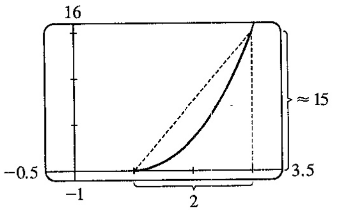
8 ☐ FURTHER APPLICATIONS OF INTEGRATION

8.1 Arc Length

1. $y = 2 - 3x \;\Rightarrow\; L = \int_{-2}^{1} \sqrt{1 + (dy/dx)^2}\,dx = \int_{-2}^{1} \sqrt{1 + (-3)^2}\,dx = \sqrt{10}\,[1 - (-2)] = 3\sqrt{10}.$
The arc length can be calculated using the distance formula, since the curve is a line segment, so

$$L = [\text{distance from } (-2, 8) \text{ to } (1, -1)] = \sqrt{[1 - (-2)]^2 + [(-1) - 8]^2} = \sqrt{90} = 3\sqrt{10}$$

3.



From the figure, the length of the curve is slightly larger than the hypotenuse of the triangle formed by the points $(1, 0)$, $(3, 0)$, and $(3, f(3)) \approx (3, 15)$, where $y = f(x) = \frac{2}{3}(x^2 - 1)^{3/2}$. This length is about $\sqrt{15^2 + 2^2} \approx 15$, so we might estimate the length to be 15.5. $y = \frac{2}{3}(x^2 - 1)^{3/2} \;\Rightarrow\; y' = (x^2 - 1)^{1/2}(2x) \;\Rightarrow$ $1 + (y')^2 = 1 + 4x^2(x^2 - 1) = 4x^4 - 4x^2 + 1 = (2x^2 - 1)^2$, so, using the fact that $2x^2 - 1 > 0$ for $1 \le x \le 3$,

$$L = \int_1^3 \sqrt{(2x^2 - 1)^2}\,dx = \int_1^3 \left|2x^2 - 1\right|dx = \int_1^3 (2x^2 - 1)\,dx = \left[\tfrac{2}{3}x^3 - x\right]_1^3$$
$$= (18 - 3) - \left(\tfrac{2}{3} - 1\right) = \tfrac{46}{3} = 15.\overline{3}$$

5. $y = 1 + 6x^{3/2} \;\Rightarrow\; dy/dx = 9x^{1/2} \;\Rightarrow\; 1 + (dy/dx)^2 = 1 + 81x$. So

$L = \int_0^1 \sqrt{1 + 81x}\,dx = \int_1^{82} u^{1/2}\left(\frac{1}{81}\,du\right)$ [where $u = 1 + 81x$ and $du = 81\,dx$]

$= \frac{1}{81} \cdot \frac{2}{3}\left[u^{3/2}\right]_1^{82} = \frac{2}{243}\left(82\sqrt{82} - 1\right)$

7. $y = \dfrac{x^5}{6} + \dfrac{1}{10x^3} \;\Rightarrow\; \dfrac{dy}{dx} = \dfrac{5}{6}x^4 - \dfrac{3}{10}x^{-4} \;\Rightarrow$

$1 + (dy/dx)^2 = 1 + \frac{25}{36}x^8 - \frac{1}{2} + \frac{9}{100}x^{-8} = \frac{25}{36}x^8 + \frac{1}{2} + \frac{9}{100}x^{-8} = \left(\frac{5}{6}x^4 + \frac{3}{10}x^{-4}\right)^2$. So

$$L = \int_1^2 \sqrt{\left(\tfrac{5}{6}x^4 + \tfrac{3}{10}x^{-4}\right)^2}\,dx = \int_1^2 \left(\tfrac{5}{6}x^4 + \tfrac{3}{10}x^{-4}\right)dx = \left[\tfrac{1}{6}x^5 - \tfrac{1}{10}x^{-3}\right]_1^2$$
$$= \left(\tfrac{32}{6} - \tfrac{1}{80}\right) - \left(\tfrac{1}{6} - \tfrac{1}{10}\right) = \tfrac{31}{6} + \tfrac{7}{80} = \tfrac{1261}{240}$$

9. $x = \frac{1}{3}\sqrt{y}\,(y - 3) = \frac{1}{3}y^{3/2} - y^{1/2} \;\Rightarrow\; dx/dy = \frac{1}{2}y^{1/2} - \frac{1}{2}y^{-1/2} \;\Rightarrow$

$1 + (dx/dy)^2 = 1 + \frac{1}{4}y - \frac{1}{2} + \frac{1}{4}y^{-1} = \frac{1}{4}y + \frac{1}{2} + \frac{1}{4}y^{-1} = \left(\frac{1}{2}y^{1/2} + \frac{1}{2}y^{-1/2}\right)^2$. So

$$L = \int_1^9 \left(\tfrac{1}{2}y^{1/2} + \tfrac{1}{2}y^{-1/2}\right)dy = \tfrac{1}{2}\left[\tfrac{2}{3}y^{3/2} + 2y^{1/2}\right]_1^9 = \tfrac{1}{2}\left[\left(\tfrac{2}{3}\cdot 27 + 2\cdot 3\right) - \left(\tfrac{2}{3}\cdot 1 + 2\cdot 1\right)\right]$$
$$= \tfrac{1}{2}\left(24 - \tfrac{8}{3}\right) = \tfrac{1}{2}\left(\tfrac{64}{3}\right) = \tfrac{32}{3}$$

11. $y = \ln(\sec x) \;\Rightarrow\; \dfrac{dy}{dx} = \dfrac{\sec x \tan x}{\sec x} = \tan x \;\Rightarrow\; 1 + \left(\dfrac{dy}{dx}\right)^2 = 1 + \tan^2 x = \sec^2 x$, so

$$L = \int_0^{\pi/4} \sqrt{\sec^2 x}\,dx = \int_0^{\pi/4} |\sec x|\,dx = \int_0^{\pi/4} \sec x\,dx = \Big[\ln(\sec x + \tan x)\Big]_0^{\pi/4}$$
$$= \ln\left(\sqrt{2} + 1\right) - \ln(1 + 0) = \ln\left(\sqrt{2} + 1\right)$$

13. $y = \cosh x \Rightarrow y' = \sinh x \Rightarrow 1 + (y')^2 = 1 + \sinh^2 x = \cosh^2 x$.
So $L = \int_0^1 \cosh x\, dx = [\sinh x]_0^1 = \sinh 1 = \frac{1}{2}(e - 1/e)$.

15. $y = e^x \Rightarrow y' = e^x \Rightarrow 1 + (y')^2 = 1 + e^{2x}$. So

$$\begin{aligned} L &= \int_0^1 \sqrt{1+e^{2x}}\, dx = \int_1^e \sqrt{1+u^2}\frac{du}{u} \qquad [u = e^x,\text{ so } x = \ln u,\ dx = du/u] \\ &= \int_1^e \frac{\sqrt{1+u^2}}{u^2}u\, du = \int_{\sqrt{2}}^{\sqrt{1+e^2}} \frac{v}{v^2-1}v\, dv \qquad \left[v = \sqrt{1+u^2},\text{ so } v^2 = 1+u^2,\ v\, dv = u\, du\right] \\ &= \int_{\sqrt{2}}^{\sqrt{1+e^2}} \left(1 + \frac{1/2}{v-1} - \frac{1/2}{v+1}\right) dv = \left[v + \frac{1}{2}\ln\frac{v-1}{v+1}\right]_{\sqrt{2}}^{\sqrt{1+e^2}} \\ &= \sqrt{1+e^2} + \frac{1}{2}\ln\frac{\sqrt{1+e^2}-1}{\sqrt{1+e^2}+1} - \sqrt{2} - \frac{1}{2}\ln\frac{\sqrt{2}-1}{\sqrt{2}+1} \\ &= \sqrt{1+e^2} - \sqrt{2} + \ln\left(\sqrt{1+e^2}-1\right) - 1 - \ln\left(\sqrt{2}-1\right) \end{aligned}$$

Or: Use Formula 23 for $\int \left(\sqrt{1+u^2}/u\right) du$, or substitute $u = \tan\theta$.

17. $y = \cos x \Rightarrow dy/dx = -\sin x \Rightarrow 1 + (dy/dx)^2 = 1 + \sin^2 x$. So $L = \int_0^{2\pi} \sqrt{1+\sin^2 x}\, dx$.

19. $x = y + y^3 \Rightarrow dx/dy = 1 + 3y^2 \Rightarrow 1 + (dx/dy)^2 = 1 + (1+3y^2)^2 = 9y^4 + 6y^2 + 2$.
So $L = \int_1^4 \sqrt{9y^4 + 6y^2 + 2}\, dy$.

21. $y = xe^{-x} \Rightarrow dy/dx = e^{-x} - xe^{-x} = e^{-x}(1-x) \Rightarrow 1 + (dy/dx)^2 = 1 + e^{-2x}(1-x)^2$. Let $f(x) = \sqrt{1 + (dy/dx)^2} = \sqrt{1 + e^{-2x}(1-x)^2}$. Then $L = \int_0^5 f(x)\, dx$. Since $n = 10$, $\Delta x = \frac{5-0}{10} = \frac{1}{2}$. Now

$$\begin{aligned} L \approx S_{10} = \tfrac{1/2}{3}[&f(0) + 4f(\tfrac{1}{2}) + 2f(1) + 4f(\tfrac{3}{2}) + 2f(2) + 4f(\tfrac{5}{2}) + 2f(3) \\ &+ 4f(\tfrac{7}{2}) + 2f(4) + 4f(\tfrac{9}{2}) + f(5)] \approx 5.115840 \end{aligned}$$

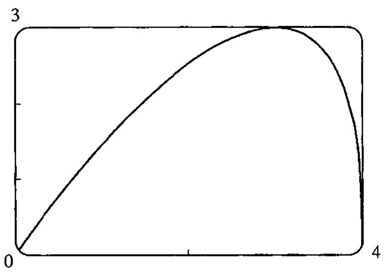
The value of the integral produced by a calculator is 5.113568 (to six decimal places).

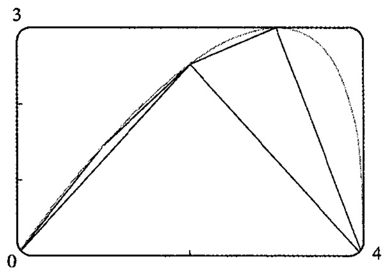
23. $y = \sec x \Rightarrow dy/dx = \sec x \tan x \Rightarrow L = \int_0^{\pi/3} f(x)\, dx$, where $f(x) = \sqrt{1 + \sec^2 x \tan^2 x}$.
Since $n = 10$, $\Delta x = \dfrac{\pi/3 - 0}{10} = \dfrac{\pi}{30}$. Now

$$\begin{aligned} L \approx S_{10} = \frac{\pi/30}{3}\Bigg[&f(0) + 4f\left(\frac{\pi}{30}\right) + 2f\left(\frac{2\pi}{30}\right) + 4f\left(\frac{3\pi}{30}\right) + 2f\left(\frac{4\pi}{30}\right) + 4f\left(\frac{5\pi}{30}\right) \\ &+ 2f\left(\frac{6\pi}{30}\right) + 4f\left(\frac{7\pi}{30}\right) + 2f\left(\frac{8\pi}{30}\right) + 4f\left(\frac{9\pi}{30}\right) + f\left(\frac{\pi}{3}\right)\Bigg] \approx 1.569619. \end{aligned}$$

The value of the integral produced by a calculator is 1.569259 (to six decimal places).

25. (a)



(b) 

Let $f(x) = y = x\sqrt[3]{4-x}$. The polygon with one side is just the line segment joining the points $(0, f(0)) = (0,0)$ and $(4, f(4)) = (4,0)$, and its length is 4. The polygon with two sides joins the points $(0,0)$, $(2, f(2)) = \left(2, 2\sqrt[3]{2}\right)$ and $(4,0)$.

Its length is

$$\sqrt{(2-0)^2 + \left(2\sqrt[3]{2}-0\right)^2} + \sqrt{(4-2)^2 + \left(0-2\sqrt[3]{2}\right)^2} = 2\sqrt{4+2^{8/3}} \approx 6.43$$

Similarly, the inscribed polygon with four sides joins the points $(0,0)$, $\left(1, \sqrt[3]{3}\right)$, $\left(2, 2\sqrt[3]{2}\right)$, $(3,3)$, and $(4,0)$, so its length is

$$\sqrt{1+\left(\sqrt[3]{3}\right)^2} + \sqrt{1+\left(2\sqrt[3]{2}-\sqrt[3]{3}\right)^2} + \sqrt{1+\left(3-2\sqrt[3]{2}\right)^2} + \sqrt{1+9} \approx 7.50$$

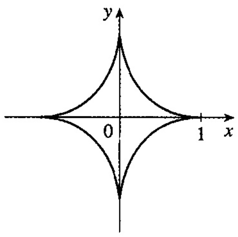
(c) Using the arc length formula with $\dfrac{dy}{dx} = x\left[\frac{1}{3}(4-x)^{-2/3}(-1)\right] + \sqrt[3]{4-x} = \dfrac{12-4x}{3(4-x)^{2/3}}$, the length of the curve is $L = \displaystyle\int_0^4 \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx = \int_0^4 \sqrt{1+\left[\frac{12-4x}{3(4-x)^{2/3}}\right]^2}\,dx$.

(d) According to a CAS, the length of the curve is $L \approx 7.7988$. The actual value is larger than any of the approximations in part (b). This is always true, since any approximating straight line between two points on the curve is shorter than the length of the curve between the two points.

