12 Differential Equations and Euler's Method

Key to Text Coverage

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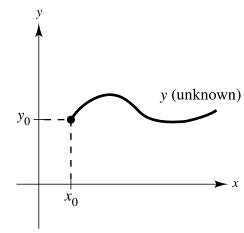
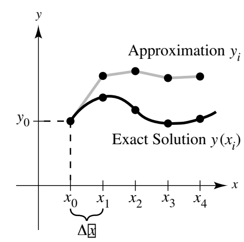
Summary

Many differential equations, such as $y' = e^{x^2}$ and $y' = x^2 + y^2$, cannot be solved by traditional pencil and paper methods. In fact, most differential equations that arise in real life applications are solved on computers using approximation techniques. The simplest such technique is called Euler's Method and is based on a tangent line approximation.



Let $dy/dx = y' = f(x, y)$ be a differential equation with initial condition $y = y_0$ when $x = x_0$. This means that the solution y is only known at the initial point (x_0, y_0), and that the derivative of y is known at all points (x, y), as indicated in the figure at the top right.



Partition the x-axis to the right of x_0 into subintervals $x_0 < x_1 < x_2 < \cdots$ of length $\Delta x = x_{i+1} - x_i$. Euler's Method approximates the value of the exact solution at these points x_i, as indicated in the figure at the bottom right.

The algorithm uses the known slope of the tangent line at (x_0, y_0) to estimate the solution y_1 at x_1:

$$\frac{y_1 - y_0}{x_1 - x_0} = f(x_0, y_0)$$

$$y_1 = y_0 + f(x_0, y_0)\,\Delta x$$

The procedure is repeated using (x_1, y_1) to estimate the solution y_2 at x_2, and so on. Each of these steps is called one **iteration.**

$$y_1 = y_0 + f(x_0, y_0)\,\Delta x$$

$$y_2 = y_1 + f(x_1, y_1)\,\Delta x$$

$$y_3 = y_2 + f(x_2, y_2)\,\Delta x$$

$$\vdots$$

$$y_{i+1} = y_i + f(x_i, y_i)\,\Delta x$$

If $y_1, y_2, \ldots$ are the approximations from Euler's Method, and $y(x_1), y(x_2), \ldots$ are the exact values of the (unknown) solution, then the error at iteration i is the absolute value of the difference between the approximation and the exact value,

$$\text{Error} = |y_i - y(x_i)|.$$

In general, the error increases as i increases.

A *TI-83* calculator program for Euler's Method is given at the right.

Store the differential equation in Y1.

```
:Program Euler
:Disp "INITIAL X"
:Input x
:Disp "INITIAL Y"
:Input y
:Disp "DELTA X"
:Input D
:Disp "ITERATIONS"
:Input N
:0 → I
:While I < N
: y+Y1*D → YNEW
:Disp YNEW
:Pause
: x+D → x
: YNEW → y
:Disp x, y
: I+1 → I
:End
```

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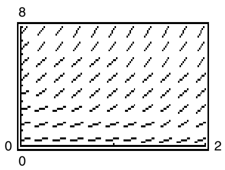
Worked Example

Consider the differential equation $y' = x + y$ with initial condition $y = 1$ when $x = 0$.

(a) Do three iterations of Euler's Method with $\Delta x = 0.1$.

(b) Sketch the solution to the differential equation on the given slope field.



(c) The exact solution to this differential equation is $y = 2e^x - x - 1$. Calculate the error of the Euler approximations y_1, y_2, and y_3. What do you observe?

SOLUTION

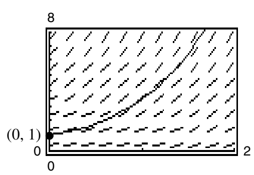
(a) You can use Euler's Method $y_{i+1} = y_i + f(x_i, y_1)\,\Delta x$ with $x_0 = 0$, $y_0 = 1$, $f(x, y) = x + y$ and $\Delta x = 0.1$.

$$y_1 = y_0 + f(x_0, y_0)\,\Delta x = 1 + (0 + 1)(0.1) = 1.1$$

$$y_2 = y_1 + f(x_1, y_1)\,\Delta x = 1.1 + (0.1 + 1.1)(0.1) = 1.22$$

$$y_3 = y_2 + f(x_2, y_2)\,\Delta x = 1.22 + (0.2 + 1.22)(0.1) = 1.362$$

(b) The solution follows the direction of the tick marks, and must pass through the point (0, 1).



(c) The following table shows the approximations y_i, the exact solutions $y(x_i)$ and the errors $|y_i - y(x_i)|$ at the four points x_0, x_1, x_2, and x_3.

x	$x_0 = 0$	$x_1 = 0.1$	$x_2 = 0.2$	$x_3 = 0.3$
y_i	1.0	1.1	1.22	1.362
$y(x_i)$	1.0	1.11034	1.24281	1.39972
Error	0	0.01034	0.02281	0.03772

The errors get larger as i increases.

