# 7 Volumes with Known Cross Sections and Other Applications of Integration

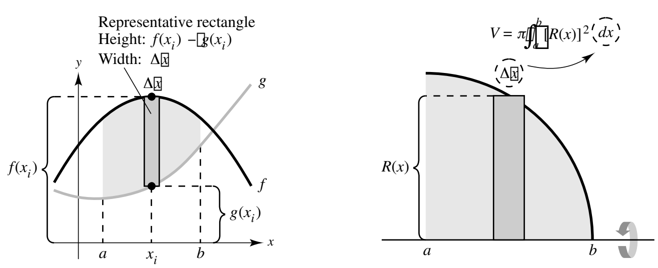
### Key to Text Coverage

| Section | Examples | Exercises | Topics |
|---|---|---|---|
| 4.3 | 1–6 | 1–71 | *Riemann sums, definite integrals and area* |
| 6.1 | 1–5 | 1–52, 57–81 | *Area between two curves* |
| 6.2 | 1–7 | 1–67 | *Volumes by disks and known cross sections* |
| 6.3 | 1–5 | 1–43 | *Volumes by the shell method* |
| 6.4 | 1–7 | 1–56 | *Arc length and surfaces of revolution* |
| 6.5 | 1–6 | 1–46 | *Work* |

## Summary

Most of the applications of integration are based on the summation interpretation of definite integrals as limits of Riemann Sums. This idea is highlighted on page 412 in the context of finding the area between two curves $f(x)$ and $g(x)$, for $a \le x \le b$.

$$\text{Area} = \sum(\text{Height})(\text{Width}) = \int_a^b [f(x) - g(x)]\,dx$$




The disk method for finding volumes can be analyzed similarly (page 422). For a horizontal axis of rotation, the volume of a representative disk is $\pi R(x)^2\,\Delta x$, as indicated in the figure on the right.

$$\text{Volume} = \sum(\text{Area of disk})(\text{Thickness}) = \int_a^b \pi[R(x)]^2\,dx$$

The concept of work can also be thought of as a summation of increments (page 453)

$$\text{Work} = \sum(\text{Force})(\text{Distance}) = \int_a^b F(x)\,dx.$$

## Worked Example

An open tank has the shape of a right circular cone, obtained by revolving the curve $y = 3x/2$, $0 \le x \le 2$ about the $y$-axis. Water enters the empty tank at the rate of 1 cubic foot per hour.

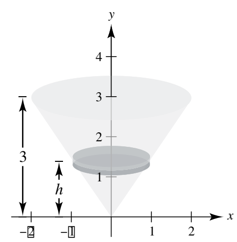
(a) Find the volume of water in the tank as a function of the height $h$ of the water.

(b) How fast is the level of the water rising when $h = 1$ and when $h = 2$?

(c) How much work is done in emptying the full tank by pumping the water over the top edge? Water weighs 62.4 pounds per cubic foot.

### SOLUTION

(a) Imagine the volume as a sum of horizontal disks:



$$\text{Volume} = \sum(\text{Area})(\text{Thickness})$$

$$V = \int_0^h \pi x^2\,dy$$

$$= \int_0^h \pi\left(\frac{2y}{3}\right)^2 dy$$

$$= \frac{4\pi}{9}\left[\frac{y^3}{3}\right]_0^h = \frac{4\pi h^3}{27} \text{ cubic feet}$$

(b) Differentiate both sides of the equation $V = 4\pi h^3/27$ with respect to $t$.

$$\frac{dV}{dt} = 1 = \frac{4\pi h^2}{9}\frac{dh}{dt} \Longrightarrow \frac{dh}{dt} = \frac{9}{4\pi h^2}$$

At $h = 1$, $dh/dt = 9/(4\pi) \approx 0.716$ feet per hour, and at $h = 2$, $dh/dt = 9/(16\pi) \approx 0.179$ feet per hour.

(c) $W = \sum(\text{Force})(\text{Distance}) = \sum(\text{Volume})(\text{Density})(\text{Distance})$

$$= \sum(\pi x^2\,dy)(62.4)(3 - y)$$

$$W = \int_0^3 \pi\left(\frac{2y}{3}\right)^2 (62.4)(3 - y)\,dy$$

$$= 62.4\pi\frac{4}{9}\int_0^3 y^2(3 - y)\,dy$$

$$= 62.4\pi\frac{4}{9}\left[y^3 - \frac{y^4}{4}\right]_0^3$$

$$= 62.4\pi\frac{4}{9}\left[27\left(1 - \frac{3}{4}\right)\right] \text{ foot-pounds}$$

$$= 187.2\pi \text{ foot-pounds}$$

### Notes

(a) This answer holds when the cone is full of water, $h = 3$. In this case, you have just the volume of a cone of radius 2 and height 3, $V = \frac{1}{3}\pi(2^2)(3) = 4\pi$.

