

# 14 Polar Coordinates and Graphing

## Key to Text Coverage

Section	Examples	Exercises	Topics
9.4	1–6	1–116	Polar coordinates and polar graphs
9.5	1–5	1–65	Area and arc length in polar coordinates
9.6	1–4	1–63	Polar equations of conics

## Summary

Polar coordinates are related to rectangular coordinates by the equations

$$x = r \cos \theta \quad \tan \theta = \frac{y}{x}$$

$$y = r \sin \theta \quad r^2 = x^2 + y^2$$

Keep in mind that the polar coordinates for a point are not unique. For instance, the point  $(x, y) = (0, 1)$  has polar coordinates  $(r, \theta) = (1, \pi/2) = (-1, 3\pi/2) = (1, 5\pi/2) = \dots$

This multiple representation of the same point often makes it difficult to determine all the points of intersection of polar graphs. For example, to determine the points of intersection of the two polar equations  $r = 1$  and  $r = 1 - 2 \sin \theta$  you begin by solving  $1 = 1 - 2 \sin \theta$  to obtain  $\theta = 0, \pi$ . However, there is a third point of intersection,  $(r, \theta) = (-1, \pi/2)$ . The graphs intersect at this point, but for different values of  $\theta$ . This can be seen by graphing the two curves in **simultaneous mode**.

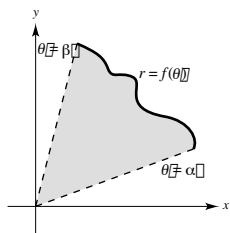
If  $r = f(\theta)$  is a polar curve, then  $x = r \cos \theta = f(\theta) \cos \theta$  and  $y = r \sin \theta = f(\theta) \sin \theta$ . So, the slope of the curve,  $dy/dx$  is given by

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{f'(\theta) \sin \theta + f(\theta) \cos \theta}{f'(\theta) \cos \theta - f(\theta) \sin \theta}$$

The area and arc length formulas in polar coordinates are

$$\text{Area} = \int_{\alpha}^{\beta} \frac{1}{2} r^2 d\theta$$

$$\text{Arc length} = \int_{\alpha}^{\beta} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} d\theta.$$



**Worked Example**

Consider the polar curves  $r = 6 \sin \theta$  and  $r = 2 + 2 \sin \theta$ , for  $0 \leq \theta \leq 2\pi$ .

- Find all points of intersection of the two curves.
- Graph the two curves and indicate their points of intersection.
- Find the area inside the first curve and outside the second.

**SOLUTION**

- Begin by solving the equations simultaneously.

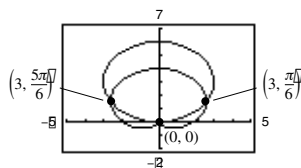
$$6 \sin \theta = 2 + 2 \sin \theta$$

$$4 \sin \theta = 2$$

$$\sin \theta = 1/2$$

So,  $\theta = \pi/6, 5\pi/6$ . However, the origin is a third point of intersection. So, the two polar graphs intersect at  $(r, \theta) = (3, \pi/6), (3, 5\pi/6), (0, 0)$ .

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- The area is given by the following integral.

$$\begin{aligned} \text{Area} &= \frac{1}{2} \int_{\pi/6}^{5\pi/6} [(6 \sin \theta)^2 - (2 + 2 \sin \theta)^2] d\theta \\ &= \int_{\pi/6}^{5\pi/6} [16 \sin^2 \theta - 2 - 4 \sin \theta] d\theta \\ &= \int_{\pi/6}^{5\pi/6} [8 - 8 \cos 2\theta - 2 - 4 \sin \theta] d\theta \\ &= \left[ 6\theta - 4 \sin 2\theta + 4 \cos \theta \right]_{\pi/6}^{5\pi/6} = 4\pi \end{aligned}$$

**Notes**

- The integrals  $\int \sin^2 \theta d\theta$  and  $\int \cos^2 \theta d\theta$  occur frequently in the calculus of polar coordinates. You should memorize the half-angle formulas needed to integrate these functions.

$$\sin^2 \theta = \frac{1 - \cos 2\theta}{2} \qquad \cos^2 \theta = \frac{1 + \cos 2\theta}{2}$$

Name \_\_\_\_\_

Date \_\_\_\_\_

**Sample Questions**

Show all your work on a separate sheet of paper. Indicate clearly the methods you use because you will be graded on the correctness of your methods as well as on the accuracy of your answers.