27. $x = \ln\left(1-y^2\right) \Rightarrow \dfrac{dx}{dy} = \dfrac{-2y}{1-y^2} \Rightarrow 1+\left(\dfrac{dx}{dy}\right)^2 = 1+\dfrac{4y^2}{\left(1-y^2\right)^2} = \dfrac{\left(1+y^2\right)^2}{\left(1-y^2\right)^2}$. So

$$L = \int_0^{1/2}\sqrt{\frac{\left(1+y^2\right)^2}{\left(1-y^2\right)^2}}\,dy = \int_0^{1/2}\frac{1+y^2}{1-y^2}\,dy = \ln 3 - \tfrac{1}{2}\ \text{[from a CAS]} \approx 0.599$$

29. $y^{2/3} = 1 - x^{2/3} \Rightarrow y = \left(1-x^{2/3}\right)^{3/2} \Rightarrow$



$\dfrac{dy}{dx} = \frac{3}{2}\left(1-x^{2/3}\right)^{1/2}\left(-\frac{2}{3}x^{-1/3}\right) = -x^{-1/3}\left(1-x^{2/3}\right)^{1/2} \Rightarrow$

$\left(\dfrac{dy}{dx}\right)^2 = x^{-2/3}\left(1-x^{2/3}\right) = x^{-2/3} - 1$. Thus

$L = 4\int_0^1 \sqrt{1+\left(x^{-2/3}-1\right)}\,dx = 4\int_0^1 x^{-1/3}\,dx = 4\lim\limits_{t\to 0^+}\left[\frac{3}{2}x^{2/3}\right]_t^1 = 6.$

31. $y = 2x^{3/2} \Rightarrow y' = 3x^{1/2} \Rightarrow 1+(y')^2 = 1+9x$. The arc length function with starting point $P_0(1,2)$ is

$$s(x) = \int_1^x \sqrt{1+9t}\,dt = \left[\tfrac{2}{27}(1+9t)^{3/2}\right]_1^x = \tfrac{2}{27}\left[(1+9x)^{3/2} - 10\sqrt{10}\right]$$

... use CAS] $\frac{\pi}{3}\left[\frac{1}{2}u\sqrt{1+u^2} + \frac{1}{2}\ln\left(u+\sqrt{1+u^2}\right)\right]_0^3$ ... $ax]$

$= \frac{\pi}{3}\left[\frac{3}{2}\sqrt{10} + \frac{1}{2}\ln\left(3+\sqrt{10}\right)\right] = \frac{\pi}{6}\left[3\sqrt{10} + \ln\left(3+\sqrt{10}\right)\right]$

33. The prey hits the ground when $y = 0 \iff 180 - \frac{1}{45}x^2 = 0 \iff x^2 = 45 \cdot 180 \Rightarrow x = \sqrt{8100} = 90$, since x must be positive. $y' = -\frac{2}{45}x \Rightarrow 1 + (y')^2 = 1 + \frac{4}{45^2}x^2$, so the distance traveled by the prey is

$$L = \int_0^{90} \sqrt{1 + \tfrac{4}{45^2}x^2}\,dx = \int_0^4 \sqrt{1+u^2}\left(\tfrac{45}{2}\,du\right) \qquad \left[u = \tfrac{2}{45}x,\ du = \tfrac{2}{45}\,dx\right]$$
$$\overset{21}{=} \tfrac{45}{2}\left[\tfrac{1}{2}u\sqrt{1+u^2} + \tfrac{1}{2}\ln\left(u + \sqrt{1+u^2}\right)\right]_0^4$$
$$= \tfrac{45}{2}\left[2\sqrt{17} + \tfrac{1}{2}\ln\left(4+\sqrt{17}\right)\right] = 45\sqrt{17} + \tfrac{45}{4}\ln\left(4+\sqrt{17}\right) \approx 209.1 \text{ m}$$

35. The sine wave has amplitude 1 and period 14, since it goes through two periods in a distance of 28 in., so its equation is $y = 1\sin\left(\frac{2\pi}{14}x\right) = \sin\left(\frac{\pi}{7}x\right)$. The width w of the flat metal sheet needed to make the panel is the arc length of the sine curve from $x = 0$ to $x = 28$. We set up the integral to evaluate w using the arc length formula with $\frac{dy}{dx} = \frac{\pi}{7}\cos\left(\frac{\pi}{7}x\right)$: $L = \int_0^{28}\sqrt{1 + \left[\frac{\pi}{7}\cos\left(\frac{\pi}{7}x\right)\right]^2}\,dx = 2\int_0^{14}\sqrt{1 + \left[\frac{\pi}{7}\cos\left(\frac{\pi}{7}x\right)\right]^2}\,dx$. This integral would be very difficult to evaluate exactly, so we use a CAS, and find that $L \approx 29.36$ inches.

37. $y = \int_1^x \sqrt{t^3 - 1}\,dt \Rightarrow \frac{dy}{dx} = \sqrt{x^3 - 1}$ [by FTC1] $\Rightarrow 1 + \left(\frac{dy}{dx}\right)^2 = 1 + \left(\sqrt{x^3-1}\right)^2 = x^3 \Rightarrow$

$$L = \int_1^4 \sqrt{x^3}\,dx = \int_1^4 x^{3/2}\,dx = \tfrac{2}{5}\left[x^{5/2}\right]_1^4 = \tfrac{2}{5}(32-1) = \tfrac{62}{5} = 12.4$$

8.2 Area of a Surface of Revolution

1. $y = \ln x \Rightarrow ds = \sqrt{1 + (dy/dx)^2}\,dx = \sqrt{1 + (1/x)^2}\,dx \Rightarrow S = \int_1^3 2\pi(\ln x)\sqrt{1+(1/x)^2}\,dx$ [by (7)]

3. $y = \sec x \Rightarrow ds = \sqrt{1 + (dy/dx)^2}\,dx = \sqrt{1 + (\sec x \tan x)^2}\,dx \Rightarrow$
$S = \int_0^{\pi/4} 2\pi x\sqrt{1 + (\sec x \tan x)^2}\,dx$ [by (8)]

5. $y = x^3 \Rightarrow y' = 3x^2$. So

$$S = \int_0^2 2\pi y\sqrt{1 + (y')^2}\,dx = 2\pi\int_0^2 x^3\sqrt{1 + 9x^4}\,dx \qquad \left[u = 1 + 9x^4,\ du = 36x^3\,dx\right]$$
$$= \tfrac{2\pi}{36}\int_1^{145}\sqrt{u}\,du = \tfrac{\pi}{18}\left[\tfrac{2}{3}u^{3/2}\right]_1^{145} = \tfrac{\pi}{27}\left(145\sqrt{145} - 1\right)$$

7. $y = \sqrt{x} \Rightarrow 1 + (dy/dx)^2 = 1 + \left[1/(2\sqrt{x})\right]^2 = 1 + 1/(4x)$. So

$$S = \int_4^9 2\pi y\sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx = \int_4^9 2\pi\sqrt{x}\sqrt{1 + \frac{1}{4x}}\,dx = 2\pi\int_4^9\sqrt{x + \tfrac{1}{4}}\,dx$$
$$= 2\pi\left[\tfrac{2}{3}\left(x + \tfrac{1}{4}\right)^{3/2}\right]_4^9 = \tfrac{4\pi}{3}\left[\tfrac{1}{8}(4x+1)^{3/2}\right]_4^9 = \tfrac{\pi}{6}\left(37\sqrt{37} - 17\sqrt{17}\right)$$

9. $y = \cosh x \Rightarrow 1 + (dy/dx)^2 = 1 + \sinh^2 x = \cosh^2 x$. So

$$S = 2\pi\int_0^1 \cosh x\cosh x\,dx = 2\pi\int_0^1 \tfrac{1}{2}(1 + \cosh 2x)\,dx = \pi\left[x + \tfrac{1}{2}\sinh 2x\right]_0^1$$
$$= \pi\left(1 + \tfrac{1}{2}\sinh 2\right) \quad \text{or} \quad \pi\left[1 + \tfrac{1}{4}\left(e^2 - e^{-2}\right)\right]$$

11. $x = \frac{1}{3}\left(y^2+2\right)^{3/2} \Rightarrow dx/dy = \frac{1}{2}\left(y^2+2\right)^{1/2}(2y) = y\sqrt{y^2+2} \Rightarrow$
$1 + (dx/dy)^2 = 1 + y^2\left(y^2+2\right) = \left(y^2+1\right)^2$. So

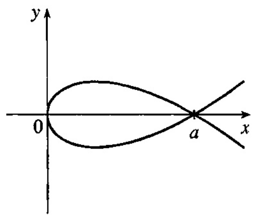
$$S = 2\pi\int_1^2 y\left(y^2+1\right)dy = 2\pi\left[\tfrac{1}{4}y^4 + \tfrac{1}{2}y^2\right]_1^2 = 2\pi\left(4 + 2 - \tfrac{1}{4} - \tfrac{1}{2}\right) = \tfrac{21\pi}{2}$$

25. $S = 2\pi \int_1^\infty y\sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = 2\pi \int_1^\infty \frac{1}{x}\sqrt{1 + \frac{1}{x^4}}\, dx = 2\pi \int_1^\infty \frac{\sqrt{x^4+1}}{x^3}\, dx$. Rather than trying to evaluate this integral, note that $\sqrt{x^4+1} > \sqrt{x^4} = x^2$ for $x > 0$. Thus, if the area is finite,

$$S = 2\pi \int_1^\infty \frac{\sqrt{x^4+1}}{x^3}\, dx > 2\pi \int_1^\infty \frac{x^2}{x^3}\, dx = 2\pi \int_1^\infty \frac{1}{x}\, dx$$

But we know that this integral diverges, so the area S is infinite.

27. Since $a > 0$, the curve $3ay^2 = x(a-x)^2$ only has points with $x \geq 0$. ($3ay^2 \geq 0 \;\Rightarrow\; x(a-x)^2 \geq 0 \;\Rightarrow\; x \geq 0$.) The curve is symmetric about the x-axis (since the equation is unchanged when y is replaced by $-y$). $y = 0$ when $x = 0$ or a, so the curve's loop extends from $x = 0$ to $x = a$.



$$\frac{d}{dx}(3ay^2) = \frac{d}{dx}\left[x(a-x)^2\right] \;\Rightarrow\; 6ay\frac{dy}{dx} = x \cdot 2(a-x)(-1) + (a-x)^2 \;\Rightarrow$$

$$\frac{dy}{dx} = \frac{(a-x)[-2x+a-x]}{6ay} \;\Rightarrow\; \left(\frac{dy}{dx}\right)^2 = \frac{(a-x)^2(a-3x)^2}{36a^2y^2} = \frac{(a-x)^2(a-3x)^2}{36a^2} \cdot \frac{3a}{x(a-x)^2}$$

$$\begin{bmatrix}\text{the last fraction} \\ \text{is } 1/y^2\end{bmatrix} = \frac{(a-3x)^2}{12ax} \;\Rightarrow$$

$$1 + \left(\frac{dy}{dx}\right)^2 = 1 + \frac{a^2 - 6ax + 9x^2}{12ax} = \frac{12ax}{12ax} + \frac{a^2 - 6ax + 9x^2}{12ax} = \frac{a^2 + 6ax + 9x^2}{12ax} = \frac{(a+3x)^2}{12ax} \text{ for } x \neq 0.$$

(a) $$S = \int_{x=0}^a 2\pi y\, ds = 2\pi \int_0^a \frac{\sqrt{x}(a-x)}{\sqrt{3a}} \cdot \frac{a+3x}{\sqrt{12ax}}\, dx = 2\pi \int_0^a \frac{(a-x)(a+3x)}{6a}\, dx$$
$$= \frac{\pi}{3a}\int_0^a (a^2 + 2ax - 3x^2)\, dx = \frac{\pi}{3a}\left[a^2x + ax^2 - x^3\right]_0^a = \frac{\pi}{3a}(a^3 + a^3 - a^3) = \frac{\pi}{3a} \cdot a^3 = \frac{\pi a^2}{3}.$$

Note that we have rotated the top half of the loop about the x-axis. This generates the full surface.

(b) We must rotate the full loop about the y-axis, so we get double the area obtained by rotating the top half of the loop:

$$S = 2 \cdot 2\pi \int_{x=0}^a x\, ds = 4\pi \int_0^a x\frac{a+3x}{\sqrt{12ax}}\, dx = \frac{4\pi}{2\sqrt{3a}}\int_0^a x^{1/2}(a+3x)\, dx$$
$$= \frac{2\pi}{\sqrt{3a}}\int_0^a (ax^{1/2} + 3x^{3/2})\, dx = \frac{2\pi}{\sqrt{3a}}\left[\frac{2}{3}ax^{3/2} + \frac{6}{5}x^{5/2}\right]_0^a = \frac{2\pi\sqrt{3}}{3\sqrt{a}}\left(\frac{2}{3}a^{5/2} + \frac{6}{5}a^{5/2}\right)$$
$$= \frac{2\pi\sqrt{3}}{3}\left(\frac{2}{3} + \frac{6}{5}\right)a^2 = \frac{2\pi\sqrt{3}}{3}\left(\frac{28}{15}\right)a^2 = \frac{56\pi\sqrt{3}\,a^2}{45}$$

