Theme 3

3 The Derivative at a Point and the Derivative as a Function

Key to Text Coverage

Section	Examples	Exercises	Topics
2.1	1–7	1–94	*Tangent lines and the derivative*
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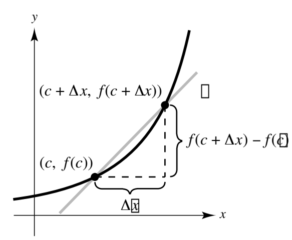
2.4	1–12	1–114	*The Chain Rule*
2.6	1–6	1–54	*Related Rates*

Summary

The definition of the slope m of the tangent line to a graph of a function f at the point $(c, f(c))$ is given on page 95 as follows.

$$m = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(c + \Delta x) - f(c)}{\Delta x}$$

This is the definition of the **derivative at a point.** It is common practice to use other letters, such as h, for Δx. The geometric significance of this definition as a limit of slopes of secant lines makes it easy to remember.



If a function has a derivative at the point $(c, f(c))$, then the graph of f will be "locally linear." If you zoom in at the point, then the graph will appear to be a straight line. For example, try graphing the two functions

$$y_1 = |x| + 1 \quad \text{and} \quad y_2 = \sqrt{x^2 + 0.0025} + 0.95$$

in the viewing window $[-6, 6] \times [-3, 5]$. By zooming in near the point $(0, 1)$ you can see that y_1 is never "locally linear," whereas y_2 is. This property of local linearity forms the basis of the concept of differentials (Section 3.9).

As developed on page 97, the derivative of a function is itself a function

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}.$$

For example, if $f(x) = x^{1/3}$, then the derivative of f, $f'(x) = 1/(3x^{2/3})$, is another function of the variable x. Notice that 0 is not in the domain of f'. Although f is continuous for all x, it is not differentiable at $x = 0$ because it has a vertical tangent at the point $(0, 0)$. In general, differentiability implies continuity, but the converse is false (Theorem 2.1, page 101).

In applications, the *instantaneous rate of change* of a function at a point is equivalent to the function's derivative at that point. To see this, observe that the *average rate of change* of a function f on an interval $[c, c + h]$ is the change in f divided by the change in x, or

$$\frac{f(c + \Delta x) - f(c)}{(c + \Delta x) - c}.$$

Taking the limit as h approaches zero, you have the instantaneous rate of change of f at c

$$\lim_{\Delta x \to 0} \frac{f(c + \Delta x) - f(c)}{\Delta x} = f'(c).$$

Worked Example

Consider the function

$f(x) = e^{3x}$.

(a) Find the equation of the tangent line $y = T(x)$ to the graph of f at the point $(0, 1)$.

(b) Graph f and the tangent line in the same viewing window. Describe the graphs as you zoom in near the point of tangency.

(c) Fill in the following table of values and explain how these values confirm your conclusion to part (b).

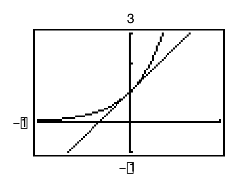
x	-1	-0.1	-0.01	0	0.01	0.1	1
$f(x)$							
$T(x)$							

(d) Use the equation of the tangent line to approximate $e^{0.0003}$.

SOLUTION

(a) $f'(x) = 3e^{3x}$ and $f'(0) = 3$ is the slope of the tangent line. The equation of the tangent line is

$$y - 1 = 3(x - 0)$$
$$y = 3x + 1$$
$$T(x) = 3x + 1$$



(b) Near the point of tangency $(0, 1)$, the graphs are nearly identical. $T(x)$ is the linear approximation to $f(x)$ at this point.

(c) You can use the *table* feature of a graphing utility to fill in the table.

x	-1	-0.1	-0.01	0	0.01	0.1	1
$f(x)$	0.0498	0.7408	0.9704	1	1.0305	1.3499	20.0855
$T(x)$	-2.0	0.7	0.97	1	1.03	1.3	4.0

Notice that the values of f and T are nearly identical near $x = 0$.

(d) $e^{0.0003} = e^{3(0.0001)} = f(0.0001) \approx T(0.0001) = 3(0.0001) + 1 = 1.0003$.

Notes

(b) This is an illustration of the concept of "local linearity." Near the point of tangency, the graph of f is nearly equal to that of the tangent line. If you were to zoom in at this point, you would observe that the graph of f appears to be a straight line.

(c) Notice how poor the approximation becomes as you move further away from the point of tangency.

(d) This linear approximation is very close to the value of $e^{0.0003} \approx 1.000300045$.

