

8 Rectilinear Motion

Key to Text Coverage

Section	Examples	Exercises	Topics
2.2	10	91–98	<i>Defines position, velocity and speed</i>
2.3	10	101–103	<i>Adds the concept of acceleration</i>
2 Review		34–36, 39, 40, 58	<i>Reinforces ideas of Chapter 2</i>
3.2		25, 43	<i>Rolle's Theorem/Mean Value Theorem</i>
4.1	8	67–88	<i>Incorporates integration</i>
4.4		103–105	<i>Fundamental Theorem of Calculus</i>
4 Review		11–14	<i>Reinforces ideas of Chapter 4</i>

Formulas

The formulas for the position, velocity, acceleration and speed of a moving object are given by the following derivatives.

$x(t)$	Position
$v(t) = x'(t)$	Velocity
$a(t) = v'(t) = x''(t)$	Acceleration
$ v(t) $	Speed

Summary

Although some of the examples and exercises considered motion in a vertical direction, it will be convenient here to assume that the particle (or object) moves horizontally along the x -axis, with the positive direction to the right.

Given the position of the particle, you can use differentiation to find its velocity and acceleration. Conversely, given the velocity (or acceleration), use integration to find the position function. If $v(t) > 0$, then the particle is moving to the right in the positive direction along the x -axis. The velocity is zero at points where the particle changes direction.

The total distance a particle travels between time $t = a$ and time $t = b$ is the integral of the speed

$$\text{total distance} = \int_a^b |v(t)| \, dt.$$

If $v(t) \geq 0$, then

$$\int_a^b |v(t)| \, dt = \int_a^b v(t) \, dt = \int_a^b x'(t) \, dt = x(b) - x(a).$$

If $v(t) < 0$ on some t -intervals, then you must split up the integral, as indicated in Question 2.

Worked Example

The position function of a particle moving along the x -axis is given by $x(t) = t^3 - 12t^2 + 36t - 20$, $0 \leq t \leq 8$.

- Find the velocity and acceleration of the particle.
- Find the open t -intervals when the particle is moving to the left.
- Find the velocity of the particle when the acceleration is 0.
- Describe the motion of the particle.

SOLUTION

- $x(t) = t^3 - 12t^2 + 36t - 20$ Position
 $v(t) = 3t^2 - 24t + 36$ Velocity
 $a(t) = 6t - 24$ Acceleration
- $v(t) = 3t^2 - 24t + 36 = 3(t - 2)(t - 6) < 0$ when $2 < t < 6$.
- $a(t) = 6t - 24 = 0$ when $t = 4$. So, $v(4) = 3(4)^2 - 24(4) + 36 = -12$.
- You can analyze the motion of the particle by building a table of values for $x(t)$, $v(t)$, and $a(t)$ at $t = 0, 1, \dots, 8$.

t	0	1	2	3	4	5	6	7	8
$x(t)$	-20	5	12	7	-4	-15	-20	-13	12
$v(t)$	36	15	0	-9	-12	-9	0	15	36
$a(t)$	-24	-18	-12	-6	0	6	12	18	24

From this table you can see that the particle starts at $x = -20$ and moves to the right to $x = 12$ when $t = 2$. The velocity is zero when $t = 2$ and the particle reverses direction and returns to $x = -20$ when $t = 6$. Again, the velocity is zero here, and the particle reverses direction once more and moves off to the right.

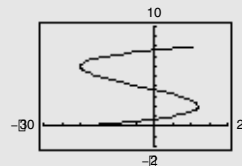
Notes

- You can obtain the velocity and acceleration functions by differentiating the given position function, $v(t) = x'(t)$ and $a(t) = x''(t) = v'(t)$.
- The particle is moving to the left when $v(t) < 0$, so you have to solve a quadratic inequality. Note that the particle is moving to the right when $0 < t < 2$ and $6 < t < 8$.
- The velocity is negative because the particle is moving to the left.
- The *table* feature on the TI-83 is especially useful for constructing a table of values. You can obtain a schematic graph of the position of the particle by plotting (in parametric mode) the curve

$$x_1(t) = t^3 - 12t^2 + 36t - 20$$

$$y_1(t) = t$$

for $0 \leq t \leq 8$, using the viewing window $[-30, 20] \times [-2, 10]$.