Notes

(c) At the initial point $(x_0, y_0) = (1, 0)$ there is no error because y_0 is the given exact solution of the differential equation. However, at each iteration the errors seem to increase. One way to see this is to note that the calculation of y_2 uses y_1 as its initial condition, which is already in error.

Name ______________________________

Date ______________________________

Sample Questions

Show all your work on a separate sheet of paper. Indicate clearly the methods you use because you will be graded on the correctness of your methods as well as on the accuracy of your answers.

Multiple Choice

1. Let $y' = 2xy$ be a differential equation with initial condition $y_0 = 2$ when $x_0 = 1$. Approximate the solution at $x_1 = 1.1$ by performing one iteration of Euler's Method with $\Delta x = 0.1$.

   (a) 2.0 (b) 2.1 (c) 2.2 (d) 2.4 (e) 6.0

2. Let

   $$\frac{dy}{dt} = \frac{1}{2}y$$

   be a differential equation with initial condition $y_0 = 4$ when $t_0 = 0$. Approximate the solution at $t_2 = 0.4$ by performing two iterations of Euler's Method with $\Delta t = 0.2$.

   (a) 4.4 (b) 4.44 (c) 4.84 (d) 4.8 (e) 8.84

3. Let $y' = 2x/y$ be a differential equation with initial condition $y_0 = 1$ when $x_0 = 1$. Use a graphing utility to do ten iterations of Euler's Method with $\Delta x = 0.1$ and find the difference between the exact solution and the Euler approximation at $x_{10} = 2.0$. You may use the program given on page 45.

   (a) 0.01468 (b) 0.01646 (c) 0.01684 (d) 0.01692 (e) 0.01741

Free Response

Consider the logistics differential equation $y' = y(1 - y)$ with initial condition $y_0 = 1/2$ when $x_0 = 0$.

(a) Do two iterations of Euler's Method using $\Delta x = 0.05$.

(b) Solve differential equation. $\left(\text{Hint: } \frac{1}{y(1-y)} = \frac{1}{y} + \frac{1}{1-y}.\right)$

(c) Graph the solution on the interval $0 \le x \le 10$ and determine the horizontal asymptote to the right.

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SOLUTIONS

Multiple Choice

1. Answer (d). Use Euler's Method with $x_0 = 1$, $y_0 = 2$, $f(x, y) = 2xy$, and $\Delta x = 0.1$.

$$y_1 = y_0 + f(x_0, y_0)\,\Delta x = 2 + 2(1)(2)(0.1) = 2 + 0.4 = 2.4$$

2. Answer (c). Two iterations of Euler's Method with $t_0 = 0$, $y_0 = 4$, $f(t, y) = y/2$, and $\Delta t = 0.2$ are

$$y_1 = y_0 + f(t_0, y_0)\,\Delta t = 4 + \frac{1}{2}(4.0)(0.2) = 4.4$$

$$y_2 = y_1 + f(t_1, y_1)\,\Delta t = 4.4 + \frac{1}{2}(4.4)(0.2) = 4.84.$$

3. Answer (a). The Euler approximation at x_{10} is 2.66043. The exact solution is obtained by separation of variables.

$$\frac{dy}{dx} = \frac{2x}{y} \implies \int y\,dy = \int 2x\,dx \implies \frac{y^2}{2} = x^2 + C_1 \implies y = \sqrt{2x^2 + C}$$

The initial condition $y(1) = 1$ gives $C = -1$ and the particular solution is $y = \sqrt{2x^2 - 1}$. At $x = 2$, $y = \sqrt{2(2^2) - 1} = 2.64575$ and the error is $|2.66043 - 2.64575| = 0.01468$.

Free Response

(a) Two iterations of Euler's Method with $x_0 = 0$, $y_0 = 1/2$, $f(x, y) = y(1 - y)$, and $\Delta x = 0.05$ are

$$y_1 = y_0 + f(x_0, y_0)\,\Delta x = 0.5 + 0.5(1 - 0.5)(0.05) = 0.5125$$

$$y_2 = y_1 + f(x_1, y_1)\,\Delta x = 0.5125 + 0.5125(1 - 0.5125)(0.05) = 0.5250.$$

(b) Use the method of separation of variables.

$$\frac{dy}{dx} = y(1 - y)$$

$$\frac{1}{y(1 - y)}\,dy = dx$$

$$\int \left(\frac{1}{y} + \frac{1}{1 - y}\right) dy = \int dx$$

$$\ln \frac{y}{1 - y} = x + C_1$$

$$\frac{y}{1 - y} = e^{x + C_1} = Ce^x$$

$$y + yCe^x = Ce^x$$

$$y = \frac{Ce^x}{1 + Ce^x}$$

The initial condition $y = 1/2$ when $x = 0$ gives $C = 1$. So, the exact solution is

$$y = \frac{e^x}{1 + e^x} = \frac{1}{1 + e^{-x}}.$$

(c) $y = 1$ is the horizontal asymptote to the right.