1}{2} = 0.$ Convergent

15. $\int_{2\pi}^\infty \sin\theta\,d\theta = \lim_{t\to\infty}\int_{2\pi}^t \sin\theta\,d\theta = \lim_{t\to\infty}\left[-\cos\theta\right]_{2\pi}^t = \lim_{t\to\infty}(-\cos t+1)$. This limit does not exist, so the integral is divergent. Divergent

17. $$\int_1^\infty \frac{x+1}{x^2+2x}\,dx = \lim_{t\to\infty}\int_1^t \frac{\frac{1}{2}(2x+2)}{x^2+2x}\,dx = \tfrac{1}{2}\lim_{t\to\infty}\left[\ln(x^2+2x)\right]_1^t = \tfrac{1}{2}\lim_{t\to\infty}\left[\ln(t^2+2t)-\ln 3\right]$$
$= \infty.$ Divergent

19. $$\int_0^\infty se^{-5s}\,ds = \lim_{t\to\infty}\int_0^t se^{-5s}\,ds = \lim_{t\to\infty}\left[-\frac{1}{5}se^{-5s}-\frac{1}{25}e^{-5s}\right]_0^t \quad \left[\begin{array}{l}\text{by integration by}\\ \text{parts with } u=s\end{array}\right]$$
$$= \lim_{t\to\infty}\left(-\tfrac{1}{5}te^{-5t}-\tfrac{1}{25}e^{-5t}+\tfrac{1}{25}\right) = 0-0+\tfrac{1}{25} \quad \text{[by l'Hospital's Rule]}$$
$= \tfrac{1}{25}.$ Convergent

21. $$\int_1^\infty \frac{\ln x}{x}\,dx = \lim_{t\to\infty}\left[\frac{(\ln x)^2}{2}\right]_1^t \text{ (by substitution with } u=\ln x,\ du=dx/x) = \lim_{t\to\infty}\frac{(\ln t)^2}{2} = \infty. \quad \text{Divergent}$$

23. $$\int_{-\infty}^\infty \frac{x^2}{9+x^6}\,dx = \int_{-\infty}^0 \frac{x^2}{9+x^6}\,dx + \int_0^\infty \frac{x^2}{9+x^6}\,dx = 2\int_0^\infty \frac{x^2}{9+x^6}\,dx \quad \text{[since the integrand is even].}$$

Now $$\int\frac{x^2\,dx}{9+x^6}\ \left[\begin{array}{l}u=x^3\\ du=3x^2dx\end{array}\right] = \int\frac{\frac{1}{3}\,du}{9+u^2}\ \left[\begin{array}{l}u=3v\\ du=3\,dv\end{array}\right] = \int\frac{\frac{1}{3}(3\,dv)}{9+9v^2} = \frac{1}{9}\int\frac{dv}{1+v^2}$$
$$= \frac{1}{9}\tan^{-1}v + C = \frac{1}{9}\tan^{-1}\left(\frac{u}{3}\right)+C = \frac{1}{9}\tan^{-1}\left(\frac{x^3}{3}\right)+C,$$

so $$2\int_0^\infty \frac{x^2}{9+x^6}\,dx = 2\lim_{t\to\infty}\int_0^t \frac{x^2}{9+x^6}\,dx = 2\lim_{t\to\infty}\left[\frac{1}{9}\tan^{-1}\left(\frac{x^3}{3}\right)\right]_0^t$$
$$= 2\lim_{t\to\infty}\frac{1}{9}\tan^{-1}\left(\frac{t^3}{3}\right) = \frac{2}{9}\cdot\frac{\pi}{2} = \frac{\pi}{9}. \quad \text{Convergent}$$

25. Integrate by parts with $u=\ln x$, $dv = dx/x^2 \Rightarrow du = dx/x$, $v=-1/x$.

$$\int_1^\infty \frac{\ln x}{x^2}\,dx = \lim_{t\to\infty}\int_1^t \frac{\ln x}{x^2}\,dx = \lim_{t\to\infty}\left[-\frac{\ln x}{x}-\frac{1}{x}\right]_1^t = \lim_{t\to\infty}\left(-\frac{\ln t}{t}-\frac{1}{t}+0+1\right)$$
$$= -0-0+0+1 = 1$$

since $\lim_{t\to\infty}\frac{\ln t}{t} \overset{\text{H}}{=} \lim_{t\to\infty}\frac{1/t}{1} = 0.$ Convergent

27. There is an infinite discontinuity at the left endpoint of $[0,3]$.

$$\int_0^3 \frac{dx}{\sqrt{x}} = \lim_{t\to 0^+}\int_t^3 \frac{dx}{\sqrt{x}} = \lim_{t\to 0^+}\left[2\sqrt{x}\right]_t^3 = \lim_{t\to 0^+}\left(2\sqrt{3}-2\sqrt{t}\right) = 2\sqrt{3}.\quad \text{Convergent}$$

29. There is an infinite discontinuity at the right endpoint of $[-1,0]$.

$$\int_{-1}^0 \frac{dx}{x^2} = \lim_{t\to 0^-}\int_{-1}^t \frac{dx}{x^2} = \lim_{t\to 0^-}\left[\frac{-1}{x}\right]_{-1}^t = \lim_{t\to 0^-}\left[-\frac{1}{t}+\frac{1}{-1}\right] = \infty.\quad \text{Divergent}$$

31. $\displaystyle\int_{-2}^3 \frac{dx}{x^4} = \int_{-2}^0 \frac{dx}{x^4} + \int_0^3 \frac{dx}{x^4}$, but $\displaystyle\int_{-2}^0 \frac{dx}{x^4} = \lim_{t\to 0^-}\left[-\frac{x^{-3}}{3}\right]_{-2}^t = \lim_{t\to 0^-}\left[-\frac{1}{3t^3}-\frac{1}{24}\right] = \infty.$ Divergent

33. There is an infinite discontinuity at $x=1$. $\int_0^{33}(x-1)^{-1/5}\,dx = \int_0^1(x-1)^{-1/5}\,dx + \int_1^{33}(x-1)^{-1/5}\,dx$. Here

$$\int_0^1(x-1)^{-1/5}\,dx = \lim_{t\to 1^-}\int_0^t(x-1)^{-1/5}\,dx = \lim_{t\to 1^-}\left[\tfrac{5}{4}(x-1)^{4/5}\right]_0^t = \lim_{t\to 1^-}\left[\tfrac{5}{4}(t-1)^{4/5}-\tfrac{5}{4}\right] = -\tfrac{5}{4} \text{ and}$$

$$\int_1^{33}(x-1)^{-1/5}\,dx = \lim_{t\to 1^+}\int_t^{33}(x-1)^{-1/5}\,dx = \lim_{t\to 1^+}\left[\tfrac{5}{4}(x-1)^{4/5}\right]_t^{33} = \lim_{t\to 1^+}\left[\tfrac{5}{4}\cdot 16-\tfrac{5}{4}(t-1)^{4/5}\right] = 20.$$

Thus, $\int_0^{33}(x-1)^{-1/5}\,dx = -\frac{5}{4}+20 = \frac{75}{4}$. Convergent

35. $\int_0^{\pi}\sec x\,dx = \int_0^{\pi/2}\sec x\,dx + \int_{\pi/2}^{\pi}\sec x\,dx$. $\int_0^{\pi/2}\sec x\,dx = \lim\limits_{t\to\pi/2^-}\int_0^t \sec x\,dx$

$$= \lim_{t\to\pi/2^-}\Big[\ln|\sec x+\tan x|\Big]_0^t = \lim_{t\to\pi/2^-}\ln|\sec t+\tan t| = \infty.\quad \text{Divergent}$$

37. There is an infinite discontinuity at $x=0$. $\displaystyle\int_{-1}^1 \frac{e^x}{e^x-1}\,dx = \int_{-1}^0 \frac{e^x}{e^x-1}\,dx + \int_0^1 \frac{e^x}{e^x-1}\,dx.$

$$\int_{-1}^0 \frac{e^x}{e^x-1}\,dx = \lim_{t\to 0^-}\int_{-1}^t \frac{e^x}{e^x-1}\,dx = \lim_{t\to 0^-}\Big[\ln\left|e^x-1\right|\Big]_{-1}^t = \lim_{t\to 0^-}\Big[\ln\left|e^t-1\right|-\ln\left|e^{-1}-1\right|\Big] = -\infty,$$

so $\displaystyle\int_{-1}^1 \frac{e^x}{e^x-1}\,dx$ is divergent. The integral $\displaystyle\int_0^1 \frac{e^x}{e^x-1}\,dx$ also diverges since

$$\int_0^1 \frac{e^x}{e^x-1}\,dx = \lim_{t\to 0^+}\int_t^1 \frac{e^x}{e^x-1}\,dx = \lim_{t\to 0^+}\Big[\ln\left|e^x-1\right|\Big]_t^1 = \lim_{t\to 0^+}\Big[\ln|e-1|-\ln\left|e^t-1\right|\Big] = \infty.$$

Divergent

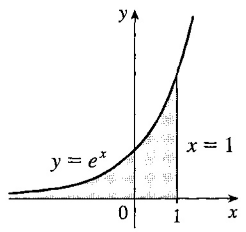
39. $I = \int_0^2 z^2\ln z\,dz = \lim\limits_{t\to 0^+}\int_t^2 z^2\ln z\,dz \overset{101}{=} \lim\limits_{t\to 0^+}\left[\dfrac{z^3}{3^2}(3\ln z-1)\right]_t^2$

$$= \lim_{t\to 0^+}\left[\tfrac{8}{9}(3\ln 2-1)-\tfrac{1}{9}t^3(3\ln t-1)\right] = \tfrac{8}{3}\ln 2-\tfrac{8}{9}-\tfrac{1}{9}\lim_{t\to 0^+}\left[t^3(3\ln t-1)\right] = \tfrac{8}{3}\ln 2-\tfrac{8}{9}-\tfrac{1}{9}L.$$

Now $L = \lim\limits_{t\to 0^+}\left[t^3(3\ln t-1)\right] = \lim\limits_{t\to 0^+}\dfrac{3\ln t-1}{t^{-3}} \overset{H}{=} \lim\limits_{t\to 0^+}\dfrac{3/t}{-3/t^4} = \lim\limits_{t\to 0^+}\left(-t^3\right) = 0$. Thus, $L=0$ and

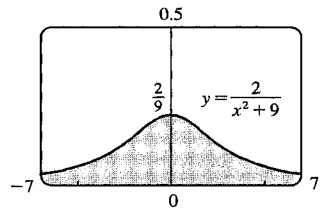
$I = \frac{8}{3}\ln 2-\frac{8}{9}$. Convergent

41.



