9 Functions and Their Inverses

Theme 9

Key to Text Coverage

Section	Examples	Exercises	Topics
5.3	1–6	1–107	*Inverse functions and their derivatives*
5.4	1–10	1–131	*Exponential function is inverse of logarithmic function*
5.8	1–7	1–82	*Inverse trigonometric functions*
5.10	5–6	55–62, 64–66	*Inverse hyperbolic functions*

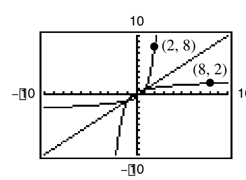
Formulas

Let $y = f(x)$ be a function. Then

- $g = f^{-1}$ is the inverse of f if $f(g(x)) = x$ and $g(f(x)) = x$.
- The domain of f is the range of f^{-1}, and the domain of f^{-1} is the range of f.
- The point (a, b) is on the graph of f if and only if (b, a) is on the graph of f^{-1}.
- f has an inverse if and only if f is one-to-one.
- If $g = f^{-1}$ is the inverse of f, then $g'(x) = 1/f'(g(x))$.

Summary

Let $y = g(x)$ be the inverse of the function $y = f(x)$. Then the point (a, b) is on the graph of f if and only if the corresponding point (b, a) is on the graph of g. Graphically, this means that the graphs of f and g are symmetric about the line $y = x$. Furthermore, if the slope of the graph of f at (a, b) is m, then the slope at the corresponding point (b, a) on the graph of g is $1/m$.



It is instructive to verify these facts with a simple example. Let $f(x) = x^3$ and $g(x) = x^{1/3}$. Graph f, g and the line $y = x$ in the same viewing window. The point $(2, 8)$ on f corresponds to $(8, 2)$ on g. Notice that the derivatives $f'(2) = 12$ and $g'(8) = 1/12$ are reciprocals of each other. What can you say about $f'(0)$ and $g'(0)$?

You can use implicit differentiation to find the formula for the derivative of an inverse function. For the function $y = x^3$, first interchange x and y, and then differentiate.

$$x = y^3$$

$$1 = 3y^2 \frac{dy}{dx}$$

$$\frac{dy}{dx} = \frac{1}{3y^2}$$

$$= \frac{1}{3x^{2/3}}$$

This is the derivative of $y = x^{1/3}$, the inverse of $y = x^3$.

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Worked Example

Let $f(x) = \sqrt{x^2 - 9}$ be defined on the domain $x < -3$.

(a) What is the range of the function f ?

(b) Calculate $f'(x)$ and explain why f has an inverse.

(c) Find a formula for the inverse g of f and indicate its domain and range.

(d) Sketch the graph of f and g on the same set of axes. Discuss the relationship between the graphs.

(e) Find the slope of the tangent line to f at $(-5, 4)$ and the slope of the tangent line to g at $(4, -5)$.

SOLUTION

(a) $f(x) > 0$ for all $x < -3$. The range is $(0, \infty)$.

(b) $f'(x) = \frac{1}{2}(x^2 - 9)^{-1/2}(2x) = x/\sqrt{x^2 - 9}$. Since $f'(x) < 0$ for all x in the domain of f, the graph of f is decreasing. So, f is one-to-one and has an inverse.

(c) You can solve for x in terms of y.

$$y = \sqrt{x^2 - 9}$$

$$y^2 = x^2 - 9$$

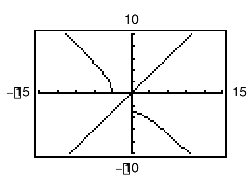
$$x^2 = y^2 + 9$$

$$x = -\sqrt{y^2 + 9} \qquad \text{(select the negative root)}$$

$$y = g(x) = -\sqrt{x^2 + 9} \qquad \text{(interchange variables)}$$

The domain of g is the range of f: $(0, \infty)$.
The range of g is the domain of f: $(-\infty, -3)$.

(d) The graphs of f and g are symmetric about the line $y = x$, as indicated in the graph.



(e) The slope of the tangent line to f at $(-5, 4)$ is

$$f'(-5) = \frac{-5}{\sqrt{5^2 - 9}} = -\frac{5}{4}.$$

The slope of the tangent line to g at the corresponding point $(4, -5)$ is therefore the reciprocal, $-\frac{4}{5}$.