Name ____________________

Date ____________________

Sample Questions

Show all your work on a separate sheet of paper. Indicate clearly the methods you use because you will be graded on the correctness of your methods as well as on the accuracy of your answers.

Multiple Choice

1. Find the derivative of $h(x) = x|x|$ at the point $(0, 0)$.

   (a) $h'(0) = 0$ (b) $h'(0) = 1$ (c) $h'(0) = -1$

   (d) $h'(0) = 2x$ (e) $h'(0)$ does not exist.

2. Which of the following expressions is the definition of the derivative of $g(x) = \ln x$ at the point $(2, \ln 2)$?

   (a) $\lim\limits_{h\to 0} \dfrac{\ln(x+h) - \ln x}{h}$ (b) $\lim\limits_{h\to 0} \dfrac{\ln(2+h) + \ln h}{h}$ (c) $\lim\limits_{h\to 0} \dfrac{\ln(2+h) + \ln 2}{h}$

   (d) $\lim\limits_{h\to 0} \dfrac{\ln(2+h) - \ln 2}{h}$ (e) $\lim\limits_{h\to 0} \dfrac{\ln(2-h) - \ln 4}{h}$

3. Use the linear approximation to $f(x) = \tan x$ at the point $(\pi/4, 1)$ to approximate tan 1.

   (a) 1.5574 (b) 1.7720 (c) 1.3428 (d) 1.4292 (e) 1.3789

Free Response

A baseball is dropped from a height of 40 meters. Its height s at time t is given by the position function

$$s(t) = -4.9t^2 + 40$$

where t is measured in seconds and s in meters.

(a) Find the average rate of change of s over the time interval $[1, 2]$. What does this represent?

(b) Find the velocity of the baseball at time $t = 1$ as a limit of its average velocity.

(c) Use a derivative to find the velocity function.

(d) Find the velocity at time $t = 1.5$.

(e) What is the velocity of the baseball when it hits the ground?

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SOLUTIONS

Multiple Choice

1. Answer (a). One way to solve the problem is use the definition of derivative.

$$h'(0) = \lim_{\Delta x \to 0} \frac{h(0 + \Delta x) - h(0)}{\Delta x} = \lim_{\Delta x \to 0} \frac{\Delta x|\Delta x| - 0}{\Delta x} = \lim_{\Delta x \to 0} |\Delta x| = 0$$

Another way to solve the problem is to rewrite h as follows.

$$h(x) = \begin{cases} x^2, & x \geq 0 \\ -x^2, & x < 0 \end{cases}$$

From this formula, you can see that the derivatives from the left and the right at $x = 0$ are both equal to 0. You can verify this by graphing h and noticing that it is flat at the origin.

2. Answer (d). This is a direct use of the definition of derivative in which h has replaced Δx and $c = 2$.

3. Answer (d). First find the equation of the tangent line to the graph of f at the given point. Since $f'(x) = \sec^2 x$, $f'(\pi/4) = 2$.

$$y - 1 = 2\left(x - \frac{\pi}{4}\right)$$

$$y = 2x - \frac{\pi}{2} + 1$$

At $x = 1$, $y = 2(1) - \pi/2 + 1 = 3 - \pi/2 \approx 1.4292$.

Free Response

(a) The average rate of change is the average velocity of the baseball during the time between one and two seconds after it is dropped. It is given by the following difference quotient.

$$\frac{s(2) - s(1)}{2 - 1} = \frac{[(-4.9)4 + 40] - [(-4.9)1 + 40]}{1} = -3(4.9) = -14.7 \text{ m/sec}$$

(b) $$v(1) = \lim_{\Delta t \to 0} \frac{s(1 + \Delta t) - s(1)}{\Delta t} = \lim_{\Delta t \to 0} \frac{-4.9[1 + 2\Delta t + (\Delta t)^2] + 40 - [(-4.9)(1^2) + 40]}{\Delta t}$$

$$= \lim_{\Delta t \to 0} \frac{-4.9 - 9.8\Delta t - 4.9(\Delta t)^2 + 4.9}{\Delta t} = \lim_{\Delta t \to 0} -9.8 - 4.9\Delta t = -9.8 \text{ m/sec}$$

(c) The velocity function is the derivative of the position function, $v(t) = s'(t) = -9.8t$.

(d) The velocity at time $t = 1.5$ is $v(1.5) = (-9.8)(1.5) = -14.7$ m/sec.

(e) First find the time when the baseball hits the ground.

$s(t) = -4.9t^2 + 40 = 0$ which implies $t = \sqrt{40/4.9} = 20/7$

Substituting this value of t into the velocity equation gives

$v(20/7) = (-9.8)(20/7) = -28$ m/sec.