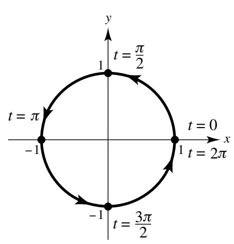
15 Parametric Equations and Vector Functions

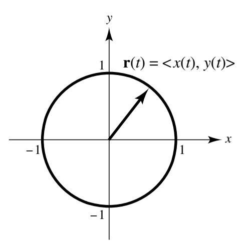
Key to Text Coverage

Section	Examples	Exercises	Topics
9.2	1–5	1–76	*Parametric equations and plane curves*
9.3	1–6	1–78	*Parametric equations and calculus*
10.1	1–8	1–104	*Vectors in the plane*
11.1	1–5	1–90	*Vector-valued functions*
11.2	1–7	1–74	*Calculus of vector-valued functions*
11.3	1–6	1–56	*Velocity and acceleration*

Summary

Parametric equations and vector-valued functions are very similar. For instance, consider the parametric equations for the unit circle, $x(t) = \cos t$, $y(t) = \sin t$, $0 \le t \le 2\pi$. To each value of t there corresponds a point on the unit circle $x^2 + y^2 = 1$. The circle is traced out counterclockwise, starting and finishing at $(x, y) = (1, 0)$, as shown in the figure below on the left. In the language of vectors, $\mathbf{r}(t) = \langle x(t), y(t)\rangle = \langle \cos t, \sin t\rangle$ is a vector-valued function. To each value of t there corresponds a vector having its tail at the origin and its head on the unit circle, as shown in the figure below on the right.





Given the parametric equations $x(t)$, $y(t)$ on the interval $a \le t \le b$, or equivalently, the vector-valued function $\mathbf{r}(t) = \langle x(t), y(t)\rangle$, the length s of the curve traced out by these equations is given on page 678.

$$s = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$$

If you interpret the equation $\mathbf{r}(t) = \langle x(t), y(t)\rangle$ as the position vector of a particle, then its velocity is $\mathbf{v}(t) = \mathbf{r}'(t) = \langle x'(t), y'(t)\rangle$. The speed is the magnitude of the velocity

$$\text{Speed} = \|\mathbf{v}(t)\| = \sqrt{[x'(t)]^2 + [y'(t)]^2}.$$

The total distance traveled by such a particle on the interval $a \le t \le b$ is given by the integral

$$\text{Total Distance} = \int_a^b \sqrt{[x'(t)]^2 + [y'(t)]^2}\, dt = \int_a^b \|\mathbf{v}(t)\|\, dt.$$

Worked Example

The position of a particle at time $t \geq 0$ is given by the parametric equations $x(t) = (t - 2)^3/3 + 4$, $y(t) = t^2 - 4t + 4$.

(a) Find the magnitude of the velocity vector at $t = 1$.

(b) Use a graphing utility to approximate the total distance traveled by the particle from $t = 0$ to $t = 1$.

(c) When is the particle at rest and what is its position at that time?

SOLUTION

(a) Begin by calculating the derivatives with respect to t.

$$\frac{dx}{dt} = (t - 2)^2$$

$$\frac{dy}{dt} = 2t - 4 = 2(t - 2)$$

The magnitude of the velocity vector is

$$\|\mathbf{v}(t)\| = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = \sqrt{(t - 2)^4 + 4(t - 2)^2} = |t - 2|\sqrt{t^2 - 4t + 8}.$$

So, the magnitude of the velocity at $t = 1$ is $\|\mathbf{v}(1)\| = \sqrt{5}$.

(b) The total distance traveled is given by the integral

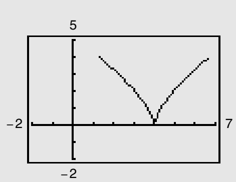
$$\int_0^1 |t - 2|\sqrt{t^2 - 4t + 8}\, dt \approx 3.8157.$$

(c) The particle is at rest when $\mathbf{v} = \mathbf{0}$, which occurs when $t = 2$. The particle is located at $(x(2), y(2)) = (4, 0)$.

Notes

(b) You can verify this result by using the arc length feature of a graphing utility.

(c) Try graphing the parametric equations on your graphing utility in the viewing window indicated. Let tStep $= 0.01$ and watch closely as the curve is traced out near $(x, y) = (4, 0)$. Do you see how the movement seems to stop there?



Name ____________________

Date ____________________

Theme 15

Sample Questions

Show all your work on a separate sheet of paper. Indicate clearly the methods you use because you will be graded on the correctness of your methods as well as on the accuracy of your answers.

