6 The Integral as an Accumulation Function

Key to Text Coverage

Section	Examples	Exercises	Topics
4.4	6	94	*Definite Integral as a Function*
4.4	7, 8	69–93, Project	*Second Fundamental Theorem of Calculus*
5.1	1, 2	1, 2	*Definition of Logarithm as an Integral*

Formulas

$F(x) = \int_a^x f(t)\, dt$ $\qquad$ $F(x)$ is an accumulation function.

$F'(x) = f(x)$ $\qquad$ Second Fundamental Theorem of Calculus

If the upper limit of integration is a function u of x, then

$$F(x) = \int_a^{u(x)} f(t)\, dt \Longrightarrow F'(x) = f(u(x))\, u'(x).$$

Summary

One of the most important examples of how an integral can be used to define another function is the definition of the natural logarithm.

$\ln x = \int_1^x \frac{1}{t}\, dt$ $\qquad$ Section 5.1

As x moves to the right from $t = 1$, the integral "accumulates" the area under the curve $f(t) = 1/t$. By the Second Fundamental Theorem of Calculus, the derivative of the logarithm function is

$$(\ln x)' = \frac{1}{x}.$$

Notice how this illustrates the inverse relationship between the operations of differentiation and integration.

If the upper limit of integration is a function of x, say $u(x)$, then the Second Fundamental Theorem of Calculus can be combined with the Chain Rule to give

$$F(x) = \int_a^{u(x)} f(t)\, dt \Longrightarrow F'(x) = \frac{dF}{du}\frac{du}{dx} = f(u(x))\, u'(x).$$

Worked Example

Let h be defined as the following integral.

$$h(x) = \int_{-1}^{x} \cos(t^2)\,dt.$$

(a) Find $h(-1)$ and approximate $h(1)$.

(b) Find $h'(x)$.

(c) Is the graph of h increasing, decreasing, or neither at $x = 0$? Show the analysis that leads to your conclusion.

(d) Is the graph of h concave upwards, concave downwards, or neither at $x = 1$? Show the analysis that leads to your conclusion.

SOLUTION

(a) $h(-1) = \int_{-1}^{-1} \cos(t^2)\,dt = 0$. You can use the numerical integration capability of a graphing utility to obtain

$$h(1) = \int_{-1}^{1} \cos(t^2)\,dt \approx 1.809.$$

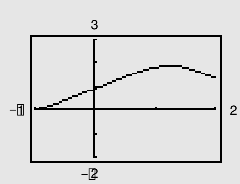
(b) By the Second Fundamental Theorem of Calculus, $h'(x) = \cos(x^2)$.

(c) At $x = 0$, $h'(0) = \cos(0) = 1 > 0$. So, the graph of h is increasing at $x = 0$.

(d) $h''(x) = -2x\sin(x^2)$, which is less than 0 at $x = 1$. So, the graph of h is concave downwards at $x = 1$.

Notes

The function $f(t) = \cos(t^2)$ does not have an elementary antiderivative (see page 300), and so you must use the Second Fundamental Theorem of Calculus in this problem. Even though you do not know the explicit antiderivative of $f(t) = \cos(t^2)$, you can still produce its graph with a graphing utility. For example, on a *TI-83,* you can obtain the graph of h by plotting Y1 = fnInt(cos(X ^ 2),X,−1,X), as indicated below.



(a) If the limits of integration are the same, then the definite integral equals 0.

(b) The Second Fundamental Theorem of Calculus applies here. The bottom limit of integration is constant, and the top is simply x. So, the derivative of the integral is equal to the integrand expressed as a function of x.

(c), (d) Notice how the graph above confirms our answer.

Name ________________________________

Date ________________________________

Theme 6

Sample Questions

Show all your work on a separate sheet of paper. Indicate clearly the methods you use because you will be graded on the correctness of your methods as well as on the accuracy of your answers.