Name _____

Date _____

Sample Questions

Show all your work on a separate sheet of paper. Indicate clearly the methods you use because you will be graded on the correctness of your methods as well as on the accuracy of your answers.

Multiple Choice

1. A particle is moving along the x -axis so that its position at time $t \geq 0$ is given by $x(t) = t \ln 2t$. Find the acceleration of the particle when the velocity is 0.

(a) $2e^2$ (b) $2e$ (c) e (d) $1/e$ (e) $1/e^2$

2. A particle is moving along the x -axis with velocity $v(t) = \sin 2t$, $t \geq 0$. At time $t = \pi/2$ it has position $x = 3$. Find the total distance traveled by this particle on the interval $0 \leq t \leq \pi$.

(a) 0 (b) 1 (c) 2 (d) 3 (e) π

3. The acceleration of a particle moving along the x -axis is given by $a(t) = te^{2t}$, for $t \geq 0$. If $v(0) = -1/4$, find the speed of the particle when $t = 1/4$.

(a) $\sqrt{e}/2$ (b) $-\sqrt{e}/4$ (c) $\sqrt{e}/4$ (d) $-\sqrt{e}/8$ (e) $\sqrt{e}/8$

Free Response

A particle is moving along the x -axis so that at time t its acceleration is $a(t) = \pi \cos \pi t$. At time $t = 1/2$, the velocity of the particle is $v(1/2) = 1/2$.

- (a) Find the velocity of the particle at any time t .
(b) Find the minimum velocity of the particle.
(c) Find the equation for the position $x(t)$ if $x(0) = 0$.
(d) What is the first time $t > 0$ that the particle returns to the origin?

SOLUTIONS

Multiple Choice

1. Answer (b). Set the derivative of the position function equal to zero: $v(t) = x'(t) = 1 + \ln 2t = 0$. This gives $\ln 2t = -1$ or $t = 1/(2e)$. The acceleration is $a(t) = v'(t) = 1/t$ and so $a(1/(2e)) = 2e$.
2. Answer (c). The total distance traveled by the particle is the integral of the speed. Because the graph of $v(t)$ is nonnegative on $[0, \pi/2]$, and negative on $(\pi/2, \pi)$ you must split up the integral as follows.

$$\text{total distance} = \int_0^{\pi} |\sin 2t| dt = \int_0^{\pi/2} \sin 2t dt + \int_{\pi/2}^{\pi} (-\sin 2t) dt = 1 + 1 = 2.$$

You can verify this answer by integrating $y = |\sin 2t|$ from $t = 0$ to $t = \pi$ with a graphing utility. Notice that the particle has returned to its original position.

3. Answer (e). You can obtain the velocity by integrating the acceleration function using integration by parts.

$$v(t) = \int a(t) dt = \int te^{2t} dt = \frac{1}{2}te^{2t} - \frac{1}{4}e^{2t} + C$$

Since $v(0) = -1/4$, $C = 0$. The speed at $t = 1/4$ is the absolute value of the velocity.

$$\text{speed} = \left| v\left(\frac{1}{4}\right) \right| = \left| \frac{1}{8}e^{1/2} - \frac{1}{4}e^{1/2} \right| = \left| -\frac{\sqrt{e}}{8} \right| = \frac{\sqrt{e}}{8}$$

Free Response

- (a) $v(t) = \int a(t) dt = \int \pi \cos \pi t dt = \sin \pi t + C$. Since $v(1/2) = 1/2$, $1/2 = \sin(\pi/2) + C$, which gives $C = -1/2$. So, $v(t) = \sin \pi t - 1/2$.
- (b) Since $\sin \pi t \geq -1$, the minimum velocity is $-3/2$.
- (c) Integrating the velocity function, you have

$$x(t) = \int v(t) dt = \frac{-\cos \pi t}{\pi} - \frac{t}{2} + C_1.$$

Because $x(0) = 0$, $C_1 = 1/\pi$ and

$$x(t) = \frac{-\cos \pi t}{\pi} - \frac{t}{2} + \frac{1}{\pi}.$$

- (d) The particle returns to the origin when

$$\frac{-\cos \pi t}{\pi} - \frac{t}{2} + \frac{1}{\pi} = 0.$$

By graphing the function

$$y = \frac{-\cos \pi t}{\pi} - \frac{t}{2} + \frac{1}{\pi}$$

on the interval $[0, 2]$, you see that the first positive zero of y is approximately $t = 0.353$.