Multiple Choice

1. Find the equation of the tangent line at $t = 1$ to the curve given by the parametric equations $x(t) = 3t^2 - 4t + 2$, $y(t) = t^3 - 4t$.

   (a) $x + 2y = 5$ (b) $x + 2y = -5$ (c) $2x - y = 5$

   (d) $2x - y = -5$ (e) $2x + y = 5$

2. Let $x(t) = e^t + 1$ and $y(t) = 2e^{2t}$ be the equations of the path of a particle moving in the xy-plane. Find an equation for the path of the particle in terms of only x and y.

   (a) $y = 2x^2 - 4x + 2$ (b) $y = x^2 - 2x + 1$ (c) $y = 2x - 2$

   (d) $y = x^2 + 2x + 1$ (e) $y = 2x^2$

3. A particle is moving in the plane according to the position equations $x(t) = e^{-t}\cos t$, $y(t) = e^{-t}\sin t$, $0 \le t \le \pi/2$. Find the total distance traveled by the particle.

   (a) $\sqrt{2}(\pi/2 - 1)$ (b) $\sqrt{2}(e^{-\pi/2})$ (c) $\sqrt{2}(1 + e^{-\pi/2})$

   (d) $\sqrt{2}(1 - e^{-\pi/2})$ (e) $\sqrt{2}(1 - e^{\pi/2})$

Free Response

Two particles are moving in the xy-plane. The position of particle A at time t is given by the parametric equations $x(t) = \sin t$, $y(t) = \cos 2t$, $-\pi/2 \le t \le \pi/2$. The position of particle B is given by $x(t) = t$, $y(t) = 1 - t$, $-\pi/2 \le t \le \pi/2$.

(a) Find the velocity vector for each particle at time $t = \pi/4$.

(b) Sketch the path of each particle, indicating their directions of motion.

(c) Do the particles collide? If so, find the time at collision.

Theme 15

SOLUTIONS

Multiple Choice

1. Answer (b).

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{3t^2 - 4}{6t - 4}$$

At $t = 1$, $x(1) = 1$, $y(1) = -3$, and $dy/dx = -1/2$. Using the point-slope formula for a line

$$y + 3 = -\frac{1}{2}(x - 1) \implies 2y + x = -5.$$

2. Answer (a). Eliminate the parameter t as follows.

$$\begin{aligned} x - 1 &= e^t \\ x^2 - 2x + 1 &= e^{2t} \\ 2x^2 - 4x + 2 &= 2e^{2t} = y \end{aligned}$$

So, $y = 2x^2 - 4x + 2$.

3. Answer (d). $dx/dt = -e^{-t}\cos t - e^{-t}\sin t$ and $dy/dt = -e^{-t}\sin t + e^{-t}\cos t$. So

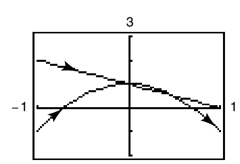
$$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = e^{-2t}\cos^2 t + e^{-2t}\sin^2 t + 2e^{-2t}\cos t \sin t + e^{-2t}\sin^2 t$$
$$+ e^{-2t}\cos^2 t - 2e^{-2t}\cos t \sin t = 2e^{-2t}.$$

$$s = \int_0^{\pi/2} \sqrt{2e^{-2t}}\, dt = \int_0^{\pi/2} \sqrt{2}e^{-t}\, dt = -\sqrt{2}e^{-t}\Big]_0^{\pi/2} = -\sqrt{2}(e^{-\pi/2} - 1)$$
$$= \sqrt{2}(1 - e^{-\pi/2}) \approx 1.1202$$

Free Response

(a) For particle A, $x'(t) = \cos t$ and $y'(t) = -2\sin 2t$. At time $t = \pi/4$ the velocity vector is $\langle x'(\pi/4), y'(\pi/4)\rangle = \left\langle \sqrt{2}/2, -2\right\rangle$. For particle B, $x'(t) = 1$, $y'(t) = -1$, and $\langle x'(\pi/4), y'(\pi/4)\rangle = \langle 1, -1\rangle$.

(b)



(c) Yes, the particles collide at $t = 0$. You can see this by tracing along the two curves in part (b). Or, you can observe that the x-components of the two sets of parametric equations are equal when $t = 0$. Furthermore, the y-components are the same at this t-value. So, the particles collide at the point $(x, y) = (0, 1)$.