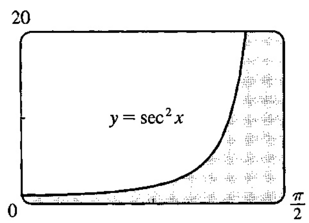
$$\text{Area} = \int_{-\infty}^{1} e^x\,dx = \lim_{t\to-\infty} \left[e^x\right]_t^1 = e - \lim_{t\to-\infty} e^t = e$$

43.



$$\text{Area} = \int_{-\infty}^{\infty} \frac{2}{x^2+9}\,dx = 2\cdot 2\int_0^{\infty}\frac{1}{x^2+9}\,dx = 4\lim_{t\to\infty}\int_0^t \frac{1}{x^2+9}\,dx = 4\lim_{t\to\infty}\left[\frac{1}{3}\tan^{-1}\frac{x}{3}\right]_0^t = \frac{4}{3}\lim_{t\to\infty}\left[\tan^{-1}\frac{t}{3} - 0\right] = \frac{4}{3}\cdot\frac{\pi}{2} = \frac{2\pi}{3}$$

45.



$$\text{Area} = \int_0^{\pi/2} \sec^2 x\,dx = \lim_{t\to(\pi/2)^-}\int_0^t \sec^2 x\,dx = \lim_{t\to(\pi/2)^-}\left[\tan x\right]_0^t = \lim_{t\to(\pi/2)^-}(\tan t - 0) = \infty$$

Infinite area

47. (a)

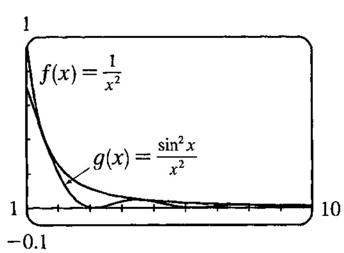
t	$\int_1^t g(x)\,dx$
2	0.447453
5	0.577101
10	0.621306
100	0.668479
1000	0.672957
10,000	0.673407

$g(x) = \dfrac{\sin^2 x}{x^2}$.

It appears that the integral is convergent.

(b) $-1 \le \sin x \le 1 \;\Rightarrow\; 0 \le \sin^2 x \le 1 \;\Rightarrow\; 0 \le \dfrac{\sin^2 x}{x^2} \le \dfrac{1}{x^2}$. Since $\displaystyle\int_1^{\infty}\frac{1}{x^2}\,dx$ is convergent (Equation 2 with $p = 2 > 1$), $\displaystyle\int_1^{\infty}\frac{\sin^2 x}{x^2}\,dx$ is convergent by the Comparison Theorem.

(c)



Since $\int_1^{\infty} f(x)\,dx$ is finite and the area under $g(x)$ is less than the area under $f(x)$ on any interval $[1, t]$, $\int_1^{\infty} g(x)\,dx$ must be finite; that is, the integral is convergent.

49. For $x \geq 1$, $\dfrac{\cos^2 x}{1+x^2} \leq \dfrac{1}{1+x^2} < \dfrac{1}{x^2}$. $\displaystyle\int_1^\infty \frac{1}{x^2}\,dx$ is convergent by Equation 2 with $p = 2 > 1$, so $\displaystyle\int_1^\infty \frac{\cos^2 x}{1+x^2}\,dx$ is convergent by the Comparison Theorem.

51. For $x \geq 1$, $x + e^{2x} > e^{2x} > 0 \;\Rightarrow\; \dfrac{1}{x+e^{2x}} \leq \dfrac{1}{e^{2x}} = e^{-2x}$ on $[1, \infty)$.

$\displaystyle\int_1^\infty e^{-2x}\,dx = \lim_{t\to\infty}\left[-\frac{1}{2}e^{-2x}\right]_1^t = \lim_{t\to\infty}\left[-\frac{1}{2}e^{-2t} + \frac{1}{2}e^{-2}\right] = \frac{1}{2}e^{-2}$. Therefore, $\displaystyle\int_1^\infty e^{-2x}\,dx$ is convergent, and by the Comparison Theorem, $\displaystyle\int_1^\infty \frac{dx}{x+e^{2x}}$ is also convergent.

53. $\dfrac{1}{x\sin x} \geq \dfrac{1}{x}$ on $\left(0, \frac{\pi}{2}\right]$ since $0 \leq \sin x \leq 1$. $\displaystyle\int_0^{\pi/2} \frac{dx}{x} = \lim_{t\to 0^+}\int_t^{\pi/2}\frac{dx}{x} = \lim_{t\to 0^+}\left[\ln x\right]_t^{\pi/2}$.

But $\ln t \to -\infty$ as $t \to 0^+$, so $\displaystyle\int_0^{\pi/2}\frac{dx}{x}$ is divergent, and by the Comparison Theorem, $\displaystyle\int_0^{\pi/2}\frac{dx}{x\sin x}$ is also divergent.

55. $\displaystyle\int_0^\infty \frac{dx}{\sqrt{x}\,(1+x)} = \int_0^1 \frac{dx}{\sqrt{x}\,(1+x)} + \int_1^\infty \frac{dx}{\sqrt{x}\,(1+x)} = \lim_{t\to 0^+}\int_t^1 \frac{dx}{\sqrt{x}\,(1+x)} + \lim_{t\to\infty}\int_1^t \frac{dx}{\sqrt{x}\,(1+x)}$. Now

$$\begin{aligned}\int \frac{dx}{\sqrt{x}\,(1+x)} &= \int \frac{2u\,du}{u(1+u^2)} \quad [u = \sqrt{x}, x = u^2, dx = 2u\,du] \\ &= 2\int \frac{du}{1+u^2} = 2\tan^{-1}u + C = 2\tan^{-1}\sqrt{x} + C,\end{aligned}$$

so

$$\begin{aligned}\int_0^\infty \frac{dx}{\sqrt{x}\,(1+x)} &= \lim_{t\to 0^+}\left[2\tan^{-1}\sqrt{x}\right]_t^1 + \lim_{t\to\infty}\left[2\tan^{-1}\sqrt{x}\right]_1^t \\ &= \lim_{t\to 0^+}\left[2\left(\tfrac{\pi}{4}\right) - 2\tan^{-1}\sqrt{t}\right] + \lim_{t\to\infty}\left[2\tan^{-1}\sqrt{t} - 2\left(\tfrac{\pi}{4}\right)\right] = \tfrac{\pi}{2} - 0 + 2\left(\tfrac{\pi}{2}\right) - \tfrac{\pi}{2} = \pi.\end{aligned}$$

57. If $p = 1$, then $\displaystyle\int_0^1 \frac{dx}{x^p} = \lim_{t\to 0^+}\int_t^1 \frac{dx}{x} = \lim_{t\to 0^+}\left[\ln x\right]_t^1 = \infty$. Divergent.

If $p \neq 1$, then $\displaystyle\int_0^1 \frac{dx}{x^p} = \lim_{t\to 0^+}\int_t^1 \frac{dx}{x^p}$ (note that the integral is not improper if $p < 0$)

$$= \lim_{t\to 0^+}\left[\frac{x^{-p+1}}{-p+1}\right]_t^1 = \lim_{t\to 0^+}\frac{1}{1-p}\left[1 - \frac{1}{t^{p-1}}\right]$$

If $p > 1$, then $p - 1 > 0$, so $\dfrac{1}{t^{p-1}} \to \infty$ as $t \to 0^+$, and the integral diverges.

If $p < 1$, then $p - 1 < 0$, so $\dfrac{1}{t^{p-1}} \to 0$ as $t \to 0^+$ and $\displaystyle\int_0^1 \frac{dx}{x^p} = \frac{1}{1-p}\left[\lim_{t\to 0^+}\left(1 - t^{1-p}\right)\right] = \frac{1}{1-p}$.

Thus, the integral converges if and only if $p < 1$, and in that case its value is $\dfrac{1}{1-p}$.