29. $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1 \;\Rightarrow\; \dfrac{y\,(dy/dx)}{b^2}=-\dfrac{x}{a^2} \;\Rightarrow\; \dfrac{dy}{dx}=-\dfrac{b^2x}{a^2y} \;\Rightarrow$

$$1+\left(\frac{dy}{dx}\right)^2=1+\frac{b^4x^2}{a^4y^2}=\frac{b^4x^2+a^4y^2}{a^4y^2}=\frac{b^4x^2+a^4b^2\left(1-x^2/a^2\right)}{a^4b^2\left(1-x^2/a^2\right)}=\frac{a^4b^2+b^4x^2-a^2b^2x^2}{a^4b^2-a^2b^2x^2}$$

$$=\frac{a^4+b^2x^2-a^2x^2}{a^4-a^2x^2}=\frac{a^4-(a^2-b^2)x^2}{a^2(a^2-x^2)}$$

The ellipsoid's surface area is twice the area generated by rotating the first quadrant portion of the ellipse about the x-axis. Thus,

$$S=2\int_0^a 2\pi y\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx=4\pi\int_0^a \frac{b}{a}\sqrt{a^2-x^2}\,\frac{\sqrt{a^4-(a^2-b^2)x^2}}{a\sqrt{a^2-x^2}}\,dx$$

$$=\frac{4\pi b}{a^2}\int_0^a\sqrt{a^4-(a^2-b^2)x^2}\,dx=\frac{4\pi b}{a^2}\int_0^{a\sqrt{a^2-b^2}}\sqrt{a^4-u^2}\,\frac{du}{\sqrt{a^2-b^2}}\qquad\left[u=\sqrt{a^2-b^2}\,x\right]$$

$$\overset{30}{=}\frac{4\pi b}{a^2\sqrt{a^2-b^2}}\left[\frac{u}{2}\sqrt{a^4-u^2}+\frac{a^4}{2}\sin^{-1}\frac{u}{a^2}\right]_0^{a\sqrt{a^2-b^2}}$$

$$=\frac{4\pi b}{a^2\sqrt{a^2-b^2}}\left[\frac{a\sqrt{a^2-b^2}}{2}\sqrt{a^4-a^2(a^2-b^2)}+\frac{a^4}{2}\sin^{-1}\frac{\sqrt{a^2-b^2}}{a}\right]=2\pi\left[b^2+\frac{a^2b\sin^{-1}\dfrac{\sqrt{a^2-b^2}}{a}}{\sqrt{a^2-b^2}}\right]$$

31. The analogue of $f(x_i^*)$ in the derivation of (4) is now $c-f(x_i^*)$, so

$$S=\lim_{n\to\infty}\sum_{i=1}^n 2\pi[c-f(x_i^*)]\sqrt{1+[f'(x_i^*)]^2}\,\Delta x=\int_a^b 2\pi[c-f(x)]\sqrt{1+[f'(x)]^2}\,dx.$$

33. For the upper semicircle, $f(x)=\sqrt{r^2-x^2}$, $f'(x)=-x/\sqrt{r^2-x^2}$. The surface area generated is

$$S_1=\int_{-r}^r 2\pi\left(r-\sqrt{r^2-x^2}\right)\sqrt{1+\frac{x^2}{r^2-x^2}}\,dx=4\pi\int_0^r\left(r-\sqrt{r^2-x^2}\right)\frac{r}{\sqrt{r^2-x^2}}\,dx$$

$$=4\pi\int_0^r\left(\frac{r^2}{\sqrt{r^2-x^2}}-r\right)dx$$

For the lower semicircle, $f(x)=-\sqrt{r^2-x^2}$ and $f'(x)=\dfrac{x}{\sqrt{r^2-x^2}}$, so $S_2=4\pi\displaystyle\int_0^r\left(\frac{r^2}{\sqrt{r^2-x^2}}+r\right)dx$.

Thus, the total area is $S=S_1+S_2=8\pi\displaystyle\int_0^r\left(\frac{r^2}{\sqrt{r^2-x^2}}\right)dx=8\pi\left[r^2\sin^{-1}\left(\frac{x}{r}\right)\right]_0^r=8\pi r^2\left(\frac{\pi}{2}\right)=4\pi^2r^2$.

35. In the derivation of (4), we computed a typical contribution to the surface area to be $2\pi\,\dfrac{y_{i-1}+y_i}{2}\,|P_{i-1}P_i|$, the area of a frustum of a cone. When $f(x)$ is not necessarily positive, the approximations $y_i=f(x_i)\approx f(x_i^*)$ and $y_{i-1}=f(x_{i-1})\approx f(x_i^*)$ must be replaced by $y_i=|f(x_i)|\approx|f(x_i^*)|$ and $y_{i-1}=|f(x_{i-1})|\approx|f(x_i^*)|$. Thus, $2\pi\,\dfrac{y_{i-1}+y_i}{2}\,|P_{i-1}P_i|\approx 2\pi\,|f(x_i^*)|\sqrt{1+[f'(x_i^*)]^2}\,\Delta x$. Continuing with the rest of the derivation as before, we obtain $S=\int_a^b 2\pi\,|f(x)|\sqrt{1+[f'(x)]^2}\,dx$.

8.3 Applications to Physics and Engineering

1. The weight density of water is $\delta = 62.5\ \text{lb/ft}^3$.

(a) $P = \delta d \approx (62.5\ \text{lb/ft}^3)(3\ \text{ft}) = 187.5\ \text{lb/ft}^2$

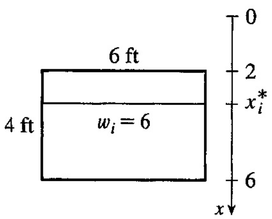
(b) $F = PA \approx (187.5\ \text{lb/ft}^2)(5\ \text{ft})(2\ \text{ft}) = 1875\ \text{lb}$. ($A$ is the area of the bottom of the tank.)

(c) As in Example 1, the area of the ith strip is $2\,(\Delta x)$ and the pressure is $\delta d = \delta x_i$. Thus,

$$F = \int_0^3 \delta x \cdot 2\,dx \approx (62.5)(2)\int_0^3 x\,dx = 125\left[\tfrac{1}{2}x^2\right]_0^3 = 125\left(\tfrac{9}{2}\right) = 562.5\ \text{lb}$$

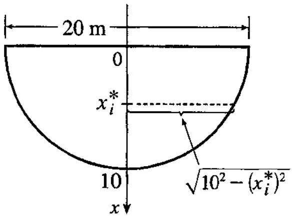
In Exercises 3–9, n is the number of subintervals of length Δx and x_i^* is a sample point in the ith subinterval $[x_{i-1}, x_i]$.

3. Set up a vertical x-axis as shown, with $x = 0$ at the water's surface and x increasing in the downward direction. Then the area of the ith rectangular strip is $6\,\Delta x$ and the pressure on the strip is δx_i^* (where $\delta \approx 62.5\ \text{lb/ft}^3$). Thus, the hydrostatic force on the strip is $\delta x_i^* \cdot 6\,\Delta x$ and the total hydrostatic force $\approx \sum_{i=1}^{n} \delta x_i^* \cdot 6\,\Delta x$. The total force



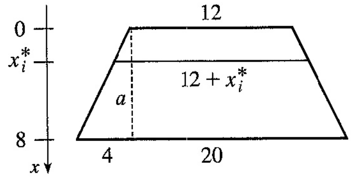
$$F = \lim_{n\to\infty} \sum_{i=1}^{n} \delta x_i^* \cdot 6\,\Delta x = \int_2^6 \delta x \cdot 6\,dx = 6\delta \int_2^6 x\,dx$$
$$= 6\delta\left[\tfrac{1}{2}x^2\right]_2^6 = 6\delta(18-2) = 96\delta \approx 6000\ \text{lb}$$

5. Since an equation for the shape is $x^2 + y^2 = 10^2$ ($x \geq 0$), we have $y = \sqrt{100 - x^2}$. Thus, the area of the ith strip is $2\sqrt{100 - (x_i^*)^2}\,\Delta x$ and the pressure on the strip is $\rho g x_i^*$, so the hydrostatic force on the strip is $\rho g x_i^* \cdot 2\sqrt{100 - (x_i^*)^2}\,\Delta x$ and the total force on the plate $\approx \sum_{i=1}^{n} \rho g x_i^* \cdot 2\sqrt{100 - (x_i^*)^2}\,\Delta x$. The total force



$$F = \lim_{n\to\infty} \sum_{i=1}^{n} \rho g x_i^* \cdot 2\sqrt{100 - (x_i^*)^2}\,\Delta x = \int_0^{10} 2\rho g x\sqrt{100 - x^2}\,dx$$
$$= -\rho g \int_0^{10} (100 - x^2)^{1/2}(-2x)\,dx = -\rho g\left[\tfrac{2}{3}(100 - x^2)^{3/2}\right]_0^{10} = -\tfrac{2}{3}\rho g(0 - 1000)$$
$$= \tfrac{2000}{3}\rho g \approx \tfrac{2000}{3} \cdot 1000 \cdot 9.8 \approx 6.5 \times 10^6\ \text{N} \quad [\rho \approx 1000\ \text{kg/m}^3 \text{ and } g \approx 9.8\ \text{m/s}^2.]$$

7. Using similar triangles, $\dfrac{4 \text{ ft wide}}{8 \text{ ft high}} = \dfrac{a \text{ ft wide}}{x_i^* \text{ ft high}}$, so $a = \frac{1}{2}x_i^*$ and the width of the ith rectangular strip is $12 + 2a = 12 + x_i^*$. The area of the strip is $(12 + x_i^*)\,\Delta x$. The pressure on the strip is δx_i^*.

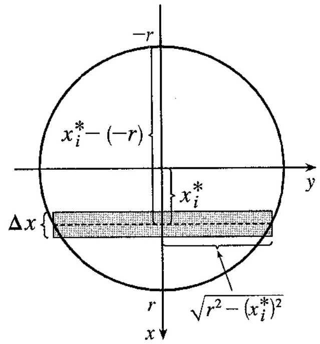


$$F = \lim_{n\to\infty} \sum_{i=1}^{n} \delta x_i^*(12 + x_i^*)\,\Delta x = \int_0^8 \delta x \cdot (12 + x)\,dx$$

$$= \delta \int_0^8 (12x + x^2)\,dx = \delta\left[6x^2 + \tfrac{x^3}{3}\right]_0^8 = \delta\left(384 + \tfrac{512}{3}\right)$$

$$= (62.5)\tfrac{1664}{3} \approx 3.47 \times 10^4 \text{ lb}$$

9. From the figure, the area of the ith rectangular strip is $2\sqrt{r^2 - (x_i^*)^2}\,\Delta x$ and the pressure on it is $\rho g(x_i^* + r)$.



$$F = \lim_{n\to\infty} \sum_{i=1}^{n} \rho g(x_i^* + r)\, 2\sqrt{r^2 - (x_i^*)^2}\,\Delta x$$

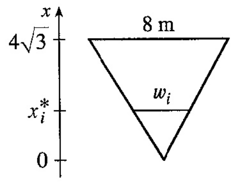
$$= \int_{-r}^{r} \rho g(x + r) \cdot 2\sqrt{r^2 - x^2}\,dx$$

$$= \rho g \int_{-r}^{r} \sqrt{r^2 - x^2}\, 2x\,dx + 2\rho g r \int_{-r}^{r} \sqrt{r^2 - x^2}\,dx$$

The first integral is 0 because the integrand is an odd function. The second integral can be interpreted as the area of a semicircular disk with radius r, or we could make the trigonometric substitution $x = r\sin\theta$. Continuing:

$F = \rho g \cdot 0 + 2\rho g r \cdot \frac{1}{2}\pi r^2 = \rho g \pi r^3 = 1000 g \pi r^3$ N (SI units assumed).

11. By similar triangles, $\dfrac{8}{4\sqrt{3}} = \dfrac{w_i}{x_i^*} \quad\Rightarrow\quad w_i = \dfrac{2x_i^*}{\sqrt{3}}$. The area of the ith rectangular strip is $\dfrac{2x_i^*}{\sqrt{3}}\,\Delta x$ and the pressure on it is $\rho g\left(4\sqrt{3} - x_i^*\right)$.



$$F = \int_0^{4\sqrt{3}} \rho g\left(4\sqrt{3} - x\right)\frac{2x}{\sqrt{3}}\,dx = 8\rho g \int_0^{4\sqrt{3}} x\,dx - \frac{2\rho g}{\sqrt{3}}\int_0^{4\sqrt{3}} x^2\,dx$$

$$= 4\rho g\left[x^2\right]_0^{4\sqrt{3}} - \frac{2\rho g}{3\sqrt{3}}\left[x^3\right]_0^{4\sqrt{3}} = 192\rho g - \frac{2\rho g}{3\sqrt{3}}\,64 \cdot 3\sqrt{3}$$

$$= 192\rho g - 128\rho g = 64\rho g \approx 64(840)(9.8) \approx 5.27 \times 10^5 \text{ N}$$

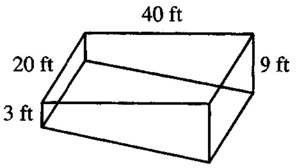
13. (a) The top of the cube has depth $d = 1\text{ m} - 20\text{ cm} = 80\text{ cm} = 0.8\text{ m}$.

$$F = \rho g d A \approx (1000)(9.8)(0.8)(0.2)^2 = 313.6 \approx 314\text{ N}$$

(b) The area of a strip is $0.2\,\Delta x$ and the pressure on it is $\rho g x_i^*$.

$$F = \int_{0.8}^{1} \rho g x(0.2)\,dx = 0.2\rho g\left[\tfrac{1}{2}x^2\right]_{0.8}^{1} = (0.2\rho g)(0.18) = 0.036\rho g = 0.036(1000)(9.8)$$
$$= 352.8 \approx 353\text{ N}$$

15. (a) The area of a strip is $20\,\Delta x$ and the pressure on it is δx_i.



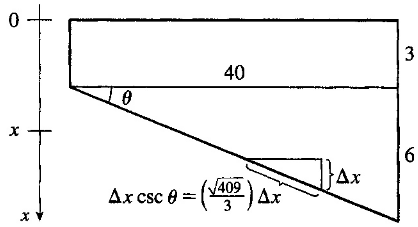
$$F = \int_0^3 \delta x 20\,dx = 20\delta\left[\frac{1}{2}x^2\right]_0^3 = 20\delta\cdot\frac{9}{2} = 90\delta$$
$$= 90(62.5) = 5625\text{ lb} \approx 5.63\times 10^3\text{ lb}$$

(b) $F = \int_0^9 \delta x 20\,dx = 20\delta\left[\frac{1}{2}x^2\right]_0^9 = 20\delta\cdot\frac{81}{2} = 810\delta = 810(62.5) = 50{,}625\text{ lb} \approx 5.06\times 10^4\text{ lb}$.