Multiple Choice

1. Find $g''(1)$ if $g(x) = \int_0^x t^3 e^t \, dt$.

   (a) e (b) $2e$ (c) $e - 1$ (d) $3e$ (e) $4e$

2. Let $h(x) = \int_{x^2}^2 \sqrt{1 + t^4} \, dt$. Find $h'(1)$.

   (a) $-\sqrt{2}$ (b) $\sqrt{2}$ (c) $-2\sqrt{2}$ (d) $2\sqrt{2}$ (e) $4\sqrt{2}$

3. Find the range of the function $F(x) = \int_{-4}^x \sqrt{16 - t^2} \, dt$.

   (a) $[-4, 4]$ (b) $[-4, 0]$ (c) $[0, 4]$ (d) $[0, 4\pi]$ (e) $[0, 8\pi]$

Free Response

The graph of the function f defined on $[-2, 2]$ is shown in the figure. Let

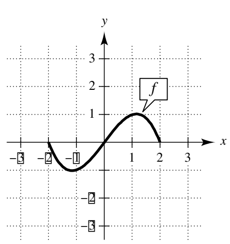
$$F(x) = \int_{-2}^{2x+1} f(t) \, dt.$$

(a) Find $F\left(-\frac{3}{2}\right)$.

(b) Find $F'(x)$ and $F'(0)$.

(c) Find the domain of F.

(d) Find the x-coordinate of the minimum of F. Show the analysis that leads to your conclusion.



SOLUTIONS

Multiple Choice

1. Answer (e). First use the Second Fundamental Theorem of Calculus to find $g'(x) = x^3e^x$. Differentiating this expression, $g''(x) = 3x^2e^x + x^3e^x$. Finally, $g''(1) = 3e + e = 4e$. You could have used integration by parts to explicitly find the antiderivative, but it would have made the problem much more difficult.

2. Answer (c).

$$h(x) = \int_{x^2}^{2} \sqrt{1 + t^4}\, dt = -\int_{2}^{x^2} \sqrt{1 + t^4}\, dt.$$

By the Second Fundamental Theorem of Calculus and the Chain Rule, $h'(x) = -\sqrt{1 + (x^2)^4}\,(2x)$ and $h'(1) = -2\sqrt{2}$. Notice that the integrand does not have an elementary antiderivative.

3. Answer (e). $F(x)$ gives the area of the region under the semicircle $y = \sqrt{16 - t^2}$ between $t = -4$ and $t = x$. As x varies from -4 to 4, $F(x)$ goes from 0 to 8π, the area of a semicircle of radius 4. Verify this answer by graphing $F(x)$ on the viewing window $[-4, 5] \times [0, 30]$.

Free Response

(a) $F\left(-\dfrac{3}{2}\right) = \displaystyle\int_{-2}^{2(-3/2)+1} f(t)\, dt = \int_{-2}^{-2} f(t)\, dt = 0.$

(b) By the Second Fundamental Theorem of Calculus and the Chain Rule, $F'(x) = f(2x + 1)(2)$ and $F'(0) = 2f(1) = 2$.

(c) Since the domain of f is $[-2, 2]$, solve the inequality $-2 \le 2x + 1 \le 2$ to obtain the domain of F, $[-3/2, 1/2]$.

(d) $F'(x) = 2f(2x + 1) = 0$ when $2x + 1$ equals -2, 0, and 2, the zeros of f. So, the critical numbers of F are $-\frac{3}{2}$, $-\frac{1}{2}$, and $\frac{1}{2}$.

By checking the two intervals determined by these three points, you can see that F is decreasing on

$$\left(-\tfrac{3}{2}, -\tfrac{1}{2}\right)\ (F'(-1) = 2f(-1) = -2 < 0)$$

and increasing on

$$\left(-\tfrac{1}{2}, \tfrac{1}{2}\right)\ (F'(0) = 2 > 0).$$

So, the minimum occurs at $x = -\frac{1}{2}$.