**Multiple Choice**

1. Find the slope  $dy/dx$  of the polar curve  $r = 2(1 - \cos \theta)$  at the point  $(r, \theta) = (2, \pi/2)$ .

(a)  $-2$                       (b)  $-1$                       (c)  $0$                       (d)  $1$                       (e)  $2$

2. Find the area inside the polar curve  $r = 3 \cos 3\theta$ .

(a)  $\frac{7\pi}{4}$                       (b)  $2\pi$                       (c)  $\frac{9\pi}{4}$                       (d)  $\frac{5\pi}{2}$                       (e)  $\frac{11\pi}{4}$

3. Use the integration capability of a graphing utility to find the arc length of the polar curve  $r = 2 + 2 \sin \theta$  from  $\theta = \pi/6$  to  $\theta = 5\pi/6$ .

(a)  $8$                       (b)  $8.4$                       (c)  $9$                       (d)  $\sqrt{85}$                       (e)  $10$

**Free Response**

Consider the polar function  $r = 2 \csc \theta + 4$ .

- (a) Sketch the curve on the domain  $0 < \theta < \pi$ .
- (b) Sketch the curve on the domain  $\pi < \theta < 2\pi$ . Find the bounds on  $\theta$  over which the curve is traced out exactly once.
- (c) Find all horizontal tangents.
- (d) Set up the integral for the area of the closed loop and use a graphing utility to approximate the integral.

## SOLUTIONS

## Multiple Choice

1. Answer (b).  $r = f(\theta) = 2 - 2 \cos \theta$ ,  $f'(\theta) = 2 \sin \theta$ ,  $f(\pi/2) = 2$ , and  $f'(\pi/2) = 2$ . So

$$\frac{dy}{dx} = \frac{f'(\theta) \sin \theta + f(\theta) \cos \theta}{f'(\theta) \cos \theta - f(\theta) \sin \theta} = \frac{2(1) + 2(0)}{2(0) - 2(1)} = -1.$$

2. Answer (c). Notice that the curve is traced out once on the interval  $0 \leq \theta \leq \pi$ . One third of the area of the rose curve is traced out from  $\theta = \pi/6$  to  $\theta = \pi/2$ .

$$\begin{aligned} \text{Area} &= 3 \int_{\pi/6}^{\pi/2} \frac{1}{2} (3 \cos 3\theta)^2 d\theta \\ &= \frac{27}{2} \int_{\pi/6}^{\pi/2} \cos^2 3\theta d\theta \\ &= \frac{27}{4} \int_{\pi/6}^{\pi/2} (1 + \cos 6\theta) d\theta \quad (\text{half-angle formula}) \\ &= \frac{27}{4} \left[ \theta + (\sin 6\theta)/6 \right]_{\pi/6}^{\pi/2} \\ &= \frac{27}{4} [\pi/2 - \pi/6] = 9\pi/4 \end{aligned}$$

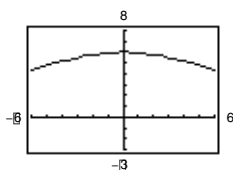
3. Answer (a).  $r = 2 + 2 \sin \theta$ ,  $dr/d\theta = 2 \cos \theta$ , and

$$r^2 + \left( \frac{dr}{d\theta} \right)^2 = 4 + 8 \sin \theta + 4 \sin^2 \theta + 4 \cos^2 \theta = 8 + 8 \sin \theta.$$

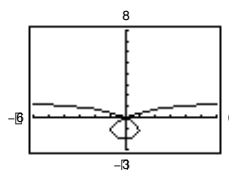
So, the arc length is given by the integral  $\int_{\pi/6}^{5\pi/6} \sqrt{8 + 8 \sin \theta} d\theta = 8$ .

## Free Response

(a)



(b)



The entire graph is traced out on the two open intervals  $0 < \theta < \pi$  and  $\pi < \theta < 2\pi$ .

- (c) Set the derivative  $dy/dx$  equal to 0.

$$\frac{dy}{dx} = \frac{f'(\theta) \sin \theta + f(\theta) \cos \theta}{f'(\theta) \cos \theta - f(\theta) \sin \theta} = \frac{-2 \csc \theta \cot \theta \sin \theta + (2 \csc \theta + 4) \cos \theta}{-2 \csc \theta \cot \theta \sin \theta - (2 \csc \theta + 4) \sin \theta} = 0.$$

So,  $-2 \csc \theta \cot \theta \sin \theta + (2 \csc \theta + 4) \cos \theta = \cos \theta [2 \csc \theta + 4 - 2 \csc \theta] = 0$ , which gives  $\theta = \pi/2$  and  $\theta = 3\pi/2$ . The graph has horizontal tangents at  $(r, \theta) = (6, \pi/2)$  and  $(2, 3\pi/2)$ .

- (d) The loop is traced out between  $\theta = 7\pi/6$  and  $\theta = 11\pi/6$ . The area is given by the integral

$$A = \frac{1}{2} \int_{7\pi/6}^{11\pi/6} (2 \csc \theta + 4)^2 d\theta \approx 2.61204.$$