59. First suppose $p = -1$. Then

$$\int_0^1 x^p \ln x\,dx = \int_0^1 \frac{\ln x}{x}\,dx = \lim_{t\to 0^+}\int_t^1 \frac{\ln x}{x}\,dx = \lim_{t\to 0^+}\left[\tfrac{1}{2}(\ln x)^2\right]_t^1 = -\tfrac{1}{2}\lim_{t\to 0^+}(\ln t)^2 = -\infty,$$

so the integral diverges. Now suppose $p \neq -1$. Then integration by parts gives

$$\int x^p \ln x\,dx = \frac{x^{p+1}}{p+1}\ln x - \int \frac{x^p}{p+1}\,dx = \frac{x^{p+1}}{p+1}\ln x - \frac{x^{p+1}}{(p+1)^2} + C.$$ If $p < -1$, then $p+1<0$, so

$$\int_0^1 x^p \ln x\,dx = \lim_{t\to 0^+}\left[\frac{x^{p+1}}{p+1}\ln x - \frac{x^{p+1}}{(p+1)^2}\right]_t^1 = \frac{-1}{(p+1)^2} - \left(\frac{1}{p+1}\right)\lim_{t\to 0^+}\left[t^{p+1}\left(\ln t - \frac{1}{p+1}\right)\right] = \infty.$$

If $p > -1$, then $p+1 > 0$ and

$$\int_0^1 x^p \ln x\,dx = \frac{-1}{(p+1)^2} - \left(\frac{1}{p+1}\right)\lim_{t\to 0^+}\frac{\ln t - 1/(p+1)}{t^{-(p+1)}} \overset{\text{H}}{=} \frac{-1}{(p+1)^2} - \left(\frac{1}{p+1}\right)\lim_{t\to 0^+}\frac{1/t}{-(p+1)t^{-(p+2)}}$$

$$= \frac{-1}{(p+1)^2} + \frac{1}{(p+1)^2}\lim_{t\to 0^+} t^{p+1} = \frac{-1}{(p+1)^2}$$

Thus, the integral converges to $-\dfrac{1}{(p+1)^2}$ if $p > -1$ and diverges otherwise.

61. (a) $I = \int_{-\infty}^{\infty} x\,dx = \int_{-\infty}^{0} x\,dx + \int_0^{\infty} x\,dx$, and

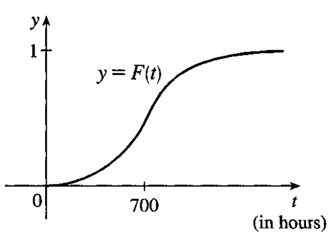
$\int_0^{\infty} x\,dx = \lim_{t\to\infty}\int_0^t x\,dx = \lim_{t\to\infty}\left[\tfrac{1}{2}x^2\right]_0^t = \lim_{t\to\infty}\left[\tfrac{1}{2}t^2 - 0\right] = \infty$, so I is divergent.

(b) $\int_{-t}^{t} x\,dx = \left[\tfrac{1}{2}x^2\right]_{-t}^{t} = \tfrac{1}{2}t^2 - \tfrac{1}{2}t^2 = 0$, so $\lim_{t\to\infty}\int_{-t}^{t} x\,dx = 0$. Therefore, $\int_{-\infty}^{\infty} x\,dx \neq \lim_{t\to\infty}\int_{-t}^{t} x\,dx$.

63. Volume $= \displaystyle\int_1^{\infty} \pi\left(\frac{1}{x}\right)^2 dx = \pi\lim_{t\to\infty}\int_1^t \frac{dx}{x^2} = \pi\lim_{t\to\infty}\left[-\frac{1}{x}\right]_1^t = \pi\lim_{t\to\infty}\left(1-\frac{1}{t}\right) = \pi < \infty.$

65. Work $= \displaystyle\int_R^{\infty} F\,dr = \lim_{t\to\infty}\int_R^t \frac{GmM}{r^2}\,dr = \lim_{t\to\infty} GmM\left(\frac{1}{R}-\frac{1}{t}\right) = \frac{GmM}{R}$. The initial kinetic energy provides the work, so $\frac{1}{2}mv_0^2 = \dfrac{GmM}{R} \Rightarrow v_0 = \sqrt{\dfrac{2GM}{R}}$.

67. (a) We would expect a small percentage of bulbs to burn out in the first few hundred hours, most of the bulbs to burn out after close to 700 hours, and a few overachievers to burn on and on.



(b) $r(t) = F'(t)$ is the rate at which the fraction $F(t)$ of burnt-out bulbs increases as t increases. This could be interpreted as a fractional burnout rate.

(c) $\int_0^{\infty} r(t)\,dt = \lim_{x\to\infty} F(x) = 1$, since all of the bulbs will eventually burn out.

69. $I = \int_a^\infty \frac{1}{x^2+1}\,dx = \lim_{t\to\infty} \int_a^t \frac{1}{x^2+1}\,dx = \lim_{t\to\infty} \left[\tan^{-1} x\right]_a^t = \lim_{t\to\infty} \left(\tan^{-1} t - \tan^{-1} a\right) = \frac{\pi}{2} - \tan^{-1} a.$

$I < 0.001 \Rightarrow \frac{\pi}{2} - \tan^{-1} a < 0.001 \Rightarrow \tan^{-1} a > \frac{\pi}{2} - 0.001 \Rightarrow a > \tan\left(\frac{\pi}{2} - 0.001\right) \approx 1000.$

71. (a) $F(s) = \int_0^\infty f(t)e^{-st}\,dt = \int_0^\infty e^{-st}\,dt = \lim_{n\to\infty} \left[-\frac{e^{-st}}{s}\right]_0^n = \lim_{n\to\infty} \left(\frac{e^{-sn}}{-s} + \frac{1}{s}\right)$. This converges to $\frac{1}{s}$ only if $s > 0$. Therefore $F(s) = \frac{1}{s}$ with domain $\{s \mid s > 0\}$.

(b) $F(s) = \int_0^\infty f(t)e^{-st}\,dt = \int_0^\infty e^t e^{-st}\,dt = \lim_{n\to\infty} \int_0^n e^{t(1-s)}\,dt = \lim_{n\to\infty} \left[\frac{1}{1-s} e^{t(1-s)}\right]_0^n$

$= \lim_{n\to\infty} \left(\frac{e^{(1-s)n}}{1-s} - \frac{1}{1-s}\right)$

This converges only if $1 - s < 0 \Rightarrow s > 1$, in which case $F(s) = \frac{1}{s-1}$ with domain $\{s \mid s > 1\}$.

(c) $F(s) = \int_0^\infty f(t)e^{-st}\,dt = \lim_{n\to\infty} \int_0^n te^{-st}\,dt$. Use integration by parts: let $u = t$, $dv = e^{-st}\,dt \Rightarrow$ $du = dt$, $v = -\frac{e^{-st}}{s}$. Then $F(s) = \lim_{n\to\infty} \left[-\frac{t}{s}e^{-st} - \frac{1}{s^2}e^{-st}\right]_0^n = \lim_{n\to\infty} \left(\frac{-n}{se^{sn}} - \frac{1}{s^2e^{sn}} + 0 + \frac{1}{s^2}\right) = \frac{1}{s^2}$ only if $s > 0$. Therefore, $F(s) = \frac{1}{s^2}$ and the domain of F is $\{s \mid s > 0\}$.

73. $G(s) = \int_0^\infty f'(t)e^{-st}\,dt$. Integrate by parts with $u = e^{-st}$, $dv = f'(t)\,dt \Rightarrow du = -se^{-st}$, $v = f(t)$:

$$G(s) = \lim_{n\to\infty} \left[f(t)e^{-st}\right]_0^n + s\int_0^\infty f(t)e^{-st}\,dt = \lim_{n\to\infty} f(n)e^{-sn} - f(0) + sF(s)$$

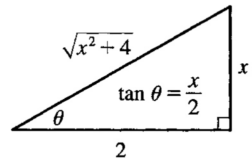
But $0 \le f(t) \le Me^{at} \Rightarrow 0 \le f(t)e^{-st} \le Me^{at}e^{-st}$ and $\lim_{t\to\infty} Me^{t(a-s)} = 0$ for $s > a$. So by the Squeeze Theorem, $\lim_{t\to\infty} f(t)e^{-st} = 0$ for $s > a \Rightarrow G(s) = 0 - f(0) + sF(s) = sF(s) - f(0)$ for $s > a$.

75. We use integration by parts: let $u = x$, $dv = xe^{-x^2}\,dx \Rightarrow du = dx$, $v = -\frac{1}{2}e^{-x^2}$. So

$$\int_0^\infty x^2e^{-x^2}\,dx = \lim_{t\to\infty} \left[-\tfrac{1}{2}xe^{-x^2}\right]_0^t + \tfrac{1}{2}\int_0^\infty e^{-x^2}\,dx$$

$$= \lim_{t\to\infty} \left[-t\Big/\left(2e^{t^2}\right)\right] + \tfrac{1}{2}\int_0^\infty e^{-x^2}\,dx = \tfrac{1}{2}\int_0^\infty e^{-x^2}\,dx$$

(The limit is 0 by l'Hospital's Rule.)



