Theme 11

11 Differential Equations, Slope Fields, and the Logistics Equation

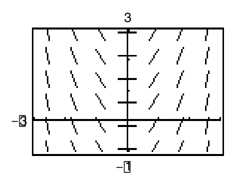
Key to Text Coverage

Section	Examples	Exercises	Topics
4.1		53, 54	*Slope fields introduced*
4.5		39, 40	*More slope fields exercises*
5.2		41, 41	*More slope fields exercises*
5.5	7	90–92	*Logistic growth model*
7.5		58	*Logistic equation developed*

Summary

Slope fields are introduced in Chapter 4, and appear throughout the textbook. Let $y' = f(x, y)$ be a differential equation. A **slope field** (or direction field) is a graph consisting of little line segments with slopes given by the differential equation. The line segments provide a visual perspective of the shape of the solution to the differential equation.

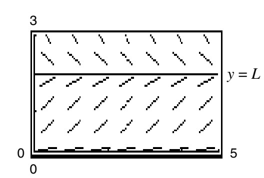
For example, consider the slope field at the right for the differential equation $y' = 2x$. At points (x, y), the calculator has indicated the slope of the solution at that point. For instance, the slope at $(1, 1)$ is 2, whereas the slope at $(2, 1)$ is 4. The solutions to this differential equation are all parabolas of the form $y = x^2 + C$.



One of the most important differential equations in population modeling is the **logistics equation**

$$\frac{dy}{dt} = ky(L - y).$$

This model indicates that the rate of growth of a population y is proportional to the product of the population and the difference $L - y$. The constant L is called the **carrying capacity** of the model, and the line $y = L$ is a horizontal asymptote of the solution. A slope field for this differential equation is shown at the right.



The general solution of the logistics equation can be obtained by separation of variables.

$$\frac{dy}{dt} = ky(L - y)$$

$$\frac{1}{y(L - y)}\,dy = k\,dt$$

$$\frac{1}{L}\int\left(\frac{1}{y} + \frac{1}{L - y}\right)dy = \int k\,dt$$

$$\ln\frac{y}{L - y} = kLt + C_1$$

$$\frac{y}{L - y} = e^{kLt + C_1} = C_2e^{kLt}$$

$$y + yC_2e^{kLt} = C_2Le^{kLt}$$

$$y = \frac{C_2Le^{kLt}}{1 + C_2e^{kLt}} = \frac{C_2L}{e^{-kLt} + C_2} = \frac{L}{1 + Ce^{-kLt}}$$

Theme 11

Worked Example

Consider the logistics differential equation

$$\frac{dy}{dt} = 2y(3 - y).$$

(a) Draw a slope field in the viewing window $0 \le t \le 5$ by $0 \le y \le 5$.

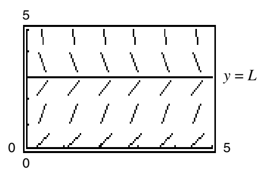
(b) Draw a solution curve on the slope field that passes through the point $(0, 1)$. Draw another solution curve on your slope field that passes through the point $(0, 4)$. What do you observe?

(c) Find the general solution to the differential equation.

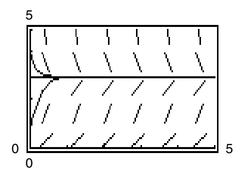
(d) At what value of y is dy/dt a maximum?

SOLUTION

(a)



(b)



Both solution curves tend to $y = 3$ as $t \to \infty$.
$y = 3$ is the horizontal asymptote to the right.