(c) For the first 3 ft, the length of the side is constant at 40 ft. For $3 < x \le 9$, we can use similar triangles to find the length a: $\dfrac{a}{40} = \dfrac{9-x}{6} \;\Rightarrow\; a = 40\cdot\dfrac{9-x}{6}$.

$$F = \int_0^3 \delta x 40\,dx + \int_3^9 \delta x(40)\tfrac{9-x}{6}\,dx = 40\delta\left[\tfrac{1}{2}x^2\right]_0^3 + \tfrac{20}{3}\delta\int_3^9 (9x - x^2)\,dx$$
$$= 180\delta + \tfrac{20}{3}\delta\left[\tfrac{9}{2}x^2 - \tfrac{1}{3}x^3\right]_3^9 = 180\delta + \tfrac{20}{3}\delta\left[\left(\tfrac{729}{2} - 243\right) - \left(\tfrac{81}{2} - 9\right)\right]$$
$$= 180\delta + 600\delta = 780\delta = 780(62.5) = 48{,}750\text{ lb} \approx 4.88\times 10^4\text{ lb}$$

(d) For any right triangle with hypotenuse on the bottom,



$$\csc\theta = \frac{\Delta x}{\text{hypotenuse}} \;\Rightarrow$$
$$\text{hypotenuse} = \Delta x\csc\theta = \Delta x\,\frac{\sqrt{40^2+6^2}}{6} = \frac{\sqrt{409}}{3}\,\Delta x.$$
$$F = \int_3^9 \delta x 20\,\tfrac{\sqrt{409}}{3}\,dx = \tfrac{1}{3}\left(20\sqrt{409}\right)\delta\left[\tfrac{1}{2}x^2\right]_3^9$$
$$= \tfrac{1}{3}\cdot 10\sqrt{409}\,\delta(81-9)$$
$$\approx 303{,}356\text{ lb} \approx 3.03\times 10^5\text{ lb}$$

17. $F = \int_2^5 \rho g x\cdot w(x)\,dx$, where $w(x)$ is the width of the plate at depth x. Since $n = 6$, $\Delta x = \frac{5-2}{6} = \frac{1}{2}$, and

$$F \approx S_6 = \rho g\cdot\tfrac{1/2}{3}\left[2\cdot w(2) + 4\cdot 2.5\cdot w(2.5) + 2\cdot 3\cdot w(3) + 4\cdot 3.5\cdot w(3.5)\right.$$
$$\left.+\, 2\cdot 4\cdot w(4) + 4\cdot 4.5\cdot w(4.5) + 5\cdot w(5)\right]$$
$$= \tfrac{1}{6}\rho g(2\cdot 0 + 10\cdot 0.8 + 6\cdot 1.7 + 14\cdot 2.4 + 8\cdot 2.9 + 18\cdot 3.3 + 5\cdot 3.6)$$
$$= \tfrac{1}{6}(1000)(9.8)(152.4) \approx 2.5\times 10^5\text{ N}$$

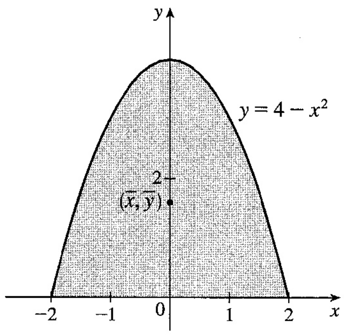
19. The moment M of the system about the origin is $M = \sum_{i=1}^{2} m_i x_i = m_1 x_1 + m_2 x_2 = 40 \cdot 2 + 30 \cdot 5 = 230$.

The mass m of the system is $m = \sum_{i=1}^{2} m_i = m_1 + m_2 = 40 + 30 = 70$. The center of mass of the system is $M/m = \frac{230}{70} = \frac{23}{7}$.

21. $m = \sum_{i=1}^{3} m_i = 6 + 5 + 10 = 21$. $M_x = \sum_{i=1}^{3} m_i y_i = 6(5) + 5(-2) + 10(-1) = 10$;

$M_y = \sum_{i=1}^{3} m_i x_i = 6(1) + 5(3) + 10(-2) = 1$. $\overline{x} = \dfrac{M_y}{m} = \dfrac{1}{21}$ and $\overline{y} = \dfrac{M_x}{m} = \dfrac{10}{21}$, so the center of mass of the system is $\left(\frac{1}{21}, \frac{10}{21}\right)$.

23. Since the region in the figure is symmetric about the y-axis, we know that $\overline{x} = 0$. The region is "bottom-heavy," so we know that $\overline{y} < 2$, and we might guess that $\overline{y} = 1.5$.



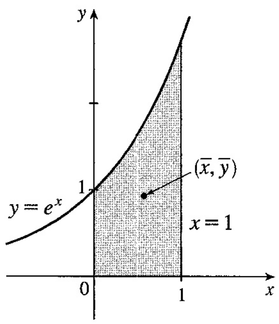
$$A = \int_{-2}^{2} (4 - x^2)\, dx = 2 \int_0^2 (4 - x^2)\, dx = 2\left[4x - \tfrac{1}{3}x^3\right]_0^2$$
$$= 2\left(8 - \tfrac{8}{3}\right) = \tfrac{32}{3}$$

$\overline{x} = \dfrac{1}{A} \displaystyle\int_{-2}^{2} x(4 - x^2)\, dx = 0$ since $f(x) = x(4 - x^2)$ is an odd function (or since the region is symmetric about the y-axis).

$$\overline{y} = \frac{1}{A} \int_{-2}^{2} \frac{1}{2}(4 - x^2)^2\, dx = \frac{3}{32} \cdot \frac{1}{2} \cdot 2 \int_0^2 (16 - 8x^2 + x^4)\, dx = \frac{3}{32}\left[16x - \frac{8}{3}x^3 + \frac{1}{5}x^5\right]_0^2$$
$$= \tfrac{3}{32}\left(32 - \tfrac{64}{3} + \tfrac{32}{5}\right) = 3\left(1 - \tfrac{2}{3} + \tfrac{1}{5}\right) = 3\left(\tfrac{8}{15}\right) = \tfrac{8}{5}$$

Thus, the centroid is $(\overline{x}, \overline{y}) = \left(0, \frac{8}{5}\right)$.

25. The region in the figure is "right-heavy" and "bottom-heavy," so we know $\overline{x} > 0.5$ and $\overline{y} < 1$, and we might guess that $\overline{x} = 0.6$ and $\overline{y} = 0.9$.



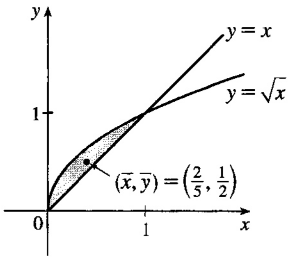
$$A = \int_0^1 e^x\, dx = \left[e^x\right]_0^1 = e - 1,$$

$$\overline{x} = \tfrac{1}{A} \int_0^1 x e^x\, dx = \tfrac{1}{e-1}\left[x e^x - e^x\right]_0^1 \quad \text{[by parts]}$$
$$= \tfrac{1}{e-1}\left[0 - (-1)\right] = \tfrac{1}{e-1},$$

$$\overline{y} = \tfrac{1}{A} \int_0^1 \tfrac{1}{2}(e^x)^2\, dx = \tfrac{1}{e-1} \cdot \tfrac{1}{4}\left[e^{2x}\right]_0^1 = \tfrac{1}{4(e-1)}\left(e^2 - 1\right) = \tfrac{e+1}{4}.$$

Thus, the centroid is $(\overline{x}, \overline{y}) = \left(\frac{1}{e-1}, \frac{e+1}{4}\right) \approx (0.58, 0.93)$.

27. $A=\int_0^1(\sqrt{x}-x)\,dx=\left[\frac{2}{3}x^{3/2}-\frac{1}{2}x^2\right]_0^1=\frac{2}{3}-\frac{1}{2}=\frac{1}{6}.$

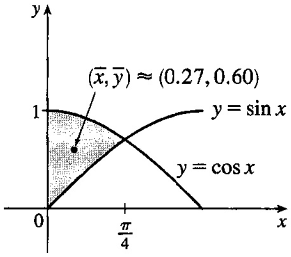


$$\begin{aligned}\overline{x}&=\frac{1}{A}\int_0^1 x(\sqrt{x}-x)\,dx=6\int_0^1(x^{3/2}-x^2)\,dx\\&=6\left[\frac{2}{5}x^{5/2}-\frac{1}{3}x^3\right]_0^1=6\left(\frac{2}{5}-\frac{1}{3}\right)=6\left(\frac{1}{15}\right)=\frac{2}{5};\end{aligned}$$

$$\begin{aligned}\overline{y}&=\frac{1}{A}\int_0^1\frac{1}{2}\left[(\sqrt{x})^2-x^2\right]dx=6\cdot\frac{1}{2}\int_0^1(x-x^2)\,dx\\&=3\left[\frac{1}{2}x^2-\frac{1}{3}x^3\right]_0^1=3\left(\frac{1}{2}-\frac{1}{3}\right)=\frac{1}{2}.\end{aligned}$$

Thus, the centroid is $(\overline{x},\overline{y})=\left(\frac{2}{5},\frac{1}{2}\right)$.

29. $A=\int_0^{\pi/4}(\cos x-\sin x)\,dx=[\sin x+\cos x]_0^{\pi/4}=\sqrt{2}-1,$

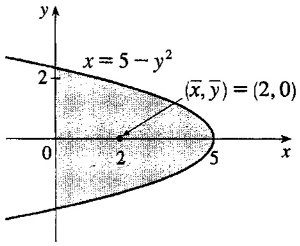


$$\begin{aligned}\overline{x}&=A^{-1}\int_0^{\pi/4}x(\cos x-\sin x)\,dx\\&=A^{-1}[x(\sin x+\cos x)+\cos x-\sin x]_0^{\pi/4}\qquad\text{[integration by parts]}\\&=A^{-1}\left(\frac{\pi}{4}\sqrt{2}-1\right)=\frac{\frac{1}{4}\pi\sqrt{2}-1}{\sqrt{2}-1}\end{aligned}$$

$$\overline{y}=A^{-1}\int_0^{\pi/4}\frac{1}{2}\left(\cos^2x-\sin^2x\right)dx=\frac{1}{2A}\int_0^{\pi/4}\cos 2x\,dx=\frac{1}{4A}[\sin 2x]_0^{\pi/4}=\frac{1}{4A}=\frac{1}{4(\sqrt{2}-1)}$$

Thus, the centroid is $(\overline{x},\overline{y})=\left(\dfrac{\pi\sqrt{2}-4}{4(\sqrt{2}-1)},\dfrac{1}{4(\sqrt{2}-1)}\right)\approx(0.27,0.60).$

31. From the figure we see that $\overline{y}=0$. Now



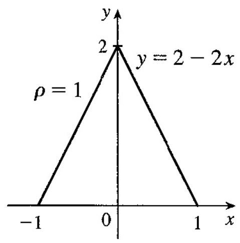
$$\begin{aligned}A&=\int_0^5 2\sqrt{5-x}\,dx=2\left[-\frac{2}{3}(5-x)^{3/2}\right]_0^5\\&=2\left(0+\frac{2}{3}\cdot 5^{3/2}\right)=\frac{20}{3}\sqrt{5}\end{aligned}$$

so

$$\begin{aligned}\overline{x}&=\frac{1}{A}\int_0^5 x\left[\sqrt{5-x}-\left(-\sqrt{5-x}\right)\right]dx=\frac{1}{A}\int_0^5 2x\sqrt{5-x}\,dx\\&=\frac{1}{A}\int_{\sqrt{5}}^0 2(5-u^2)u(-2u)\,du\qquad[u=\sqrt{5-x},\ x=5-u^2,\ u^2=5-x,\ dx=-2u\,du]\\&=\frac{4}{A}\int_0^{\sqrt{5}}u^2(5-u^2)\,du=\frac{4}{A}\left[\frac{5}{3}u^3-\frac{1}{5}u^5\right]_0^{\sqrt{5}}=\frac{3}{5\sqrt{5}}\left(\frac{25}{3}\sqrt{5}-5\sqrt{5}\right)=5-3=2\end{aligned}$$

Thus, the centroid is $(\overline{x},\overline{y})=(2,0)$.

33. By symmetry, $M_y = 0$ and $\overline{x} = 0$. $A = \frac{1}{2}bh = \frac{1}{2} \cdot 2 \cdot 2 = 2$.

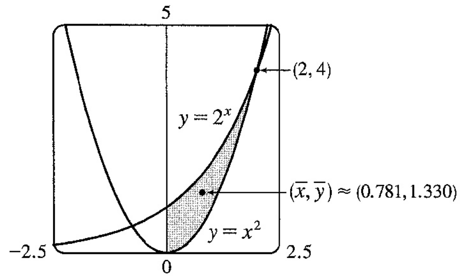


$$
\begin{aligned}
M_x &= \rho \textstyle\int_{-1}^{1} \frac{1}{2}(2-2x)^2\,dx = 2\rho \int_0^1 \frac{1}{2}(2-2x)^2\,dx \\
&= \left(2 \cdot 1 \cdot \tfrac{1}{2} \cdot 2^2\right) \textstyle\int_0^1 (1-x)^2\,dx \\
&= 4 \textstyle\int_1^0 u^2(-du) \qquad [u = 1-x,\ du = -dx] \\
&= -4\left[\tfrac{1}{3}u^3\right]_1^0 = -4\left(-\tfrac{1}{3}\right) = \tfrac{4}{3}
\end{aligned}
$$

$\overline{y} = \frac{1}{m}M_x = \frac{1}{\rho A}M_x = \frac{1}{1 \cdot 2} \cdot \frac{4}{3} = \frac{2}{3}$. Thus, the centroid is $(\overline{x}, \overline{y}) = \left(0, \frac{2}{3}\right)$.