77. For the first part of the integral, let $x = 2\tan\theta \;\Rightarrow\; dx = 2\sec^2\theta\,d\theta$.

$\int \frac{1}{\sqrt{x^2+4}}\,dx = \int \frac{2\sec^2\theta}{2\sec\theta}\,d\theta = \int \sec\theta\,d\theta = \ln|\sec\theta + \tan\theta|$. From the figure, $\tan\theta = \frac{x}{2}$, and $\sec\theta = \frac{\sqrt{x^2+4}}{2}$. So

$$I = \int_0^\infty \left(\frac{1}{\sqrt{x^2+4}} - \frac{C}{x+2}\right)dx = \lim_{t\to\infty}\left[\ln\left|\frac{\sqrt{x^2+4}}{2} + \frac{x}{2}\right| - C\ln|x+2|\right]_0^t$$

$$= \lim_{t\to\infty}\left[\ln\frac{\sqrt{t^2+4}+t}{2} - C\ln(t+2) - (\ln 1 - C\ln 2)\right]$$

$$= \lim_{t\to\infty}\left[\ln\left(\frac{\sqrt{t^2+4}+t}{2\,(t+2)^C}\right) + \ln 2^C\right]$$

$$= \ln\left(\lim_{t\to\infty}\frac{t+\sqrt{t^2+4}}{(t+2)^C}\right) + \ln 2^{C-1}$$

Now $L = \lim\limits_{t\to\infty}\frac{t+\sqrt{t^2+4}}{(t+2)^C} \overset{\text{H}}{=} \lim\limits_{t\to\infty}\frac{1+t/\sqrt{t^2+4}}{C\,(t+2)^{C-1}} = \frac{2}{C\lim\limits_{t\to\infty}(t+2)^{C-1}}$.

If $C < 1$, $L = \infty$ and I diverges. If $C = 1$, $L = 2$ and I converges to $\ln 2 + \ln 2^0 = \ln 2$. If $C > 1$, $L = 0$ and I diverges to $-\infty$.

7 Review

CONCEPT CHECK

1. See Formula 7.1.1 or 7.1.2. We try to choose $u = f(x)$ to be a function that becomes simpler when differentiated (or at least not more complicated) as long as $dv = g'(x)\,dx$ can be readily integrated to give v.

2. See the Strategy for Evaluating $\int \sin^m x\cos^n x\,dx$ on page 484.

3. If $\sqrt{a^2-x^2}$ occurs, try $x = a\sin\theta$; if $\sqrt{a^2+x^2}$ occurs, try $x = a\tan\theta$, and if $\sqrt{x^2-a^2}$ occurs, try $x = a\sec\theta$. See the Table of Trigonometric Substitutions on page 490.

4. See Equation 2 and Expressions 7, 9, and 11 in Section 7.4.

5. See the Midpoint Rule, the Trapezoidal Rule, and Simpson's Rule, as well as their associated error bounds, all in Section 7.7. We would expect the best estimate to be given by Simpson's Rule.

6. See Definitions 1(a), (b), and (c) in Section 7.8.

7. See Definitions 3(b), (a), and (c) in Section 7.8.

8. See the Comparison Theorem after Example 8 in Section 7.8.

TRUE-FALSE QUIZ

1. False. Since the numerator has a higher degree than the denominator,

$$\frac{x(x^2+4)}{x^2-4} = x + \frac{8x}{x^2-4} = x + \frac{A}{x+2} + \frac{B}{x-2}.$$

3. False. It can be put in the form $\frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-4}$.

5. False. This is an improper integral, since the denominator vanishes at $x = 1$.

$$\int_0^4 \frac{x}{x^2-1}\,dx = \int_0^1 \frac{x}{x^2-1}\,dx + \int_1^4 \frac{x}{x^2-1}\,dx \text{ and}$$

$$\int_0^1 \frac{x}{x^2-1}\,dx = \lim_{t\to 1^-} \int_0^t \frac{x}{x^2-1}\,dx = \lim_{t\to 1^-} \left[\tfrac{1}{2}\ln\left|x^2-1\right|\right]_0^t = \lim_{t\to 1^-} \tfrac{1}{2}\ln\left|t^2-1\right| = \infty$$

So the integral diverges.

7. False. See Exercise 61 in Section 7.8.

9. (a) True. See the end of Section 7.5.

(b) False. Examples include the functions $f(x) = e^{x^2}$, $g(x) = \sin(x^2)$, and $h(x) = \frac{\sin x}{x}$.

11. False. If $f(x) = 1/x$, then f is continuous and decreasing on $[1, \infty)$ with $\lim_{x\to\infty} f(x) = 0$, but $\int_1^\infty f(x)\,dx$ is divergent.

13. False. Take $f(x) = 1$ for all x and $g(x) = -1$ for all x. Then $\int_a^\infty f(x)\,dx = \infty$ [divergent] and $\int_a^\infty g(x)\,dx = -\infty$ [divergent], but $\int_a^\infty [f(x) + g(x)]\,dx = 0$ [convergent].

EXERCISES

1. $$\int_0^5 \frac{x}{x+10}\,dx = \int_0^5 \left(1 - \frac{10}{x+10}\right) dx = \Big[x - 10\ln(x+10)\Big]_0^5$$

$$= 5 - 10\ln 15 + 10\ln 10 = 5 + 10\ln\tfrac{10}{15} = 5 + 10\ln\tfrac{2}{3}$$

3. $$\int_0^{\pi/2} \frac{\cos\theta}{1+\sin\theta}\,d\theta = \Big[\ln(1+\sin\theta)\Big]_0^{\pi/2} = \ln 2 - \ln 1 = \ln 2$$

5. Let $u = \sec x$. Then $du = \sec x\tan x\,dx$, so

$$\int \tan^7 x\sec^3 x\,dx = \int \tan^6 x\sec^2 x\sec x\tan x\,dx = \int \left(u^2-1\right)^3 u^2\,du = \int \left(u^8 - 3u^6 + 3u^4 - u^2\right) du$$

$$= \tfrac{1}{9}u^9 - \tfrac{3}{7}u^7 + \tfrac{3}{5}u^5 - \tfrac{1}{3}u^3 + C = \tfrac{1}{9}\sec^9 x - \tfrac{3}{7}\sec^7 x + \tfrac{3}{5}\sec^5 x - \tfrac{1}{3}\sec^3 x + C$$

7. Let $u = \ln t$, $du = dt/t$. Then $\displaystyle\int \frac{\sin(\ln t)}{t}\,dt = \int \sin u\,du = -\cos u + C = -\cos(\ln t) + C$.

9. $$\int_1^4 x^{3/2}\ln x\,dx \quad \begin{bmatrix} u = \ln x, & dv = x^{3/2}\,dx, \\ du = dx/x & v = \frac{2}{5}x^{5/2} \end{bmatrix} \quad = \frac{2}{5}\Big[x^{5/2}\ln x\Big]_1^4 - \frac{2}{5}\int_1^4 x^{3/2}\,dx$$

$$= \tfrac{2}{5}(32\ln 4 - \ln 1) - \tfrac{2}{5}\left[\tfrac{2}{5}x^{5/2}\right]_1^4$$

$$= \tfrac{2}{5}(64\ln 2) - \tfrac{4}{25}(32-1)$$

$$= \tfrac{128}{5}\ln 2 - \tfrac{124}{25} \quad \left(\text{or } \tfrac{64}{5}\ln 4 - \tfrac{124}{25}\right)$$

11. Let $x = \sec\theta$. Then

$$\int_1^2 \frac{\sqrt{x^2-1}}{x}\,dx = \int_0^{\pi/3} \frac{\tan\theta}{\sec\theta}\sec\theta\tan\theta\,d\theta = \int_0^{\pi/3}\tan^2\theta\,d\theta = \int_0^{\pi/3}\left(\sec^2\theta - 1\right)d\theta$$
$$= [\tan\theta - \theta]_0^{\pi/3} = \sqrt{3} - \tfrac{\pi}{3}$$

13. $\displaystyle\int \frac{dx}{x^3+x} = \int\left(\frac{1}{x} - \frac{x}{x^2+1}\right)dx = \ln|x| - \tfrac{1}{2}\ln\left(x^2+1\right) + C$

15. $\int \sin^2\theta\cos^5\theta\,d\theta = \int \sin^2\theta\left(\cos^2\theta\right)^2\cos\theta\,d\theta = \int \sin^2\theta\left(1-\sin^2\theta\right)^2\cos\theta\,d\theta$

$= \int u^2\left(1-u^2\right)^2du \quad [u = \sin\theta,\ du = \cos\theta\,d\theta] \quad = \int u^2\left(1-2u^2+u^4\right)du$

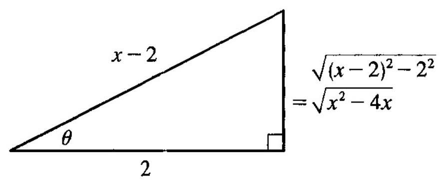
$= \int\left(u^2-2u^4+u^6\right)du = \tfrac{1}{3}u^3 - \tfrac{2}{5}u^5 + \tfrac{1}{7}u^7 + C = \tfrac{1}{3}\sin^3\theta - \tfrac{2}{5}\sin^5\theta + \tfrac{1}{7}\sin^7\theta + C$

17. Integrate by parts with $u = x$, $dv = \sec x\tan x\,dx \quad\Rightarrow\quad du = dx$, $v = \sec x$:

$\int x\sec x\,\tan x\,dx = x\sec x - \int\sec x\,dx \overset{14}{=} x\sec x - \ln|\sec x + \tan x| + C.$