Notes

(e) You can verify this answer by directly calculating the derivative of g.

$$g'(x) = -\frac{1}{2}(x^2 + 9)^{-1/2}(2x) = -\frac{x}{\sqrt{x^2 + 9}} \implies g'(4) = -\frac{4}{5}$$

Name ________________________________

Date ________________________________

Sample Questions

Show all your work on a separate sheet of paper. Indicate clearly the methods you use because you will be graded on the correctness of your methods as well as on the accuracy of your answers.

Multiple Choice

1. Let $y = mx + b$ be the equation of the tangent line to the graph of $y = f(x)$ at the point (x_0, y_0). What is the slope of the tangent line to the graph of the inverse of f at the point (y_0, x_0)?

   (a) m (b) $1/m$ (c) $-m$ (d) $-1/m$ (e) b

2. Let $f(x) = x^3 + 4x - 1$. If g is the inverse of f, find $g'(4)$.

   (a) 7 (b) $1/7$ (c) 52 (d) $1/52$ (e) 4

3. Which of the following functions defined on $(-\infty, \infty)$ have an inverse?

   I. $f(x) = 1 - x^5 - 3x^3 - x$

   II. $g(x) = \sin x$

   III. $h(x) = \arctan x$

   (a) I only (b) II only (c) III only (d) I and II only (e) I and III only

Free Response

Let f be defined by the following integral on the domain $x > 0$.

$$f(x) = \int_0^{\ln x} \frac{1}{\sqrt{4 + e^t}}\, dt$$

(a) Find $f(1)$.

(b) Find $f'(1)$.

(c) Explain why f has an inverse.

(d) Find $(f^{-1})'(0)$

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SOLUTIONS

Multiple Choice

1. Answer (b). If the slope of the tangent line to the graph of $y = f(x)$ at (x_0, y_0) is m, then the slope at the corresponding point on g is the reciprocal, $1/m$.

2. Answer (b). Notice first that $f(1) = 4$ which means that $g(4) = 1$. Now, $f'(x) = 3x^2 + 4$ and $f'(1) = 7$. So,

$$g'(4) = \frac{1}{f'(g(4))} = \frac{1}{f'(1)} = \frac{1}{7}.$$

3. Answer (e). $f'(x) = -5x^4 - 9x^2 - 1 < 0$, which implies that f is one-to-one and has an inverse. So, (I) is true. The periodic function $g(x) = \sin x$ does not have an inverse. Recall that the domain of the sine function has to be restricted in order to define the inverse sine function. So, (II) is false. Finally,

$$h'(x) = \frac{1}{1 + x^2} > 0$$

which means that the arctangent function has an inverse.

So, (III) is true. In conclusion, only (I) and (III) are true.

Free Response

(a) $f(1) = \displaystyle\int_0^{\ln 1} \frac{1}{\sqrt{4 + e^t}}\,dt = \int_0^{0} \frac{1}{\sqrt{4 + e^t}}\,dt = 0.$

(b) By the Second Fundamental Theorem of Calculus

$$f'(x) = \frac{1}{\sqrt{4 + e^{\ln x}}}\frac{1}{x} = \frac{1}{\sqrt{4 + x}}\frac{1}{x} \Longrightarrow f'(1) = \frac{1}{\sqrt{5}}$$

(c) The derivative of f is always positive, which implies that f is an increasing function. So, f is one-to-one and has an inverse.

(d) Because $f(1) = 0$, you have $(f^{-1})(0) = 1$. So, the derivative is given by the formula

$$(f^{-1})(0) = \frac{1}{f'(f^{-1}(0))} = \frac{1}{f'(1)} = \sqrt{5}.$$

It is instructive to verify these results with a graphing utility. You can graph the function f by graphing y=fnInt($1/\sqrt{4 + e^X}$, X, 0, ln X) on the viewing window $[0, 2] \times [-1, 1]$. Notice that $y(1) = 0$ confirms the answer to part (a). Furthermore, you can verify that the derivative at $x = 1$ is $1/\sqrt{5} \approx 0.44721$.