35. $A = \displaystyle\int_0^2 \left(2^x - x^2\right) dx = \left[\frac{2^x}{\ln 2} - \frac{x^3}{3}\right]_0^2$

$= \left(\dfrac{4}{\ln 2} - \dfrac{8}{3}\right) - \dfrac{1}{\ln 2} = \dfrac{3}{\ln 2} - \dfrac{8}{3} \approx 1.661418.$

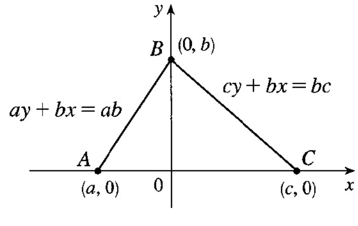


$$
\begin{aligned}
\overline{x} &= \frac{1}{A}\int_0^2 x(2^x - x^2)\,dx = \frac{1}{A}\int_0^2 (x2^x - x^3)\,dx \\
&= \frac{1}{A}\left[\frac{x2^x}{\ln 2} - \frac{2^x}{(\ln 2)^2} - \frac{x^4}{4}\right]_0^2 \qquad \text{[use parts]} \\
&= \frac{1}{A}\left[\frac{8}{\ln 2} - \frac{4}{(\ln 2)^2} - 4 + \frac{1}{(\ln 2)^2}\right] \\
&= \frac{1}{A}\left[\frac{8}{\ln 2} - \frac{3}{(\ln 2)^2} - 4\right] \approx \frac{1}{A}(1.297453) \approx 0.781
\end{aligned}
$$

$$
\begin{aligned}
\overline{y} &= \frac{1}{A}\int_0^2 \tfrac{1}{2}\left[(2^x)^2 - (x^2)^2\right] dx = \frac{1}{A}\int_0^2 \tfrac{1}{2}(2^{2x} - x^4)dx = \frac{1}{A} \cdot \frac{1}{2}\left[\frac{2^{2x}}{2\ln 2} - \frac{x^5}{5}\right]_0^2 \\
&= \frac{1}{A} \cdot \frac{1}{2}\left(\frac{16}{2\ln 2} - \frac{32}{5} - \frac{1}{2\ln 2}\right) = \frac{1}{A}\left(\frac{15}{4\ln 2} - \frac{16}{5}\right) \approx \frac{1}{A}(2.210106) \approx 1.330
\end{aligned}
$$

Since the position of a centroid is independent of density when the density is constant, we will assume for convenience that $\rho = 1$ in Exercises 36 and 37.

37. Choose x- and y-axes so that the base (one side of the triangle) lies along the x-axis with the other vertex along the positive y-axis as shown. From geometry, we know the medians intersect at a point $\frac{2}{3}$ of the way from each vertex (along the median) to the opposite side. The median from B goes to the midpoint $\left(\frac{1}{2}(a+c), 0\right)$ of side AC, so the point of intersection of the medians is $\left(\frac{2}{3} \cdot \frac{1}{2}(a+c), \frac{1}{3}b\right) = \left(\frac{1}{3}(a+c), \frac{1}{3}b\right)$.



[continued]

This can also be verified by finding the equations of two medians, and solving them simultaneously to find their point of intersection. Now let us compute the location of the centroid of the triangle. The area is $A = \frac{1}{2}(c-a)b$.

$$\begin{aligned}\overline{x} &= \tfrac{1}{A}\Big[\textstyle\int_a^0 x\cdot\frac{b}{a}(a-x)\,dx + \int_0^c x\cdot\frac{b}{c}(c-x)\,dx\Big] = \tfrac{1}{A}\Big[\frac{b}{a}\int_a^0 (ax-x^2)\,dx + \frac{b}{c}\int_0^c (cx-x^2)\,dx\Big]\\ &= \tfrac{b}{Aa}\big[\tfrac{1}{2}ax^2 - \tfrac{1}{3}x^3\big]_a^0 + \tfrac{b}{Ac}\big[\tfrac{1}{2}cx^2 - \tfrac{1}{3}x^3\big]_0^c = \tfrac{b}{Aa}\big[-\tfrac{1}{2}a^3 + \tfrac{1}{3}a^3\big] + \tfrac{b}{Ac}\big[\tfrac{1}{2}c^3 - \tfrac{1}{3}c^3\big]\\ &= \tfrac{2}{a(c-a)}\cdot\tfrac{-a^3}{6} + \tfrac{2}{c(c-a)}\cdot\tfrac{c^3}{6} = \tfrac{1}{3(c-a)}(c^2-a^2) = \tfrac{a+c}{3}\end{aligned}$$

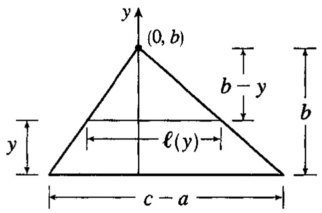
and

$$\begin{aligned}\overline{y} &= \tfrac{1}{A}\Big[\textstyle\int_a^0 \frac{1}{2}\big(\frac{b}{a}(a-x)\big)^2\,dx + \int_0^c \frac{1}{2}\big(\frac{b}{c}(c-x)\big)^2\,dx\Big]\\ &= \tfrac{1}{A}\Big[\tfrac{b^2}{2a^2}\textstyle\int_a^0 (a^2-2ax+x^2)\,dx + \tfrac{b^2}{2c^2}\int_0^c (c^2-2cx+x^2)\,dx\Big]\\ &= \tfrac{1}{A}\Big[\tfrac{b^2}{2a^2}\big[a^2x - ax^2 + \tfrac{1}{3}x^3\big]_a^0 + \tfrac{b^2}{2c^2}\big[c^2x - cx^2 + \tfrac{1}{3}x^3\big]_0^c\Big]\\ &= \tfrac{1}{A}\Big[\tfrac{b^2}{2a^2}\big(-a^3+a^3-\tfrac{1}{3}a^3\big) + \tfrac{b^2}{2c^2}\big(c^3-c^3+\tfrac{1}{3}c^3\big)\Big] = \tfrac{1}{A}\Big[\tfrac{b^2}{6}(-a+c)\Big] = \tfrac{2}{(c-a)b}\cdot\tfrac{(c-a)b^2}{6} = \tfrac{b}{3}\end{aligned}$$

Thus, the centroid is $(\overline{x},\overline{y}) = \left(\dfrac{a+c}{3},\dfrac{b}{3}\right)$, as claimed.

Remarks: Actually the computation of $\overline{y}$ is all that is needed. By considering each side of the triangle in turn to be the base, we see that the centroid is $\frac{1}{3}$ of the way from each side to the opposite vertex and must therefore be the intersection of the medians.

The computation of $\overline{y}$ in this problem (and many others) can be simplified by using horizontal rather than vertical approximating rectangles. If the length of a thin rectangle at coordinate y is $\ell(y)$, then its area is $\ell(y)\,\Delta y$, its mass is $\rho\ell(y)\,\Delta y$, and its moment about the x-axis is $\Delta M_x = \rho y\ell(y)\,\Delta y$. Thus,



$$M_x = \int \rho y\ell(y)\,dy \text{ and } \overline{y} = \frac{\int \rho y\ell(y)\,dy}{\rho A} = \tfrac{1}{A}\int y\ell(y)\,dy$$

In this problem, $\ell(y) = \dfrac{c-a}{b}(b-y)$ by similar triangles, so

$$\overline{y} = \frac{1}{A}\int_0^b \frac{c-a}{b}\,y(b-y)dy = \frac{2}{b^2}\int_0^b (by-y^2)\,dy = \frac{2}{b^2}\left[\tfrac{1}{2}by^2 - \tfrac{1}{3}y^3\right]_0^b = \frac{2}{b^2}\cdot\frac{b^3}{6} = \frac{b}{3}$$

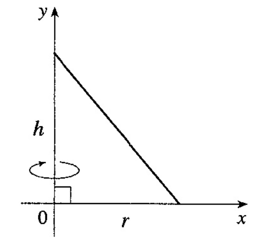
Notice that only one integral is needed when this method is used.

39. Divide the lamina into two triangles and one rectangle with respective masses of 2, 2 and 4, so that the total mass is 8. Using the result of Exercise 37, the triangles have centroids $\left(-1, \frac{2}{3}\right)$ and $\left(1, \frac{2}{3}\right)$. The centroid of the rectangle (its center) is $\left(0, -\frac{1}{2}\right)$. So, using Formulas 5 and 7, we have

$\overline{y} = \dfrac{M_x}{m} = \dfrac{1}{m}\sum\limits_{i=1}^{3} m_i y_i = \frac{1}{8}\left[2\left(\frac{2}{3}\right) + 2\left(\frac{2}{3}\right) + 4\left(-\frac{1}{2}\right)\right] = \frac{1}{8}\left(\frac{2}{3}\right) = \frac{1}{12}$, and $\overline{x} = 0$, since the lamina is symmetric about the line $x = 0$. Thus, the centroid is $(\overline{x}, \overline{y}) = \left(0, \frac{1}{12}\right)$.

41. A cone of height h and radius r can be generated by rotating a right triangle about one of its legs as shown. By Exercise 37, $\overline{x} = \frac{1}{3}r$, so by the Theorem of Pappus, the volume of the cone is

$V = Ad = \left(\frac{1}{2} \cdot \text{base} \cdot \text{height}\right) \cdot (2\pi\overline{x}) = \frac{1}{2}rh \cdot 2\pi\left(\frac{1}{3}r\right) = \frac{1}{3}\pi r^2 h.$



43. Suppose the region lies between two curves $y = f(x)$ and $y = g(x)$ where $f(x) \geq g(x)$, as illustrated in Figure 13. Choose points x_i with $a = x_0 < x_1 < \cdots < x_n = b$ and choose x_i^* to be the midpoint of the ith subinterval; that is, $x_i^* = \overline{x}_i = \frac{1}{2}(x_{i-1} + x_i)$. Then the centroid of the ith approximating rectangle R_i is its center $C_i = \left(\overline{x}_i, \frac{1}{2}[f(\overline{x}_i) + g(\overline{x}_i)]\right)$. Its area is $[f(\overline{x}_i) - g(\overline{x}_i)]\,\Delta x$, so its mass is $\rho[f(\overline{x}_i) - g(\overline{x}_i)]\,\Delta x$. Thus, $M_y(R_i) = \rho[f(\overline{x}_i) - g(\overline{x}_i)]\,\Delta x \cdot \overline{x}_i = \rho\overline{x}_i\,[f(\overline{x}_i) - g(\overline{x}_i)]\,\Delta x$ and $M_x(R_i) = \rho[f(\overline{x}_i) - g(\overline{x}_i)]\,\Delta x \cdot \frac{1}{2}[f(\overline{x}_i) + g(\overline{x}_i)] = \rho \cdot \frac{1}{2}\left[f(\overline{x}_i)^2 - g(\overline{x}_i)^2\right]\Delta x$. Summing over i and taking the limit as $n \to \infty$, we get $M_y = \lim\limits_{n\to\infty} \sum_i \rho\overline{x}_i\,[f(\overline{x}_i) - g(\overline{x}_i)]\,\Delta x = \rho\int_a^b x[f(x) - g(x)]\,dx$ and

$M_x = \lim\limits_{n\to\infty} \sum_i \rho \cdot \frac{1}{2}\left[f(\overline{x}_i)^2 - g(\overline{x}_i)^2\right]\Delta x = \rho\int_a^b \frac{1}{2}\left[f(x)^2 - g(x)^2\right]dx$. Thus,

$$\overline{x} = \frac{M_y}{m} = \frac{M_y}{\rho A} = \frac{1}{A}\int_a^b x[f(x) - g(x)]\,dx \quad \text{and} \quad \overline{y} = \frac{M_x}{m} = \frac{M_x}{\rho A} = \frac{1}{A}\int_a^b \tfrac{1}{2}\left[f(x)^2 - g(x)^2\right]dx$$

8.4 Applications to Economics and Biology

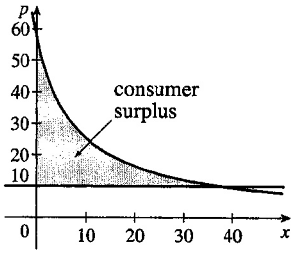
1. By the Net Change Theorem, $C(2000) - C(0) = \int_0^{2000} C'(x)\,dx \quad\Rightarrow$

$$\begin{aligned} C(2000) &= 20{,}000 + \int_0^{2000} (5 - 0.008x + 0.000009x^2)\,dx = 20{,}000 + \left[5x - 0.004x^2 + 0.000003x^3\right]_0^{2000} \\ &= 20{,}000 + 10{,}000 - 0.004(4{,}000{,}000) + 0.000003(8{,}000{,}000{,}000) = 30{,}000 - 16{,}000 + 24{,}000 \\ &= \$38{,}000 \end{aligned}$$

3. If the production level is raised from 1200 units to 1600 units, then the increase in cost is

$$\begin{aligned} C(1600) - C(1200) &= \textstyle\int_{1200}^{1600} C'(x)\,dx = \int_{1200}^{1600} \left(74 + 1.1x - 0.002x^2 + 0.00004x^3\right) dx \\ &= \left[74x + 0.55x^2 - \tfrac{0.002}{3}x^3 + 0.00001x^4\right]_{1200}^{1600} \\ &= 64{,}331{,}733.33 - 20{,}464{,}800 = \$43{,}866{,}933.33 \end{aligned}$$

5. $p(x) = 10 \quad\Rightarrow\quad \dfrac{450}{x+8} = 10 \quad\Rightarrow\quad x + 8 = 45 \quad\Rightarrow\quad x = 37.$