19. $$\int\frac{x+1}{9x^2+6x+5}\,dx = \int\frac{x+1}{(9x^2+6x+1)+4}\,dx = \int\frac{x+1}{(3x+1)^2+4}\,dx \quad \begin{bmatrix} u = 3x+1, \\ du = 3\,dx\end{bmatrix}$$
$$= \int\frac{\left[\frac{1}{3}(u-1)\right]+1}{u^2+4}\left(\frac{1}{3}\,du\right) = \frac{1}{3}\cdot\frac{1}{3}\int\frac{(u-1)+3}{u^2+4}\,du$$
$$= \frac{1}{9}\int\frac{u}{u^2+4}\,du + \frac{1}{9}\int\frac{2}{u^2+2^2}\,du = \frac{1}{9}\cdot\frac{1}{2}\ln(u^2+4) + \frac{2}{9}\cdot\frac{1}{2}\tan^{-1}\left(\frac{1}{2}u\right) + C$$
$$= \tfrac{1}{18}\ln(9x^2+6x+5) + \tfrac{1}{9}\tan^{-1}\left[\tfrac{1}{2}(3x+1)\right] + C$$

21.



$$\int\frac{dx}{\sqrt{x^2-4x}} = \int\frac{dx}{\sqrt{(x^2-4x+4)-4}} = \int\frac{dx}{\sqrt{(x-2)^2-2^2}}$$
$$= \int\frac{2\sec\theta\tan\theta\,d\theta}{2\tan\theta}\quad\begin{bmatrix} x-2 = 2\sec\theta, \\ dx = 2\sec\theta\tan\theta\,d\theta\end{bmatrix} = \int\sec\theta\,d\theta = \ln|\sec\theta+\tan\theta| + C_1$$
$$= \ln\left|\frac{x-2}{2} + \frac{\sqrt{x^2-4x}}{2}\right| + C_1 = \ln\left|x-2+\sqrt{x^2-4x}\right| + C, \text{ where } C = C_1 - \ln 2$$

23. Let $u = \cot 4x$. Then $du = -4\csc^2 4x\,dx \quad\Rightarrow$

$$\int\csc^4 4x\,dx = \int\left(\cot^2 4x + 1\right)\csc^2 4x\,dx = \int\left(u^2+1\right)\left(-\tfrac{1}{4}\,du\right)$$
$$= -\tfrac{1}{4}\left(\tfrac{1}{3}u^3 + u\right) + C = -\tfrac{1}{12}\left(\cot^3 4x + 3\cot 4x\right) + C$$

25. $\dfrac{3x^3 - x^2 + 6x - 4}{(x^2+1)(x^2+2)} = \dfrac{Ax+B}{x^2+1} + \dfrac{Cx+D}{x^2+2} \quad \Rightarrow$

$3x^3 - x^2 + 6x - 4 = (Ax+B)(x^2+2) + (Cx+D)(x^2+1)$. Equating the coefficients gives $A + C = 3$, $B + D = -1$, $2A + C = 6$, and $2B + D = -4 \quad \Rightarrow \quad A = 3$, $C = 0$, $B = -3$, and $D = 2$. Now

$$\int \frac{3x^3 - x^2 + 6x - 4}{(x^2+1)(x^2+2)}\,dx = 3\int \frac{x-1}{x^2+1}\,dx + 2\int \frac{dx}{x^2+2}$$

$$= \tfrac{3}{2}\ln(x^2+1) - 3\tan^{-1}x + \sqrt{2}\tan^{-1}\left(\tfrac{1}{\sqrt{2}}x\right) + C$$

27. $\int_0^{\pi/2} \cos^3 x \sin 2x\,dx = \int_0^{\pi/2} \cos^3 x\,(2\sin x\cos x)\,dx = \int_0^{\pi/2} 2\cos^4 x \sin x\,dx = \left[-\tfrac{2}{5}\cos^5 x\right]_0^{\pi/2} = \tfrac{2}{5}$

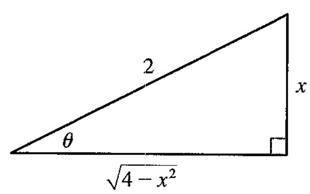
29. The product of an odd function and an even function is an odd function, so $f(x) = x^5 \sec x$ is an odd function. By Theorem 5.5.7(b), $\int_{-1}^{1} x^5 \sec x\,dx = 0$.

31. Let $u = \sqrt{e^x - 1}$. Then $u^2 = e^x - 1$ and $2u\,du = e^x\,dx$. Also, $e^x + 8 = u^2 + 9$. Thus,

$$\int_0^{\ln 10} \frac{e^x\sqrt{e^x-1}}{e^x+8}\,dx = \int_0^3 \frac{u\cdot 2u\,du}{u^2+9} = 2\int_0^3 \frac{u^2}{u^2+9}\,du = 2\int_0^3 \left(1 - \frac{9}{u^2+9}\right)du$$

$$= 2\left[u - \frac{9}{3}\tan^{-1}\left(\frac{u}{3}\right)\right]_0^3 = 2\left[(3 - 3\tan^{-1}1) - 0\right] = 2\left(3 - 3\cdot\frac{\pi}{4}\right) = 6 - \frac{3\pi}{2}$$

33. Let $x = 2\sin\theta \quad \Rightarrow \quad (4 - x^2)^{3/2} = (2\cos\theta)^3$, $dx = 2\cos\theta\,d\theta$, so



$$\int \frac{x^2}{(4-x^2)^{3/2}}\,dx = \int \frac{4\sin^2\theta}{8\cos^3\theta}\,2\cos\theta\,d\theta$$

$$= \int \tan^2\theta\,d\theta = \int (\sec^2\theta - 1)\,d\theta$$

$$= \tan\theta - \theta + C = \frac{x}{\sqrt{4-x^2}} - \sin^{-1}\left(\frac{x}{2}\right) + C$$

35. $\displaystyle\int \frac{1}{\sqrt{x + x^{3/2}}}\,dx = \int \frac{dx}{\sqrt{x(1+\sqrt{x})}} = \int \frac{dx}{\sqrt{x}\sqrt{1+\sqrt{x}}} \quad \begin{bmatrix} u = 1+\sqrt{x}, \\ du = \dfrac{dx}{2\sqrt{x}} \end{bmatrix} \quad = \int \frac{2\,du}{\sqrt{u}} = \int 2u^{-1/2}\,du$

$= 4\sqrt{u} + C = 4\sqrt{1+\sqrt{x}} + C$

37. $\int (\cos x + \sin x)^2 \cos 2x\,dx = \int \left(\cos^2 x + 2\sin x\cos x + \sin^2 x\right)\cos 2x\,dx$

$= \int (1 + \sin 2x)\cos 2x\,dx = \int \cos 2x\,dx + \tfrac{1}{2}\int \sin 4x\,dx = \tfrac{1}{2}\sin 2x - \tfrac{1}{8}\cos 4x + C$

Or: $\int (\cos x + \sin x)^2 \cos 2x\,dx = \int (\cos x + \sin x)^2\left(\cos^2 x - \sin^2 x\right)dx$

$= \int (\cos x + \sin x)^3(\cos x - \sin x)\,dx = \tfrac{1}{4}(\cos x + \sin x)^4 + C_1$

39. We'll integrate $I = \int \frac{xe^{2x}}{(1+2x)^2}\,dx$ by parts with $u = xe^{2x}$ and $dv = \frac{dx}{(1+2x)^2}$. Then

$du = (x \cdot 2e^{2x} + e^{2x} \cdot 1)\,dx$ and $v = -\frac{1}{2} \cdot \frac{1}{1+2x}$, so

$$I = -\frac{1}{2} \cdot \frac{xe^{2x}}{1+2x} - \int \left[-\frac{1}{2} \cdot \frac{e^{2x}(2x+1)}{1+2x}\right] dx = -\frac{xe^{2x}}{4x+2} + \frac{1}{2} \cdot \frac{1}{2}e^{2x} + C$$
$$= e^{2x}\left(\frac{1}{4} - \frac{x}{4x+2}\right) + C.$$

Thus, $\int_0^{1/2} \frac{xe^{2x}}{(1+2x)^2}\,dx = \left[e^{2x}\left(\frac{1}{4} - \frac{x}{4x+2}\right)\right]_0^{1/2} = e\left(\frac{1}{4} - \frac{1}{8}\right) - 1\left(\frac{1}{4} - 0\right) = \frac{1}{8}e - \frac{1}{4}$.

41. $\int_1^\infty \frac{1}{(2x+1)^3}\,dx = \lim_{t\to\infty} \int_1^t \frac{1}{(2x+1)^3}\,dx = \lim_{t\to\infty} \int_1^t \frac{1}{2}(2x+1)^{-3}\,2\,dx$

$$= \lim_{t\to\infty}\left[-\frac{1}{4(2x+1)^2}\right]_1^t = -\frac{1}{4}\lim_{t\to\infty}\left[\frac{1}{(2t+1)^2} - \frac{1}{9}\right] = -\frac{1}{4}\left(0 - \frac{1}{9}\right) = \frac{1}{36}$$

43. $\int \frac{dx}{x\ln x} \quad \left[\begin{array}{l} u = \ln x, \\ du = \frac{dx}{x} \end{array}\right] = \int \frac{du}{u} = \ln|u| + C = \ln|\ln x| + C$, so

$\int_2^\infty \frac{dx}{x\ln x} = \lim_{t\to\infty} \int_2^t \frac{dx}{x\ln x} = \lim_{t\to\infty}\Big[\ln|\ln x|\Big]_2^t = \lim_{t\to\infty}\left[\ln(\ln t) - \ln(\ln 2)\right] = \infty$, so the integral is divergent.