(b) Notice that the second rate is smaller. The water level rises slower as the depth increases.

Name ________________________________

Date ________________________________



## Sample Questions

*Show all your work on a separate sheet of paper. Indicate clearly the methods you use because you will be graded on the correctness of your methods as well as on the accuracy of your answers.*

### Multiple Choice

1. Let the region bounded by $x^2 + y^2 = 9$ be the base of a solid. Find the volume if cross sections taken perpendicular to the base are isosceles right triangles.

   (a) 30 (b) 32 (c) 34 (d) 36 (e) 38

2. A cylindrical tank 12 feet high with a radius of 8 feet is buried so that the top of the tank is 3 feet below ground level. Write down the integral that gives the amount of work done in pumping a full tank of oil up to ground level. Use 50 pounds per cubic foot as the weight-density of oil.

   (a) $(4\pi)50\int_0^{12}(15 - y)\,dy$ (b) $(16\pi)50\int_3^{15}(15 - y)\,dy$ (c) $(16\pi)50\int_0^{12}(15 - y)\,dy$

   (d) $(16\pi)\int_3^{15}(15 - y)\,dy$ (e) $(4\pi)50\int_3^{15}(12 - y)\,dy$

3. Use a graphing utility to approximate the arc length of one arch of the sine curve, $y = \sin x$, $0 \le x \le \pi$.

   (a) 2.0 (b) 2.78 (c) 2.83 (d) 3.78 (e) 3.82

### Free Response

Consider the region $R$ in the first quadrant bounded above by the curve $y = 1/x$ between $x = 1$ and $x = 5$.

(a) Find the area of $R$.

(b) Let $x = b$ be a vertical line dividing $R$ into two regions of equal area. Without doing any calculations, explain why $b < 3$.

(c) Use calculus to find the value of $b$ in part (b).

(d) Find the volume of the solid with base $R$ and whose cross sections cut by planes perpendicular to the $x$-axis are squares.

Theme 7

## SOLUTIONS

### Multiple Choice

1. Answer (d). The length of the base of the triangle is $2y = 2\sqrt{9 - x^2}$.

$$V = \int_{-3}^{3} \frac{1}{2}\left(2\sqrt{9 - x^2}\right)\left(\sqrt{9 - x^2}\right) dx = 2\int_{0}^{3} (9 - x^2)\, dx = 2\left[9x - \frac{x^3}{3}\right]_0^3 = 36$$

2. Answer (c). Work $= \sum$(Volume)(Density)(Distance).

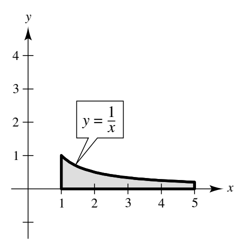
$$W = \int_0^{12} (\pi(4)^2\, dy)(50)(15 - y) = (16\pi)50\int_0^{12} (15 - y)\, dy$$

3. Answer (e). $f(x) = \sin x \Longrightarrow f'(x) = \cos x$. So,

$$s = \int_a^b \sqrt{1 + [f'(x)]^2}\, dx = \int_0^{\pi} \sqrt{1 + \cos^2 x}\, dx \approx 3.82.$$

### Free Response

(a) $A = \displaystyle\int_1^5 \frac{1}{x}\, dx = \ln x \Big|_1^5 = \ln 5.$




(b) Although $x = 3$ divides the interval $[1, 5]$ into two equal intervals, the area in the region on the interval $[1, 3]$ is greater than the area in the region on the interval $[3, 5]$.

(c) You want to find $b$ such that

$$\frac{1}{2}\ln 5 = \ln 5^{1/2} = \int_1^b \frac{1}{x}\, dx = \ln b.$$

So, $b = \sqrt{5} \approx 2.236$. Note that $b < 3$ as indicated in part (b).

(d) $V = \sum(\text{Area})(\text{Thickness}) = \displaystyle\int_1^5 \left(\frac{1}{x}\right)^2 dx = -\frac{1}{x}\Big|_1^5 = -\frac{1}{5} + 1 = \frac{4}{5}$