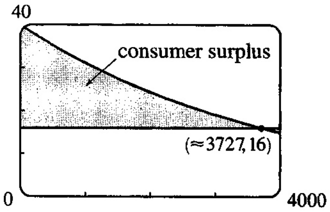
$$\begin{aligned} \text{Consumer surplus} &= \int_0^{37} [p(x) - 10]\,dx = \int_0^{37} \left(\frac{450}{x+8} - 10\right) dx \\ &= [450 \ln(x+8) - 10x]_0^{37} \\ &= (450 \ln 45 - 370) - 450 \ln 8 \\ &= 450 \ln\left(\tfrac{45}{8}\right) - 370 \approx \$407.25 \end{aligned}$$

7. $P = p_S(x) \quad\Rightarrow\quad 400 = 200 + 0.2x^{3/2} \quad\Rightarrow\quad 200 = 0.2x^{3/2} \quad\Rightarrow\quad 1000 = x^{3/2} \quad\Rightarrow\quad x = 1000^{2/3} = 100.$

$$\begin{aligned} \text{Producer surplus} &= \textstyle\int_0^{100} [P - p_S(x)]\,dx = \int_0^{100} \left[400 - \left(200 + 0.2x^{3/2}\right)\right] dx = \int_0^{100} \left(200 - \tfrac{1}{5}x^{3/2}\right) dx \\ &= \left[200x - \tfrac{2}{25}x^{5/2}\right]_0^{100} = 20{,}000 - 8{,}000 = \$12{,}000 \end{aligned}$$

9. $p(x) = \dfrac{800{,}000e^{-x/5000}}{x + 20{,}000} = 16 \quad\Rightarrow\quad x = x_1 \approx 3727.04.$

Consumer surplus $= \int_0^{x_1} [p(x) - 16]\,dx \approx \$37{,}753$



11. $f(8) - f(4) = \int_4^8 f'(t)\,dt = \int_4^8 \sqrt{t}\,dt = \left[\tfrac{2}{3}t^{3/2}\right]_4^8 = \tfrac{2}{3}\left(16\sqrt{2} - 8\right) \approx \$9.75$ million

13. $F = \dfrac{\pi P R^4}{8\eta l} = \dfrac{\pi(4000)(0.008)^4}{8(0.027)(2)} \approx 1.19 \times 10^{-4}\ \text{cm}^3/\text{s}$

15. $\int_0^{12} c(t)\,dt = \int_0^{12} \tfrac{1}{4}t(12 - t)\,dt = \int_0^{12} \left(3t - \tfrac{1}{4}t^2\right) dt = \left[\tfrac{3}{2}t^2 - \tfrac{1}{12}t^3\right]_0^{12} = (216 - 144) = 72\ \text{mg}\cdot\text{s/L}.$

Thus, the cardiac output is $F = \dfrac{A}{\int_0^{12} c(t)\,dt} = \dfrac{8\ \text{mg}}{72\ \text{mg}\cdot\text{s/L}} = \dfrac{1}{9}\ \text{L/s} = \dfrac{60}{9}\ \text{L/min}.$

8.5 Probability

1. (a) $\int_{30,000}^{40,000} f(x)\,dx$ is the probability that a randomly chosen tire will have a lifetime between 30,000 and 40,000 miles.

(b) $\int_{25,000}^{\infty} f(x)\,dx$ is the probability that a randomly chosen tire will have a lifetime of at least 25,000 miles.

3. (a) In general, we must satisfy the two conditions that are mentioned before Example 1—namely, (1) $f(x) \geq 0$ for all x, and (2) $\int_{-\infty}^{\infty} f(x)\,dx = 1$. For $0 \leq x \leq 4$, we have $f(x) = \frac{3}{64}x\sqrt{16 - x^2} \geq 0$, so $f(x) \geq 0$ for all x.

Also, $\int_{-\infty}^{\infty} f(x)\,dx = \int_0^4 \frac{3}{64}x\sqrt{16-x^2}\,dx = -\frac{3}{128}\int_0^4 (16-x^2)^{1/2}(-2x)\,dx = -\frac{3}{128}\left[\frac{2}{3}(16-x^2)^{3/2}\right]_0^4$

$$= -\tfrac{1}{64}\left[(16-x^2)^{3/2}\right]_0^4 = -\tfrac{1}{64}(0 - 64) = 1.$$

Therefore, f is a probability density function.

(b) $P(X < 2) = \int_{-\infty}^{2} f(x)\,dx = \int_0^2 \frac{3}{64}x\sqrt{16-x^2}\,dx = -\frac{3}{128}\int_0^2 (16-x^2)^{1/2}(-2x)\,dx$

$$= -\tfrac{3}{128}\left[\tfrac{2}{3}(16-x^2)^{3/2}\right]_0^2 = -\tfrac{1}{64}\left[(16-x^2)^{3/2}\right]_0^2 = -\tfrac{1}{64}(12^{3/2} - 16^{3/2})$$

$$= \tfrac{1}{64}(64 - 12\sqrt{12}) = \tfrac{1}{64}(64 - 24\sqrt{3}) = 1 - \tfrac{3}{8}\sqrt{3} \approx 0.350481$$

5. (a) In general, we must satisfy the two conditions that are mentioned before Example 1—namely, (1) $f(x) \geq 0$ for all x, and (2) $\int_{-\infty}^{\infty} f(x)\,dx = 1$. Since $f(x) = 0$ or $f(x) = 0.1$, condition (1) is satisfied. For condition (2), we see that $\int_{-\infty}^{\infty} f(x)\,dx = \int_0^{10} 0.1\,dx = \left[\frac{1}{10}x\right]_0^{10} = 1$. Thus, $f(x)$ is a probability density function for the spinner's values.

(b) Since all the numbers between 0 and 10 are equally likely to be selected, we expect the mean to be halfway between the endpoints of the interval; that is, $x = 5$.

$\mu = \int_{-\infty}^{\infty} xf(x)\,dx = \int_0^{10} x(0.1)\,dx = \left[\frac{1}{20}x^2\right]_0^{10} = \frac{100}{20} = 5$, as expected.

7. We need to find m so that $\int_m^{\infty} f(t)\,dt = \frac{1}{2} \;\Rightarrow\; \lim\limits_{x\to\infty}\int_m^x \frac{1}{5}e^{-t/5}\,dt = \frac{1}{2} \;\Rightarrow\; \lim\limits_{x\to\infty}\left[\frac{1}{5}(-5)e^{-t/5}\right]_m^x = \frac{1}{2} \;\Rightarrow$ $(-1)(0 - e^{-m/5}) = \frac{1}{2} \;\Rightarrow\; e^{-m/5} = \frac{1}{2} \;\Rightarrow\; -m/5 = \ln\frac{1}{2} \;\Rightarrow\; m = -5\ln\frac{1}{2} = 5\ln 2 \approx 3.47$ min.

9. We use an exponential density function with $\mu = 2.5$ min.

(a) $P(X > 4) = \int_4^{\infty} f(t)\,dt = \lim\limits_{x\to\infty}\int_4^x \frac{1}{2.5}e^{-t/2.5}\,dt = \lim\limits_{x\to\infty}\left[-e^{-t/2.5}\right]_4^x = 0 + e^{-4/2.5} \approx 0.202$

(b) $P(0 \leq X \leq 2) = \int_0^2 f(t)\,dt = \left[-e^{-t/2.5}\right]_0^2 = -e^{-2/2.5} + 1 \approx 0.551$

(c) We need to find a value a so that $P(X \geq a) = 0.02$, or, equivalently, $P(0 \leq X \leq a) = 0.98 \;\Leftrightarrow$ $\int_0^a f(t)\,dt = 0.98 \;\Leftrightarrow\; \left[-e^{-t/2.5}\right]_0^a = 0.98 \;\Leftrightarrow\; -e^{-a/2.5} + 1 = 0.98 \;\Leftrightarrow\; e^{-a/2.5} = 0.02 \;\Leftrightarrow$ $-a/2.5 = \ln 0.02 \;\Leftrightarrow\; a = -2.5\ln\frac{1}{50} = 2.5\ln 50 \approx 9.78 \text{ min} \approx 10$ min. The ad should say that if you aren't served within 10 minutes, you get a free hamburger.

11. $P(X \geq 10) = \int_{10}^{\infty} \frac{1}{4.2\sqrt{2\pi}} \exp\left(-\frac{(x-9.4)^2}{2 \cdot 4.2^2}\right) dx$. To avoid the improper integral we approximate it by the integral from 10 to 100. Thus, $P(X \geq 10) \approx \int_{10}^{100} \frac{1}{4.2\sqrt{2\pi}} \exp\left(-\frac{(x-9.4)^2}{2 \cdot 4.2^2}\right) dx \approx 0.443$ (using a calculator or computer to estimate the integral), so about 44 percent of the households throw out at least 10 lb of paper a week. *Note*: We can't evaluate $1 - P(0 \leq X \leq 10)$ for this problem since a significant amount of area lies to the left of $X = 0$.

13. $P(\mu - 2\sigma \leq X \leq \mu + 2\sigma) = \int_{\mu-2\sigma}^{\mu+2\sigma} \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right) dx$. Substituting $t = \frac{x-\mu}{\sigma}$ and $dt = \frac{1}{\sigma}\, dx$ gives us

$$\int_{-2}^{2} \frac{1}{\sigma\sqrt{2\pi}} e^{-t^2/2}(\sigma\, dt) = \frac{1}{\sqrt{2\pi}} \int_{-2}^{2} e^{-t^2/2}\, dt \approx 0.9545$$

15. (a) First $p(r) = \frac{4}{a_0^3} r^2 e^{-2r/a_0} \geq 0$ for $r \geq 0$. Next,

$$\int_{-\infty}^{\infty} p(r)\, dr = \int_{0}^{\infty} \frac{4}{a_0^3} r^2 e^{-2r/a_0}\, dr = \frac{4}{a_0^3} \lim_{t\to\infty} \int_0^t r^2 e^{-2r/a_0}\, dr$$

By using parts, tables, or a CAS , we find that $\int x^2 e^{bx}\, dx = (e^{bx}/b^3)(b^2x^2 - 2bx + 2)$. $(*)$

Next, we use $(*)$ (with $b = -2/a_0$) and l'Hospital's Rule to get $\frac{4}{a_0^3}\left[\frac{a_0^3}{-8}(-2)\right] = 1$. This satisfies the second condition for a function to be a probability density function.

(b) Using l'Hospital's Rule, $\frac{4}{a_0^3} \lim_{r\to\infty} \frac{r^2}{e^{2r/a_0}} = \frac{4}{a_0^3} \lim_{r\to\infty} \frac{2r}{(2/a_0)e^{2r/a_0}} = \frac{2}{a_0^2} \lim_{r\to\infty} \frac{2}{(2/a_0)e^{2r/a_0}} = 0$.

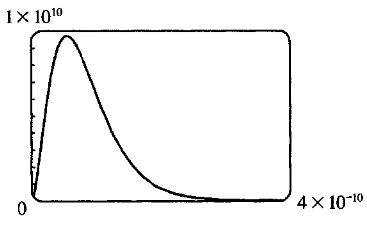
To find the maximum of p, we differentiate:

$$p'(r) = \frac{4}{a_0^3}\left[r^2 e^{-2r/a_0}\left(-\frac{2}{a_0}\right) + e^{-2r/a_0}(2r)\right] = \frac{4}{a_0^3} e^{-2r/a_0}(2r)\left(-\frac{r}{a_0} + 1\right)$$

$p'(r) = 0 \;\Leftrightarrow\; r = 0$ or $1 = \frac{r}{a_0} \;\Leftrightarrow\; r = a_0$ $[a_0 \approx 5.59 \times 10^{-11}$ m$]$. $p'(r)$ changes from positive to negative at $r = a_0$, so $p(r)$ has its maximum value at $r = a_0$.

(c) It is fairly difficult to find a viewing rectangle, but knowing the maximum value from part (b) helps.

$$p(a_0) = \frac{4}{a_0^3} a_0^2 e^{-2a_0/a_0} = \frac{4}{a_0} e^{-2} \approx 9{,}684{,}098{,}979$$



With a maximum of nearly 10 billion and a total area under the curve of 1, we know that the "hump" in the graph must be extremely narrow.

(d) $P(r) = \int_0^r \frac{4}{a_0^3} s^2 e^{-2s/a_0}\, ds \Rightarrow P(4a_0) = \int_0^{4a_0} \frac{4}{a_0^3} s^2 e^{-2s/a_0}\, ds$. Using (∗) from part (a) (with $b = -2/a_0$),

$$P(4a_0) = \frac{4}{a_0^3}\left[\frac{e^{-2s/a_0}}{-8/a_0^3}\left(\frac{4}{a_0^2}s^2 + \frac{4}{a_0}s + 2\right)\right]_0^{4a_0} = \frac{4}{a_0^3}\left(\frac{a_0^3}{-8}\right)\left[e^{-8}(64+16+2) - 1(2)\right]$$
$$= -\tfrac{1}{2}\left(82e^{-8} - 2\right) = 1 - 41e^{-8} \approx 0.986$$

(e) $\mu = \int_{-\infty}^{\infty} r p(r)\, dr = \frac{4}{a_0^3} \lim_{t\to\infty} \int_0^t r^3 e^{-2r/a_0}\, dr$. Integrating by parts three times or using a CAS, we find that $\int x^3 e^{bx}\, dx = \frac{e^{bx}}{b^4}\left(b^3x^3 - 3b^2x^2 + 6bx - 6\right)$. So with $b = -\frac{2}{a_0}$, we use l'Hospital's Rule, and get $\mu = \frac{4}{a_0^3}\left[-\frac{a_0^4}{16}(-6)\right] = \frac{3}{2}a_0$.

8 Review

CONCEPT CHECK

1. (a) The length of a curve is defined to be the limit of the lengths of the inscribed polygons, as described near Figure 3 in Section 8.1.

   (b) See Equation 8.1.2.

   (c) See Equation 8.1.4.