(c) You could find the solution by separating variables, or by using the formula on the previous page with $k = 2$ and $L = 3$. A third option is to use the substitution $y = 1/u$ and $y' = -u'/u^2$.

$$\frac{dy}{dt} = 2y(3 - y)$$

$$-\frac{1}{u^2}\frac{du}{dt} = 2\left(\frac{1}{u}\right)\left(3 - \frac{1}{u}\right)$$

$$-\frac{1}{u^2}\frac{du}{dt} = \frac{6}{u} - \frac{2}{u^2}$$

$$\frac{du}{dt} = -6u + 2$$

$$\int \frac{1}{6u - 2}\,du = -\int dt$$

$$\frac{1}{6}\ln|6u - 2| = -t + C_1$$

$$6u - 2 = C_2e^{-6t}$$

$$u = \frac{C_2e^{-6t} + 2}{6} = \frac{1}{y}$$

$$y = \frac{6}{2 + C_2e^{-6t}} = \frac{3}{1 + Ce^{-6t}}$$

(d) You know that

$$\frac{d}{dt}\left(\frac{dy}{dt}\right) = 6 - 4y = 0$$

when $y = 3/2$. By the first derivative test, dy/dt is a maximum when $y = 3/2$. In general, the maximum occurs at $y = L/2$.

Name ______________________________

Date ______________________________

Sample Questions

Show all your work on a separate sheet of paper. Indicate clearly the methods you use because you will be graded on the correctness of your methods as well as on the accuracy of your answers.

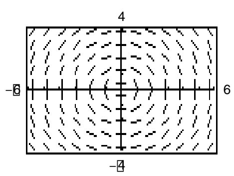
Multiple Choice

1. What is the general solution to the differential equation $dy/dx = 10y - 5y^2$?

(a) $y = \dfrac{2}{1 + Ce^{-10x}}$ (b) $y = \dfrac{5}{1 + Ce^{-10x}}$ (c) $y = \dfrac{1}{1 + Ce^{-5x}}$

(d) $y = \dfrac{5}{1 + Ce^{-2x}}$ (e) $y = \dfrac{2}{1 + Ce^{-5x}}$

2. The slope field for a differential equation is shown in the figure. Determine the general solution of this equation.

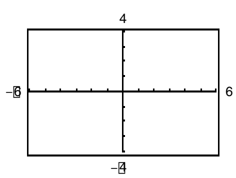


(a) $y = Cx^2$ (b) $x = Cy^2$ (c) $x^2 - y^2 = C^2$

(d) $y^2 - x^2 = C^2$ (e) $x^2 + y^2 = C^2$

Free Response

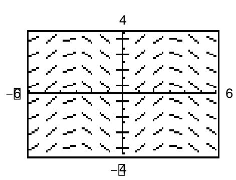
1. Sketch the slope field for the differential equation $y' = x - y$ on the viewing window below.



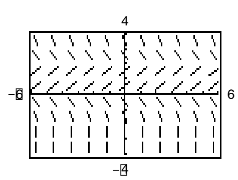
2. Match the slope field with the differential equation.

a. $y' = -x$ **b.** $y' = \sin x$ **c.** $y' = 1 - y$ **d.** $y' = y(2 - y)$

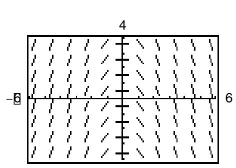
A.



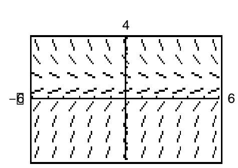
B.



C.



D.



SOLUTIONS

Multiple Choice

1. Answer (a). This is a logistics differential equation with $k = 5$ and $L = 2$.

$$\frac{dy}{dx} = 10y - 5y^2 = 5y(2 - y)$$

Using the formula developed on page 41, the general solution is

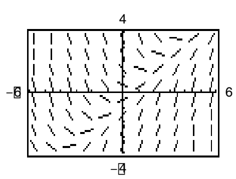
$$y = \frac{2}{1 + Ce^{-(5)(2)x}} = \frac{2}{1 + Ce^{-10x}}$$

You could also solve the differential equation by separation of variables.

2. Answer (e). The solutions have the form of concentric circles about the origin.

Free Response

1.



2. The graphs are matched as follows.

graph **A.** **b.** $y' = \sin x$

graph **B.** **d.** $y' = y(2 - y)$

graph **C.** **a.** $y' = -x$

graph **D.** **c.** $y' = 1 - y$