45. $\int_0^4 \frac{\ln x}{\sqrt{x}}\,dx = \lim_{t\to 0^+} \int_t^4 \frac{\ln x}{\sqrt{x}}\,dx \overset{*}{=} \lim_{t\to 0^+}\left[2\sqrt{x}\,\ln x - 4\sqrt{x}\right]_t^4$

$$= \lim_{t\to 0^+}\left[(2\cdot 2\ln 4 - 4\cdot 2) - (2\sqrt{t}\,\ln t - 4\sqrt{t})\right] \overset{**}{=} (4\ln 4 - 8) - (0 - 0) = 4\ln 4 - 8$$

(*) Let $u = \ln x$, $dv = \frac{1}{\sqrt{x}}\,dx \Rightarrow du = \frac{1}{x}\,dx$, $v = 2\sqrt{x}$. Then

$$\int \frac{\ln x}{\sqrt{x}}\,dx = 2\sqrt{x}\,\ln x - 2\int \frac{dx}{\sqrt{x}} = 2\sqrt{x}\,\ln x - 4\sqrt{x} + C$$

(**) $\lim_{t\to 0^+}(2\sqrt{t}\,\ln t) = \lim_{t\to 0^+} \frac{2\ln t}{t^{-1/2}} \overset{H}{=} \lim_{t\to 0^+} \frac{2/t}{-\frac{1}{2}t^{-3/2}} = \lim_{t\to 0^+}(-4\sqrt{t}) = 0$

47. $\int_0^3 \frac{dx}{x^2 - x - 2} = \int_0^3 \frac{dx}{(x+1)(x-2)} = \int_0^2 \frac{dx}{(x+1)(x-2)} + \int_2^3 \frac{dx}{(x+1)(x-2)}$, and

$$\int_2^3 \frac{dx}{x^2 - x - 2} = \lim_{t\to 2^+} \int_t^3 \left[\frac{-1/3}{x+1} + \frac{1/3}{x-2}\right] dx = \lim_{t\to 2^+}\left[\frac{1}{3}\ln\left|\frac{x-2}{x+1}\right|\right]_t^3$$
$$= \lim_{t\to 2^+}\left[\frac{1}{3}\ln\frac{1}{4} - \frac{1}{3}\ln\left|\frac{t-2}{t+1}\right|\right] = \infty$$

so $\int_0^3 \frac{dx}{x^2 - x - 2}$ diverges.

49. Let $u = 2x + 1$. Then

$$\int_{-\infty}^{\infty} \frac{dx}{4x^2 + 4x + 5} = \int_{-\infty}^{\infty} \frac{\frac{1}{2}\,du}{u^2 + 4} = \frac{1}{2}\int_{-\infty}^{0} \frac{du}{u^2 + 4} + \frac{1}{2}\int_{0}^{\infty} \frac{du}{u^2 + 4}$$

$$= \tfrac{1}{2} \lim_{t \to -\infty} \left[\tfrac{1}{2} \tan^{-1}\left(\tfrac{1}{2}u\right)\right]_t^0 + \tfrac{1}{2} \lim_{t \to \infty} \left[\tfrac{1}{2} \tan^{-1}\left(\tfrac{1}{2}u\right)\right]_0^t$$

$$= \tfrac{1}{4}\left[0 - \left(-\tfrac{\pi}{2}\right)\right] + \tfrac{1}{4}\left[\tfrac{\pi}{2} - 0\right] = \tfrac{\pi}{4}$$

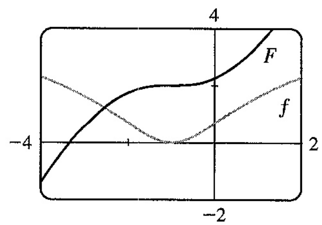
51. We first make the substitution $t = x + 1$, so $\ln(x^2 + 2x + 2) = \ln\left[(x+1)^2 + 1\right] = \ln(t^2 + 1)$. Then we use parts with $u = \ln(t^2 + 1)$, $dv = dt$:

$$\int \ln(t^2 + 1)\, dt = t\,\ln(t^2 + 1) - \int \frac{t(2t)\,dt}{t^2 + 1} = t\,\ln(t^2 + 1) - 2\int \frac{t^2\,dt}{t^2 + 1}$$

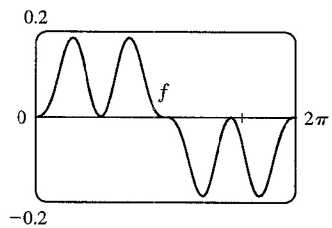
$$= t\,\ln(t^2 + 1) - 2\int \left(1 - \frac{1}{t^2 + 1}\right) dt = t\,\ln(t^2 + 1) - 2t + 2\arctan t + C$$

$$= (x+1)\,\ln(x^2 + 2x + 2) - 2x + 2\arctan(x+1) + K, \text{ where } K = C - 2$$

[Alternatively, we could have integrated by parts immediately with $u = \ln(x^2 + 2x + 2)$.] Notice from the graph that $f = 0$ where F has a horizontal tangent. Also, F is always increasing, and $f \geq 0$.



53. From the graph, it seems as though $\int_0^{2\pi} \cos^2 x \,\sin^3 x\, dx$ is equal to 0. To evaluate the integral, we write the integral as $I = \int_0^{2\pi} \cos^2 x\,(1 - \cos^2 x)\sin x\, dx$ and let $u = \cos x \Rightarrow du = -\sin x\, dx$. Thus, $I = \int_1^1 u^2(1 - u^2)(-du) = 0$.



55. $u = e^x \Rightarrow du = e^x\, dx$, so

$$\int e^x\sqrt{1 - e^{2x}}\, dx = \int \sqrt{1 - u^2}\, du \overset{30}{=} \tfrac{1}{2}u\sqrt{1 - u^2} + \tfrac{1}{2}\sin^{-1} u + C = \tfrac{1}{2}\left[e^x\sqrt{1 - e^{2x}} + \sin^{-1}(e^x)\right] + C$$

57. $\displaystyle\int \sqrt{x^2 + x + 1}\, dx = \int \sqrt{x^2 + x + \tfrac{1}{4} + \tfrac{3}{4}}\, dx = \int \sqrt{\left(x + \tfrac{1}{2}\right)^2 + \tfrac{3}{4}}\, dx$

$$= \int \sqrt{u^2 + \left(\tfrac{\sqrt{3}}{2}\right)^2}\, du \quad \left[u = x + \tfrac{1}{2},\ du = dx\right]$$

$$\overset{21}{=} \tfrac{1}{2}u\sqrt{u^2 + \tfrac{3}{4}} + \tfrac{3}{8}\ln\left(u + \sqrt{u^2 + \tfrac{3}{4}}\right) + C$$

$$= \frac{2x + 1}{4}\sqrt{x^2 + x + 1} + \tfrac{3}{8}\ln\left(x + \tfrac{1}{2} + \sqrt{x^2 + x + 1}\right) + C$$

59. (a) $\dfrac{d}{du}\left[-\dfrac{1}{u}\sqrt{a^2-u^2}-\sin^{-1}\left(\dfrac{u}{a}\right)+C\right]=\dfrac{1}{u^2}\sqrt{a^2-u^2}+\dfrac{1}{\sqrt{a^2-u^2}}-\dfrac{1}{\sqrt{1-u^2/a^2}}\cdot\dfrac{1}{a}$

$$=\left(a^2-u^2\right)^{-1/2}\left[\frac{1}{u^2}\left(a^2-u^2\right)+1-1\right]=\frac{\sqrt{a^2-u^2}}{u^2}$$

(b) Let $u=a\sin\theta \;\Rightarrow\; du=a\cos\theta\,d\theta,\ a^2-u^2=a^2\left(1-\sin^2\theta\right)=a^2\cos^2\theta.$

$$\int\frac{\sqrt{a^2-u^2}}{u^2}\,du=\int\frac{a^2\cos^2\theta}{a^2\sin^2\theta}\,d\theta=\int\frac{1-\sin^2\theta}{\sin^2\theta}\,d\theta=\int\left(\csc^2\theta-1\right)d\theta=-\cot\theta-\theta+C$$

$$=-\frac{\sqrt{a^2-u^2}}{u}-\sin^{-1}\left(\frac{u}{a}\right)+C$$

61. For $n\geq 0$, $\int_0^\infty x^n\,dx=\lim\limits_{t\to\infty}\left[x^{n+1}/(n+1)\right]_0^t=\infty$. For $n<0$, $\int_0^\infty x^n\,dx=\int_0^1 x^n\,dx+\int_1^\infty x^n\,dx$. Both integrals are improper. By (7.8.2), the second integral diverges if $-1\leq n<0$. By Exercise 7.8.57, the first integral diverges if $n\leq -1$. Thus, $\int_0^\infty x^n\,dx$ is divergent for all values of n.

63. $f(x)=\sqrt{1+x^4}$, $\Delta x=\dfrac{b-a}{n}=\dfrac{1-0}{10}=\dfrac{1}{10}$.

(a) $T_{10}=\frac{1}{10\cdot 2}\{f(0)+2\left[f(0.1)+f(0.2)+\cdots+f(0.9)\right]+f(1)\}\approx 1.090608$

(b) $M_{10}=\frac{1}{10}\left[f\left(\frac{1}{20}\right)+f\left(\frac{3}{20}\right)+f\left(\frac{5}{20}\right)+\cdots+f\left(\frac{19}{20}\right)\right]\approx 1.088840$

(c) $S_{10}=\frac{1}{10\cdot 3}\left[f(0)+4f(0.1)+2f(0.2)+\cdots+4f(0.9)+f(1)\right]\approx 1.089429$

f is concave upward, so the Trapezoidal Rule gives us an overestimate, the Midpoint Rule gives an underestimate, and we cannot tell whether Simpson's Rule gives us an overestimate or an underestimate.