2. (a) $S = \int_a^b 2\pi f(x)\sqrt{1 + [f'(x)]^2}\, dx$

   (b) If $x = g(y)$, $c \le y \le d$, then $S = \int_c^d 2\pi y \sqrt{1 + [g'(y)]^2}\, dy$.

   (c) $S = \int_a^b 2\pi x \sqrt{1 + [f'(x)]^2}\, dx$ or $S = \int_c^d 2\pi g(y) \sqrt{1 + [g'(y)]^2}\, dy$

3. Let $c(x)$ be the cross-sectional length of the wall (measured parallel to the surface of the fluid) at depth x. Then the hydrostatic force against the wall is given by $F = \int_a^b \delta x c(x)\, dx$, where a and b are the lower and upper limits for x at points of the wall and δ is the weight density of the fluid.

4. (a) The center of mass is the point at which the plate balances horizontally.

   (b) See Equations 8.3.8.

5. If a plane region $\mathfrak{R}$ that lies entirely on one side of a line ℓ in its plane is rotated about ℓ, then the volume of the resulting solid is the product of the area of $\mathfrak{R}$ and the distance traveled by the centroid of $\mathfrak{R}$.

6. See Figure 3 in Section 8.4, and the discussion which precedes it.

7. (a) See the definition in the first paragraph of the subsection *Cardiac Output* in Section 8.4.

   (b) See the discussion in the second paragraph of the subsection *Cardiac Output* in Section 8.4.

8. A probability density function f is a function on the domain of a continuous random variable X such that $\int_a^b f(x)\, dx$ measures the probability that X lies between a and b. Such a function f has nonnegative values and satisfies the relation $\int_D f(x)\, dx = 1$, where D is the domain of the corresponding random variable X. If $D = \mathbb{R}$, or if we define $f(x) = 0$ for real numbers $x \notin D$, then $\int_{-\infty}^{\infty} f(x)\, dx = 1$. (Of course, to work with f in this way, we must assume that the integrals of f exist.)

9. (a) $\int_0^{100} f(x)\,dx$ represents the probability that the weight of a randomly chosen female college student is less than 100 pounds.

(b) $\mu = \int_{-\infty}^{\infty} xf(x)\,dx = \int_0^{\infty} xf(x)\,dx$

(c) The median of f is the number m such that $\int_m^{\infty} f(x)\,dx = \frac{1}{2}$.

10. See the discussion near Equation 3 in Section 8.5.

EXERCISES

1. $y = \frac{1}{6}(x^2+4)^{3/2} \quad\Rightarrow\quad dy/dx = \frac{1}{4}(x^2+4)^{1/2}(2x) \quad\Rightarrow$

$$1 + \left(\frac{dy}{dx}\right)^2 = 1 + \left[\tfrac{1}{2}x(x^2+4)^{1/2}\right]^2 = 1 + \tfrac{1}{4}x^2(x^2+4) = \tfrac{1}{4}x^4 + x^2 + 1 = \left(\tfrac{1}{2}x^2+1\right)^2.$$

Thus, $L = \int_0^3 \sqrt{\left(\frac{1}{2}x^2+1\right)^2}\,dx = \int_0^3 \left(\frac{1}{2}x^2+1\right)dx = \left[\frac{1}{6}x^3 + x\right]_0^3 = \frac{15}{2}$.

3. (a) $y = \dfrac{x^4}{16} + \dfrac{1}{2x^2} = \dfrac{1}{16}x^4 + \dfrac{1}{2}x^{-2} \quad\Rightarrow\quad \dfrac{dy}{dx} = \dfrac{1}{4}x^3 - x^{-3} \quad\Rightarrow$

$$1 + (dy/dx)^2 = 1 + \left(\tfrac{1}{4}x^3 - x^{-3}\right)^2 = 1 + \tfrac{1}{16}x^6 - \tfrac{1}{2} + x^{-6} = \tfrac{1}{16}x^6 + \tfrac{1}{2} + x^{-6} = \left(\tfrac{1}{4}x^3 + x^{-3}\right)^2.$$

Thus, $L = \int_1^2 \left(\frac{1}{4}x^3 + x^{-3}\right)dx = \left[\frac{1}{16}x^4 - \frac{1}{2}x^{-2}\right]_1^2 = \left(1 - \frac{1}{8}\right) - \left(\frac{1}{16} - \frac{1}{2}\right) = \frac{21}{16}$.

(b) $$\begin{aligned} S &= \int_1^2 2\pi x\left(\tfrac{1}{4}x^3 + x^{-3}\right)dx = 2\pi\int_1^2 \left(\tfrac{1}{4}x^4 + x^{-2}\right)dx = 2\pi\left[\tfrac{1}{20}x^5 - \tfrac{1}{x}\right]_1^2 \\ &= 2\pi\left[\left(\tfrac{32}{20} - \tfrac{1}{2}\right) - \left(\tfrac{1}{20} - 1\right)\right] = 2\pi\left(\tfrac{8}{5} - \tfrac{1}{2} - \tfrac{1}{20} + 1\right) = 2\pi\left(\tfrac{41}{20}\right) = \tfrac{41}{10}\pi \end{aligned}$$

5. $y = e^{-x^2} \quad\Rightarrow\quad dy/dx = -2xe^{-x^2} \quad\Rightarrow\quad 1 + (dy/dx)^2 = 1 + 4x^2e^{-2x^2}$.

Let $f(x) = \sqrt{1 + 4x^2e^{-2x^2}}$. Then

$$\begin{aligned} L = \int_0^3 f(x)\,dx \approx S_6 &= \frac{(3-0)/6}{3}[f(0) + 4f(0.5) + 2f(1) + 4f(1.5) + 2f(2) + 4f(2.5) + f(3)] \\ &\approx 3.292287 \end{aligned}$$

7. $y = \displaystyle\int_1^x \sqrt{\sqrt{t} - 1}\,dt \quad\Rightarrow\quad dy/dx = \sqrt{\sqrt{x} - 1} \quad\Rightarrow\quad 1 + (dy/dx)^2 = 1 + \left(\sqrt{x} - 1\right) = \sqrt{x}$.

Thus, $L = \int_1^{16} \sqrt{\sqrt{x}}\,dx = \int_1^{16} x^{1/4}\,dx = \frac{4}{5}\left[x^{5/4}\right]_1^{16} = \frac{4}{5}(32 - 1) = \frac{124}{5}$.

9. As in Example 1 of Section 8.3, $\dfrac{a}{2-x} = \dfrac{1}{2} \quad\Rightarrow\quad 2a = 2 - x$ and

$w = 2(1.5 + a) = 3 + 2a = 3 + 2 - x = 5 - x$. Thus,

$F = \int_0^2 \rho gx(5 - x)\,dx = \rho g\left[\frac{5}{2}x^2 - \frac{1}{3}x^3\right]_0^2 = \rho g\left(10 - \frac{8}{3}\right) = \frac{22}{3}\delta \;\; [\rho g = \delta] \;\approx \frac{22}{3} \cdot 62.5 \approx 458$ lb.

11. $A = \int_{-2}^{1} \left[(4 - x^2) - (x + 2)\right] dx = \int_{-2}^{1} \left(2 - x - x^2\right) dx = \left[2x - \frac{1}{2}x^2 - \frac{1}{3}x^3\right]_{-2}^{1}$
$= \left(2 - \frac{1}{2} - \frac{1}{3}\right) - \left(-4 - 2 + \frac{8}{3}\right) = \frac{9}{2} \quad \Rightarrow$

$$\overline{x} = A^{-1} \int_{-2}^{1} x(2 - x - x^2)dx = \frac{2}{9} \int_{-2}^{1} \left(2x - x^2 - x^3\right) dx = \frac{2}{9}\left[x^2 - \frac{1}{3}x^3 - \frac{1}{4}x^4\right]_{-2}^{1}$$
$$= \frac{2}{9}\left[\left(1 - \frac{1}{3} - \frac{1}{4}\right) - \left(4 + \frac{8}{3} - 4\right)\right] = -\frac{1}{2}$$

and
$$\overline{y} = A^{-1} \int_{-2}^{1} \frac{1}{2}\left[\left(4 - x^2\right)^2 - (x + 2)^2\right] dx = \frac{1}{9} \int_{-2}^{1} \left(x^4 - 9x^2 - 4x + 12\right) dx$$
$$= \frac{1}{9}\left[\frac{1}{5}x^5 - 3x^3 - 2x^2 + 12x\right]_{-2}^{1} = \frac{1}{9}\left[\left(\frac{1}{5} - 3 - 2 + 12\right) - \left(-\frac{32}{5} + 24 - 8 - 24\right)\right] = \frac{12}{5}$$

Thus, the centroid is $(\overline{x}, \overline{y}) = \left(-\frac{1}{2}, \frac{12}{5}\right)$.

13. An equation of the line passing through $(0, 0)$ and $(3, 2)$ is $y = \frac{2}{3}x$. $A = \frac{1}{2} \cdot 3 \cdot 2 = 3$. Therefore, using Equations 8.3.8, $\overline{x} = \frac{1}{3} \int_0^3 x\left(\frac{2}{3}x\right) dx = \frac{2}{27}\left[x^3\right]_0^3 = 2$, and $\overline{y} = \frac{1}{3} \int_0^3 \frac{1}{2}\left(\frac{2}{3}x\right)^2 dx = \frac{2}{81}\left[x^3\right]_0^3 = \frac{2}{3}$.

Thus, the centroid is $(\overline{x}, \overline{y}) = \left(2, \frac{2}{3}\right)$.

15. The centroid of this circle, $(1, 0)$, travels a distance $2\pi(1)$ when the lamina is rotated about the y-axis. The area of the circle is $\pi(1)^2$. So by the Theorem of Pappus, $V = A(2\pi\overline{x}) = \pi(1)^2 2\pi(1) = 2\pi^2$.

17. $x = 100 \quad \Rightarrow \quad P = 2000 - 0.1(100) - 0.01(100)^2 = 1890$

$$\text{Consumer surplus} = \int_0^{100} [p(x) - P]\, dx = \int_0^{100} \left(2000 - 0.1x - 0.01x^2 - 1890\right) dx$$
$$= \left[110x - 0.05x^2 - \frac{0.01}{3}x^3\right]_0^{100} = 11{,}000 - 500 - \frac{10{,}000}{3} \approx \$7166.67$$

19. $f(x) = \begin{cases} \frac{\pi}{20} \sin\left(\frac{\pi}{10}x\right) & \text{if } 0 \le x \le 10 \\ 0 & \text{if } x < 0 \text{ or } x > 10 \end{cases}$

(a) $f(x) \ge 0$ for all real numbers x and

$$\int_{-\infty}^{\infty} f(x)\, dx = \int_0^{10} \frac{\pi}{20} \sin\left(\frac{\pi}{10}x\right) dx = \frac{\pi}{20} \cdot \frac{10}{\pi}\left[-\cos\left(\frac{\pi}{10}x\right)\right]_0^{10}$$
$$= \frac{1}{2}(-\cos \pi + \cos 0) = \frac{1}{2}(1 + 1) = 1$$

Therefore, f is a probability density function.

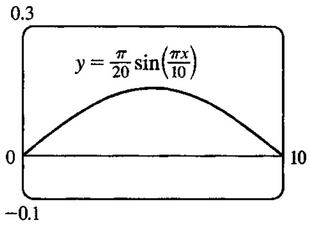
(b) $P(X < 4) = \int_{-\infty}^{4} f(x)\, dx = \int_0^4 \frac{\pi}{20} \sin\left(\frac{\pi}{10}x\right) dx = \frac{1}{2}\left[-\cos\left(\frac{\pi}{10}x\right)\right]_0^4 = \frac{1}{2}\left(-\cos \frac{2\pi}{5} + \cos 0\right)$

$\approx \frac{1}{2}(-0.309017 + 1) \approx 0.3455$

(c) $\mu = \int_{-\infty}^{\infty} xf(x)\,dx = \int_0^{10} \frac{\pi}{20}x\sin\left(\frac{\pi}{10}x\right)dx$

$= \int_0^{\pi} \frac{\pi}{20} \cdot \frac{10}{\pi}u(\sin u)\left(\frac{10}{\pi}\right)du \quad [u = \frac{\pi}{10}x,\ du = \frac{\pi}{10}\,dx]$

$= \frac{5}{\pi}\int_0^{\pi} u\sin u\,du \overset{82}{=} \frac{5}{\pi}\left[\sin u - u\cos u\right]_0^{\pi} = \frac{5}{\pi}[0 - \pi(-1)] = 5$



This answer is expected because the graph of f is symmetric about the line $x = 5$.

21. (a) The probability density function is $f(t) = \begin{cases} 0 & \text{if } t < 0 \\ \frac{1}{8}e^{-t/8} & \text{if } t \geq 0 \end{cases}$

$P(0 \leq X \leq 3) = \int_0^3 \frac{1}{8}e^{-t/8}\,dt = \left[-e^{-t/8}\right]_0^3 = -e^{-3/8} + 1 \approx 0.3127$

(b) $P(X > 10) = \int_{10}^{\infty} \frac{1}{8}e^{-t/8}\,dt = \lim\limits_{x\to\infty}\left[-e^{-t/8}\right]_{10}^{x} = \lim\limits_{x\to\infty}\left(-e^{-x/8} + e^{-10/8}\right) = 0 + e^{-5/4} \approx 0.2865$

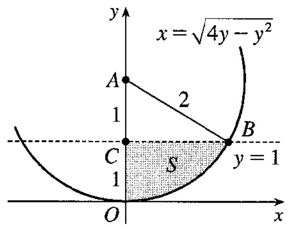
(c) We need to find m such that $P(X \geq m) = \frac{1}{2} \Rightarrow \int_m^{\infty} \frac{1}{8}e^{-t/8}\,dt = \frac{1}{2} \Rightarrow \lim\limits_{x\to\infty}\left[-e^{-t/8}\right]_m^x = \frac{1}{2} \Rightarrow$

$\lim\limits_{x\to\infty}\left(-e^{-x/8} + e^{-m/8}\right) = \frac{1}{2} \Rightarrow e^{-m/8} = \frac{1}{2} \Rightarrow -m/8 = \ln\frac{1}{2} \Rightarrow$

$m = -8\ln\frac{1}{2} = 8\ln 2 \approx 5.55$ minutes.