65. $f(x)=\left(1+x^4\right)^{1/2}$, $f'(x)=\frac{1}{2}\left(1+x^4\right)^{-1/2}\left(4x^3\right)=2x^3\left(1+x^4\right)^{-1/2}$, $f''(x)=\left(2x^6+6x^2\right)\left(1+x^4\right)^{-3/2}$. A graph of f'' on $[0,1]$ shows that it has its maximum at $x=1$, so $|f''(x)|\leq f''(1)=\sqrt{8}$ on $[0,1]$. By taking $K=\sqrt{8}$, we find that the error in Exercise 63(a) is bounded by $\dfrac{K(b-a)^3}{12n^2}=\dfrac{\sqrt{8}}{1200}\approx 0.0024$, and in (b) by about $\frac{1}{2}(0.0024)=0.0012$.

Note: Another way to estimate K is to let $x=1$ in the factor $2x^6+6x^2$ (maximizing the numerator) and let $x=0$ in the factor $\left(1+x^4\right)^{-3/2}$ (minimizing the denominator). Doing so gives us $K=8$ and errors of $0.00\overline{6}$ and $0.00\overline{3}$.

Using $K=8$ for the Trapezoidal Rule, we have $|E_T|\leq\dfrac{K(b-a)^3}{12n^2}\leq 0.00001 \;\Leftrightarrow\; \dfrac{8(1-0)^3}{12n^2}\leq\dfrac{1}{100{,}000}$

$\Leftrightarrow\; n^2\geq\dfrac{800{,}000}{12} \;\Leftrightarrow\; n\gtrsim 258.2$, so we should take $n=259$.

For the Midpoint Rule, $|E_M|\leq\dfrac{K(b-a)^3}{24n^2}\leq 0.00001 \;\Leftrightarrow\; n^2\geq\dfrac{800{,}000}{24} \;\Leftrightarrow\; n\gtrsim 182.6$, so we should take $n=183$.

67. $\Delta t = \left(\frac{10}{60} - 0\right)/\,10 = \frac{1}{60}$.

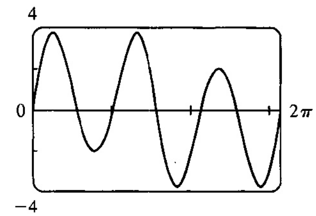
Distance traveled $= \int_0^{10} v\,dt \approx S_{10}$

$$= \tfrac{1}{60 \cdot 3}[40 + 4(42) + 2(45) + 4(49) + 2(52) + 4(54) + 2(56) + 4(57) + 2(57) + 4(55) + 56]$$

$$= \tfrac{1}{180}(1544) = 8.5\overline{7}\text{ mi}$$

69. (a) $f(x) = \sin(\sin x)$. A CAS gives

$$f^{(4)}(x) = \sin(\sin x)\left[\cos^4 x + 7\cos^2 x - 3\right] + \cos(\sin x)\left[6\cos^2 x \sin x + \sin x\right]$$



From the graph, we see that $\left|f^{(4)}(x)\right| < 3.8$ for $x \in [0, \pi]$.

(b) We use Simpson's Rule with $f(x) = \sin(\sin x)$ and $\Delta x = \frac{\pi}{10}$:

$$\int_0^\pi f(x)\,dx \approx \tfrac{\pi}{10 \cdot 3}\left[f(0) + 4f\left(\tfrac{\pi}{10}\right) + 2f\left(\tfrac{2\pi}{10}\right) + \cdots + 4f\left(\tfrac{9\pi}{10}\right) + f(\pi)\right] \approx 1.786721$$

From part (a), we know that $\left|f^{(4)}(x)\right| < 3.8$ on $[0, \pi]$, so we use Theorem 7.7.4 with $K = 3.8$, and estimate the error as $|E_S| \le \dfrac{3.8(\pi - 0)^5}{180(10)^4} \approx 0.000646$.

(c) If we want the error to be less than 0.00001, we must have $|E_S| \le \frac{3.8\pi^5}{180n^4} \le 0.00001$, so $n^4 \ge \frac{3.8\pi^5}{180(0.00001)} \approx 646{,}041.6 \;\Rightarrow\; n \ge 28.35$. Since n must be even for Simpson's Rule, we must have $n \ge 30$ to ensure the desired accuracy.

71. $\dfrac{x^3}{x^5+2} \le \dfrac{x^3}{x^5} = \dfrac{1}{x^2}$ for x in $[1, \infty)$. $\displaystyle\int_1^\infty \frac{1}{x^2}\,dx$ is convergent by (7.8.2) with $p = 2 > 1$. Therefore, $\displaystyle\int_1^\infty \frac{x^3}{x^5+2}\,dx$ is convergent by the Comparison Theorem.

73. For x in $\left[0, \frac{\pi}{2}\right]$, $0 \le \cos^2 x \le \cos x$. For x in $\left[\frac{\pi}{2}, \pi\right]$, $\cos x \le 0 \le \cos^2 x$. Thus,

$$\text{area} = \int_0^{\pi/2}\left(\cos x - \cos^2 x\right)dx + \int_{\pi/2}^{\pi}\left(\cos^2 x - \cos x\right)dx$$

$$= \left[\sin x - \tfrac{1}{2}x - \tfrac{1}{4}\sin 2x\right]_0^{\pi/2} + \left[\tfrac{1}{2}x + \tfrac{1}{4}\sin 2x - \sin x\right]_{\pi/2}^{\pi}$$

$$= \left[\left(1 - \tfrac{\pi}{4}\right) - 0\right] + \left[\tfrac{\pi}{2} - \left(\tfrac{\pi}{4} - 1\right)\right] = 2$$

75. Using the formula for disks, the volume is

$$V = \int_0^{\pi/2} \pi\,[f(x)]^2\,dx = \pi\int_0^{\pi/2}\left(\cos^2 x\right)^2 dx = \pi\int_0^{\pi/2}\left[\tfrac{1}{2}(1 + \cos 2x)\right]^2 dx$$

$$= \tfrac{\pi}{4}\int_0^{\pi/2}(1 + \cos^2 2x + 2\cos 2x)\,dx = \tfrac{\pi}{4}\int_0^{\pi/2}\left[1 + \tfrac{1}{2}(1 + \cos 4x) + 2\cos 2x\right]dx$$

$$= \tfrac{\pi}{4}\left[\tfrac{3}{2}x + \tfrac{1}{2}\left(\tfrac{1}{4}\sin 4x\right) + 2\left(\tfrac{1}{2}\sin 2x\right)\right]_0^{\pi/2} = \tfrac{\pi}{4}\left[\left(\tfrac{3\pi}{4} + \tfrac{1}{8}\cdot 0 + 0\right) - 0\right] = \tfrac{3\pi^2}{16}$$

77. By the Fundamental Theorem of Calculus,

$$\int_0^\infty f'(x)\,dx = \lim_{t\to\infty} \int_0^t f'(x)\,dx = \lim_{t\to\infty} [f(t) - f(0)] = \lim_{t\to\infty} f(t) - f(0) = 0 - f(0) = -f(0).$$

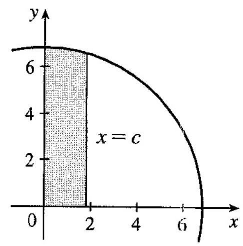
79. Let $u = 1/x \Rightarrow x = 1/u \Rightarrow dx = -\left(1/u^2\right) du$.

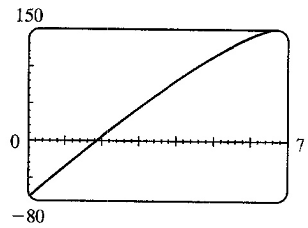
$$\int_0^\infty \frac{\ln x}{1+x^2}\,dx = \int_\infty^0 \frac{\ln(1/u)}{1+1/u^2}\left(-\frac{du}{u^2}\right) = \int_\infty^0 \frac{-\ln u}{u^2+1}(-du) = \int_\infty^0 \frac{\ln u}{1+u^2}\,du = -\int_0^\infty \frac{\ln u}{1+u^2}\,du$$

Therefore, $\displaystyle\int_0^\infty \frac{\ln x}{1+x^2}\,dx = -\int_0^\infty \frac{\ln x}{1+x^2}\,dx = 0.$

☐ PROBLEMS PLUS

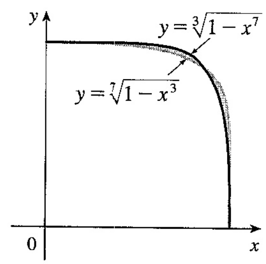
1.





By symmetry, the problem can be reduced to finding the line $x = c$ such that the shaded area is one-third of the area of the quarter-circle. The equation of the circle is $y = \sqrt{49 - x^2}$, so we require that $\int_0^c \sqrt{49 - x^2}\,dx = \frac{1}{3} \cdot \frac{1}{4}\pi(7)^2$ $\Leftrightarrow$ $\left[\frac{1}{2}x\sqrt{49 - x^2} + \frac{49}{2}\sin^{-1}(x/7)\right]_0^c = \frac{49}{12}\pi$ [by Formula 30] $\Leftrightarrow$ $\frac{1}{2}c\sqrt{49 - c^2} + \frac{49}{2}\sin^{-1}(c/7) = \frac{49}{12}\pi$. This equation would be difficult to solve exactly, so we plot the left-hand side as a function of c, and find that the equation holds for $c \approx 1.85$. So the cuts should be made at distances of about 1.85 inches from the center of the pizza.