PROBLEMS PLUS

1. $x^2 + y^2 \le 4y \quad \Leftrightarrow \quad x^2 + (y-2)^2 \le 4$, so S is part of a circle, as shown in the diagram. The area of S is

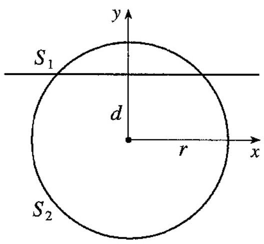


$$\begin{aligned} \int_0^1 \sqrt{4y - y^2}\, dy &\overset{113}{=} \left[\tfrac{y-2}{2}\sqrt{4y - y^2} + 2\cos^{-1}\!\left(\tfrac{2-y}{2}\right) \right]_0^1 \quad [a = 2] \\ &= -\tfrac{1}{2}\sqrt{3} + 2\cos^{-1}\!\left(\tfrac{1}{2}\right) - 2\cos^{-1} 1 \\ &= -\tfrac{\sqrt{3}}{2} + 2\left(\tfrac{\pi}{3}\right) - 2(0) = \tfrac{2\pi}{3} - \tfrac{\sqrt{3}}{2} \end{aligned}$$

Another method (without calculus): Note that $\theta = \angle CAB = \frac{\pi}{3}$, so the area is

$$(\text{area of sector } OAB) - (\text{area of } \triangle ABC) = \tfrac{1}{2}(2^2)\tfrac{\pi}{3} - \tfrac{1}{2}(1)\sqrt{3} = \tfrac{2\pi}{3} - \tfrac{\sqrt{3}}{2}$$

3. (a) The two spherical zones, whose surface areas we will call S_1 and S_2, are generated by rotation about the y-axis of circular arcs, as indicated in the figure. The arcs are the upper and lower portions of the circle $x^2 + y^2 = r^2$ that are obtained when the circle is cut with the line $y = d$. The portion of the upper arc in the first quadrant is sufficient to generate the upper spherical zone. That portion of the arc can be described by the relation $x = \sqrt{r^2 - y^2}$ for $d \le y \le r$. Thus, $dx/dy = -y/\sqrt{r^2 - y^2}$ and



$$ds = \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy = \sqrt{1 + \frac{y^2}{r^2 - y^2}}\, dy = \sqrt{\frac{r^2}{r^2 - y^2}}\, dy = \frac{r\, dy}{\sqrt{r^2 - y^2}}$$

From Formula 8.2.8 we have

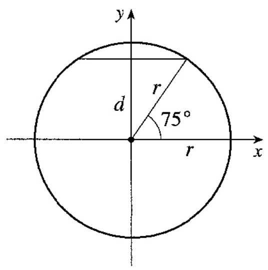
$$S_1 = \int_d^r 2\pi x \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy = \int_d^r 2\pi \sqrt{r^2 - y^2}\, \frac{r\, dy}{\sqrt{r^2 - y^2}} = \int_d^r 2\pi r\, dy = 2\pi r(r - d)$$

Similarly, we can compute $S_2 = \int_{-r}^{d} 2\pi x \sqrt{1 + (dx/dy)^2}\, dy = \int_{-r}^{d} 2\pi r\, dy = 2\pi r(r + d)$. Note that $S_1 + S_2 = 4\pi r^2$, the surface area of the entire sphere.

(b) $r = 3960$ mi and $d = r\,(\sin 75^\circ) \approx 3825$ mi,

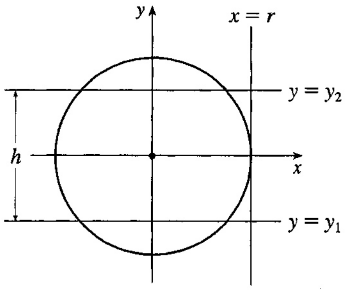
so the surface area of the Arctic Ocean is about

$2\pi r(r - d) \approx 2\pi(3960)(135) \approx 3.36 \times 10^6 \text{ mi}^2$.



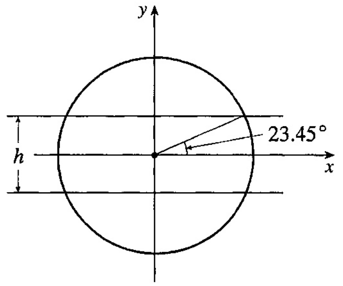
(c) The area on the sphere lies between planes $y = y_1$ and $y = y_2$, where $y_2 - y_1 = h$. Thus, we compute the surface area on the sphere to be $S = \int_{y_1}^{y_2} 2\pi x \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy = \int_{y_1}^{y_2} 2\pi r\, dy = 2\pi r(y_2 - y_1) = 2\pi rh$.

This equals the lateral area of a cylinder of radius r and height h, since such a cylinder is obtained by rotating the line $x = r$ about the y-axis, so the surface area of the cylinder between the planes $y = y_1$ and $y = y_2$ is



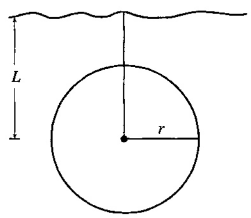
$$A = \int_{y_1}^{y_2} 2\pi x \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy = \int_{y_1}^{y_2} 2\pi r \sqrt{1 + 0^2}\, dy$$
$$= 2\pi r y\Big|_{y=y_1}^{y_2} = 2\pi r(y_2 - y_1) = 2\pi rh$$

(d) $h = 2r \sin 23.45° \approx 3152$ mi, so the surface area of the Torrid Zone is $2\pi rh \approx 2\pi(3960)(3152) \approx 7.84 \times 10^7\ \text{mi}^2$.



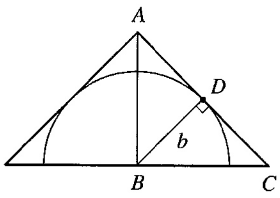
5. (a) Choose a vertical x-axis pointing downward with its origin at the surface. In order to calculate the pressure at depth z, consider n subintervals of the interval $[0, z]$ by points x_i and choose a point $x_i^* \in [x_{i-1}, x_i]$ for each i. The thin layer of water lying between depth x_{i-1} and depth x_i has a density of approximately $\rho(x_i^*)$, so the weight of a piece of that layer with unit cross-sectional area is $\rho(x_i^*)g\,\Delta x$. The total weight of a column of water extending from the surface to depth z (with unit cross-sectional area) would be approximately $\sum_{i=1}^{n} \rho(x_i^*)g\,\Delta x$. The estimate becomes exact if we take the limit as $n \to \infty$; weight (or force) per unit area at depth z is $W = \lim_{n\to\infty} \sum_{i=1}^{n} \rho(x_i^*)g\,\Delta x$. In other words, $P(z) = \int_0^z \rho(x)g\,dx$. More generally, if we make no assumptions about the location of the origin, then $P(z) = P_0 + \int_0^z \rho(x)g\,dx$, where P_0 is the pressure at $x = 0$. Differentiating, we get $dP/dz = \rho(z)g$.

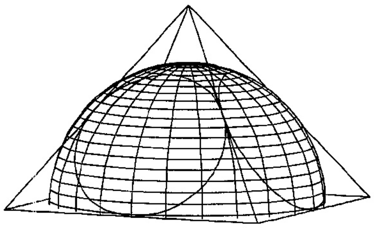
(b)

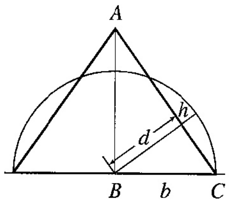


$$\begin{aligned} F &= \textstyle\int_{-r}^{r} P(L + x) \cdot 2\sqrt{r^2 - x^2}\, dx \\ &= \textstyle\int_{-r}^{r} \left(P_0 + \int_0^{L+x} \rho_0 e^{z/H} g\, dz\right) \cdot 2\sqrt{r^2 - x^2}\, dx \\ &= P_0 \textstyle\int_{-r}^{r} 2\sqrt{r^2 - x^2}\, dx + \rho_0 g H \int_{-r}^{r} \left(e^{(L+x)/H} - 1\right) \cdot 2\sqrt{r^2 - x^2}\, dx \\ &= (P_0 - \rho_0 g H) \textstyle\int_{-r}^{r} 2\sqrt{r^2 - x^2}\, dx + \rho_0 g H \int_{-r}^{r} e^{(L+x)/H} \cdot 2\sqrt{r^2 - x^2}\, dx \\ &= (P_0 - \rho_0 g H)(\pi r^2) + \rho_0 g H e^{L/H} \textstyle\int_{-r}^{r} e^{x/H} \cdot 2\sqrt{r^2 - x^2}\, dx \end{aligned}$$

7. To find the height of the pyramid, we use similar triangles. The first figure shows a cross-section of the pyramid passing through the top and through two opposite corners of the square base. Now $|BD| = b$, since it is a radius of the sphere, which has diameter $2b$ since it is tangent to the opposite sides of the square base. Also, $|AD| = b$ since $\triangle ADB$ is isosceles. So the height is $|AB| = \sqrt{b^2 + b^2} = \sqrt{2}b$.







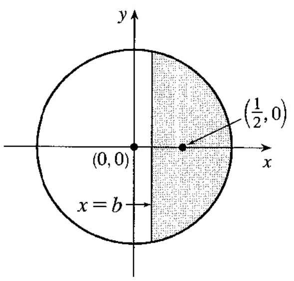
We first observe that the shared volume is equal to half the volume of the sphere, minus the sum of the four equal volumes (caps of the sphere) cut off by the triangular faces of the pyramid. See Exercise 6.2.49 for a derivation of the formula for the volume of a cap of a sphere. To use the formula, we need to find the perpendicular distance h of each triangular face from the surface of the sphere. We first find the distance d from the center of the sphere to one of the triangular faces. The third figure shows a cross-section of the pyramid through the top and through the midpoints of opposite sides of the square base. From similar triangles we find that

$$\frac{d}{b} = \frac{|AB|}{|AC|} = \frac{\sqrt{2}\,b}{\sqrt{b^2 + \left(\sqrt{2}\,b\right)^2}} \quad \Rightarrow \quad d = \frac{\sqrt{2}\,b^2}{\sqrt{3b^2}} = \frac{\sqrt{6}}{3}\,b$$

So $h = b - d = b - \frac{\sqrt{6}}{3}b = \frac{3-\sqrt{6}}{3}b$. So, using the formula $V = \pi h^2(r - h/3)$ from Exercise 6.2.49 with $r = b$, we find that the volume of each of the caps is

$\pi\left(\frac{3-\sqrt{6}}{3}b\right)^2\left(b - \frac{3-\sqrt{6}}{3\cdot 3}b\right) = \frac{15-6\sqrt{6}}{9}\cdot\frac{6+\sqrt{6}}{9}\pi b^3 = \left(\frac{2}{3} - \frac{7}{27}\sqrt{6}\right)\pi b^3$. So, using our first observation, the shared volume is $V = \frac{1}{2}\left(\frac{4}{3}\pi b^3\right) - 4\left(\frac{2}{3} - \frac{7}{27}\sqrt{6}\right)\pi b^3 = \left(\frac{28}{27}\sqrt{6} - 2\right)\pi b^3$.

9. We can assume that the cut is made along a vertical line $x = b > 0$, that the disk's boundary is the circle $x^2 + y^2 = 1$, and that the center of mass of the smaller piece (to the right of $x = b$) is $\left(\frac{1}{2}, 0\right)$. We wish to find b to two decimal places. We have



$\frac{1}{2} = \overline{x} = \dfrac{\int_b^1 x\cdot 2\sqrt{1-x^2}\,dx}{\int_b^1 2\sqrt{1-x^2}\,dx}$. Evaluating the numerator gives us

$-\int_b^1 (1-x^2)^{1/2}(-2x)\,dx = -\frac{2}{3}\left[\left(1-x^2\right)^{3/2}\right]_b^1 = -\frac{2}{3}\left[0 - \left(1-b^2\right)^{3/2}\right] = \frac{2}{3}(1-b^2)^{3/2}$. Using

Formula 30 in the table of integrals, we find that the denominator is

$\left[x\sqrt{1-x^2}+\sin^{-1}x\right]_b^1 = \left(0+\frac{\pi}{2}\right) - \left(b\sqrt{1-b^2}+\sin^{-1}b\right)$. Thus, we have

$\dfrac{1}{2} = \overline{x} = \dfrac{\frac{2}{3}(1-b^2)^{3/2}}{\frac{\pi}{2} - b\sqrt{1-b^2} - \sin^{-1}b}$, or, equivalently, $\frac{2}{3}(1-b^2)^{3/2} = \frac{\pi}{4} - \frac{1}{2}b\sqrt{1-b^2} - \frac{1}{2}\sin^{-1}b$. Solving this

equation numerically with a calculator or CAS, we obtain $b \approx 0.138173$, or $b = 0.14$ m to two decimal places.

11. If $h = L$, then

$$P = \frac{\text{area under } y = L\sin\theta}{\text{area of rectangle}} = \frac{\int_0^\pi L\sin\theta\,d\theta}{\pi L} = \frac{[-\cos\theta]_0^\pi}{\pi} = \frac{-(-1)+1}{\pi} = \frac{2}{\pi}$$

If $h = L/2$, then

$$P = \frac{\text{area under } y = \frac{1}{2}L\sin\theta}{\text{area of rectangle}} = \frac{\int_0^\pi \frac{1}{2}L\sin\theta\,d\theta}{\pi L} = \frac{[-\cos\theta]_0^\pi}{2\pi} = \frac{2}{2\pi} = \frac{1}{\pi}$$