3. The given integral represents the difference of the shaded areas, which appears to be 0. It can be calculated by integrating with respect to either x or y, so we find x in terms of y for each curve: $y = \sqrt[3]{1 - x^7} \;\Rightarrow\; x = \sqrt[7]{1 - y^3}$ and $y = \sqrt[7]{1 - x^3} \;\Rightarrow\; x = \sqrt[3]{1 - y^7}$, so $\int_0^1 \left(\sqrt[3]{1 - y^7} - \sqrt[7]{1 - y^3}\right) dy = \int_0^1 \left(\sqrt[7]{1 - x^3} - \sqrt[3]{1 - x^7}\right) dx$. But this equation is of the form $z = -z$. So $\int_0^1 \left(\sqrt[3]{1 - x^7} - \sqrt[7]{1 - x^3}\right) dx = 0$.



5. Recall that $\cos A \cos B = \frac{1}{2}[\cos(A + B) + \cos(A - B)]$. So

$$\begin{aligned} f(x) &= \int_0^{\pi} \cos t \cos(x - t)\,dt = \tfrac{1}{2}\int_0^{\pi} [\cos(t + x - t) + \cos(t - x + t)]\,dt \\ &= \tfrac{1}{2}\int_0^{\pi} [\cos x + \cos(2t - x)]\,dt = \tfrac{1}{2}\left[t\cos x + \tfrac{1}{2}\sin(2t - x)\right]_0^{\pi} \\ &= \tfrac{\pi}{2}\cos x + \tfrac{1}{4}\sin(2\pi - x) - \tfrac{1}{4}\sin(-x) = \tfrac{\pi}{2}\cos x + \tfrac{1}{4}\sin(-x) - \tfrac{1}{4}\sin(-x) \\ &= \tfrac{\pi}{2}\cos x \end{aligned}$$

The minimum of $\cos x$ on this domain is -1, so the minimum value of $f(x)$ is $f(\pi) = -\frac{\pi}{2}$.

7. In accordance with the hint, we let $I_k = \int_0^1 \left(1 - x^2\right)^k \, dx$, and we find an expression for I_{k+1} in terms of I_k. We integrate I_{k+1} by parts with $u = \left(1 - x^2\right)^{k+1} \;\Rightarrow\; du = (k+1)\left(1 - x^2\right)^k (-2x)$, $dv = dx \;\Rightarrow\; v = x$, and then split the remaining integral into identifiable quantities:

$$I_{k+1} = x(1 - x^2)^{k+1}\Big|_0^1 + 2(k+1)\int_0^1 x^2\left(1 - x^2\right)^k dx = (2k+2)\int_0^1 \left(1 - x^2\right)^k\left[1 - \left(1 - x^2\right)\right] dx$$

$$= (2k+2)(I_k - I_{k+1})$$

So $I_{k+1}[1 + (2k+2)] = (2k+2)I_k \;\Rightarrow\; I_{k+1} = \frac{2k+2}{2k+3} I_k$. Now to complete the proof, we use induction:

$I_0 = 1 = \dfrac{2^0(0!)^2}{1!}$, so the formula holds for $n = 0$. Now suppose it holds for $n = k$. Then

$$I_{k+1} = \frac{2k+2}{2k+3} I_k = \frac{2k+2}{2k+3}\left[\frac{2^{2k}(k!)^2}{(2k+1)!}\right] = \frac{2(k+1)2^{2k}(k!)^2}{(2k+3)(2k+1)!} = \frac{2(k+1)}{2k+2} \cdot \frac{2(k+1)2^{2k}(k!)^2}{(2k+3)(2k+1)!}$$

$$= \frac{[2(k+1)]^2\, 2^{2k}(k!)^2}{(2k+3)(2k+2)(2k+1)!} = \frac{2^{2(k+1)}\left[(k+1)!\right]^2}{[2(k+1)+1]!}$$

So by induction, the formula holds for all integers $n \geq 0$.

9. $0 < a < b$. Now

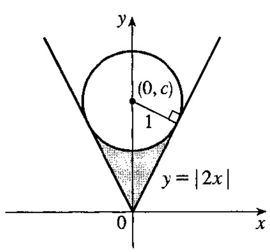
$$\int_0^1 [bx + a(1-x)]^t \, dx = \int_a^b \frac{u^t}{(b-a)} du \quad [\text{put } u = bx + a(1-x)] \quad = \left[\frac{u^{t+1}}{(t+1)(b-a)}\right]_a^b = \frac{b^{t+1} - a^{t+1}}{(t+1)(b-a)}.$$

Now let $y = \lim\limits_{t \to 0}\left[\dfrac{b^{t+1} - a^{t+1}}{(t+1)(b-a)}\right]^{1/t}$. Then $\ln y = \lim\limits_{t \to 0}\left[\dfrac{1}{t}\ln\dfrac{b^{t+1} - a^{t+1}}{(t+1)(b-a)}\right]$. This limit is of the form $0/0$, so we can apply l'Hospital's Rule to get

$$\ln y = \lim_{t \to 0}\left[\frac{b^{t+1}\ln b - a^{t+1}\ln a}{b^{t+1} - a^{t+1}} - \frac{1}{t+1}\right] = \frac{b\ln b - a\ln a}{b-a} - 1 = \frac{b\ln b}{b-a} - \frac{a\ln a}{b-a} - \ln e = \ln\frac{b^{b/(b-a)}}{ea^{a/(b-a)}}.$$

Therefore, $y = e^{-1}\left(\dfrac{b^b}{a^a}\right)^{1/(b-a)}$.

11.



An equation of the circle with center $(0, c)$ and radius 1 is $x^2 + (y-c)^2 = 1^2$, so an equation of the lower semicircle is $y = c - \sqrt{1 - x^2}$. At the points of tangency, the slopes of the line and semicircle must be equal. For $x \geq 0$, we must have

$$y' = 2 \;\Rightarrow\; \frac{x}{\sqrt{1 - x^2}} = 2 \;\Rightarrow\; x = 2\sqrt{1 - x^2} \;\Rightarrow$$

$$x^2 = 4(1 - x^2) \;\Rightarrow\; 5x^2 = 4 \;\Rightarrow\; x^2 = \tfrac{4}{5} \;\Rightarrow\; x = \tfrac{2}{5}\sqrt{5}$$

and so $y = 2\left(\frac{2}{5}\sqrt{5}\right) = \frac{4}{5}\sqrt{5}$. The slope of the perpendicular line segment is $-\frac{1}{2}$, so an equation of the line segment is $y - \frac{4}{5}\sqrt{5} = -\frac{1}{2}\left(x - \frac{2}{5}\sqrt{5}\right) \Leftrightarrow y = -\frac{1}{2}x + \frac{1}{5}\sqrt{5} + \frac{4}{5}\sqrt{5} \Leftrightarrow y = -\frac{1}{2}x + \sqrt{5}$, so $c = \sqrt{5}$ and an equation of the lower semicircle is $y = \sqrt{5} - \sqrt{1 - x^2}$. Thus, the shaded area is

$$2\int_0^{(2/5)\sqrt{5}} \left[\left(\sqrt{5} - \sqrt{1-x^2}\right) - 2x\right] dx \overset{30}{=} 2\left[\sqrt{5}\,x - \frac{x}{2}\sqrt{1-x^2} - \frac{1}{2}\sin^{-1}x - x^2\right]_0^{(2/5)\sqrt{5}}$$

$$= 2\left[2 - \frac{\sqrt{5}}{5}\cdot\frac{1}{\sqrt{5}} - \frac{1}{2}\sin^{-1}\left(\frac{2}{\sqrt{5}}\right) - \frac{4}{5}\right] - 2(0)$$

$$= 2\left[1 - \frac{1}{2}\sin^{-1}\left(\frac{2}{\sqrt{5}}\right)\right] = 2 - \sin^{-1}\left(\frac{2}{\sqrt{5}}\right)$$

13. We integrate by parts with $u = \dfrac{1}{\ln(1+x+t)}$, $dv = \sin t\,dt$, so $du = \dfrac{-1}{(1+x+t)[\ln(1+x+t)]^2}$ and $v = -\cos t$. The integral becomes

$$I = \int_0^\infty \frac{\sin t\,dt}{\ln(1+x+t)} = \lim_{b\to\infty}\left(\left[\frac{-\cos t}{\ln(1+x+t)}\right]_0^b - \int_0^b \frac{\cos t\,dt}{(1+x+t)[\ln(1+x+t)]^2}\right)$$

$$= \lim_{b\to\infty}\frac{-\cos b}{\ln(1+x+b)} + \frac{1}{\ln(1+x)} + \int_0^\infty \frac{-\cos t\,dt}{(1+x+t)[\ln(1+x+t)]^2} = \frac{1}{\ln(1+x)} + J$$

where $J = \displaystyle\int_0^\infty \frac{-\cos t\,dt}{(1+x+t)[\ln(1+x+t)]^2}$. Now $-1 \le -\cos t \le 1$ for all t; in fact, the inequality is strict except at isolated points. So $-\displaystyle\int_0^\infty \frac{dt}{(1+x+t)[\ln(1+x+t)]^2} < J < \int_0^\infty \frac{dt}{(1+x+t)[\ln(1+x+t)]^2} \Leftrightarrow$

$-\dfrac{1}{\ln(1+x)} < J < \dfrac{1}{\ln(1+x)} \Leftrightarrow 0 < I < \dfrac{2}{\ln(1+x)}$.