13 Series Investigated Graphically via Polynomial Approximation

Key to Text Coverage

Section	Topics
8.2	*Infinite series defined and developed*
8.3–8.6	*Convergence tests for series*
8.7	*Polynomial approximation: Taylor polynomials*
8.8–8.10	*Power Series*

Summary

Infinite series are completely developed in Chapter 8. Recall that a series is defined on page 567 as a limit of a sequence of partial sums; if that limit exists as a finite number, then the series is said to converge. For example, the divergence of the harmonic series

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots$$

is based on the fact that the sequence of partial sums

$$S_1 = 1$$

$$S_2 = 1 + \frac{1}{2} = 1.5$$

$$S_3 = 1 + \frac{1}{2} + \frac{1}{3} = 1.8333\ldots$$

$$\vdots$$

diverges $\left(\lim_{n\to\infty} S_n = \infty\right)$. This fundamental result is often surprising for students. Even though the individual terms of the harmonic series are approaching zero, their sum "adds up to infinity."

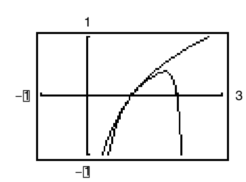
Most of the important calculus functions have power series representations, as summarized on page 638. For example

$$\ln x = (x-1) - (x-1)^2/2 + (x-1)^3/3 - \cdots \qquad 0 < x \le 2$$

You can visualize this interval of convergence by graphing the logarithm function together with the sum of the first few terms

Y1 = sum seq((-1)^(N+1) (X−1)^N/N,N,1,10,1)

as shown to the right.



Because this series converges at $x = 2$, you have $\ln 2 = 1 - 1/2 + 1/3 - 1/4 + \cdots$. If you were to use the first 10 terms to approximate ln 2, then the error formula on page 592 tells you that the approximation (0.645635) is within 1/11 of the exact value.

You can manipulate these basic power series to construct a more complicated one. For instance, differentiating the series for ln x term-by-term gives

$$1/x = 1 - (x-1) + (x-1)^2 - (x-1)^3 + \cdots \qquad 0 < x < 2$$

Notice that the radius of convergence is the same, but the interval of convergence has "lost an endpoint."

Theme 13

Worked Example

Consider the function

$$f(x) = \frac{1}{1 - 3x}.$$

(a) Find the first four terms of the Taylor series for f about $x = 0$.

(b) Find the interval of convergence for the series in part (a).

(c) Find the first four terms of the series for $g(x) = x/(1 - 3x^2)$ about $x = 0$.

(d) Find the Taylor series for f about the point $x = -1$.

SOLUTION

(a) One way to solve this problem is to calculate the derivatives of f and compute the Taylor series from the definition (Example 1, page 623). Another way is to use long division to divide $1 - 3x$ into 1 (page 626). However, the easiest way to recall that the Maclauren series for $1/(1 - x)$ is

$$\sum_{n=0}^{\infty} x^n.$$

Then, replacing x with $3x$ you obtain

$$\sum_{n=0}^{\infty} (3x)^n.$$

The first four terms are $(3x)^0 + (3x)^1 + (3x)^2 + (3x)^3 = 1 + 3x + 9x^2 + 27x^3$.

(b) You can use the ratio test to determine the radius of convergence. Notice however, that the series is a geometric series

$$\sum_{n=0}^{\infty} ar^n = \frac{a}{1 - r}, \quad 0 \le |r| < 1$$

with $a = 1$ and $r = 3x$. Then the interval of convergence immediately follows from the properties of geometric series: $0 \le |3x| < 1 \Longrightarrow -1 < 3x < 1 \Longrightarrow -1/3 < x < 1/3$ or $(-1/3, 1/3)$. You can verify the interval of convergence by graphing the first few terms of the series on your graphing utility.

(c) You can obtain the series for g by replacing x with x^2 and multiplying the resulting series by x.

$$\frac{1}{1 - 3x} = \sum_{n=0}^{\infty} (3x)^n$$

$$\frac{x}{1 - 3x^2} = \sum_{n=0}^{\infty} x(3x^2)^n$$

The first four terms are $x(3x^2)^0 + x(3x^2)^1 + x(3x^2)^2 + x(3x^2)^3 = x + 3x^3 + 9x^5 + 27x^7$.

(d) The key idea is to transform the series into a geometric series in $x + 1$.

$$\frac{1}{1 - 3x} = \frac{1}{4 - 3(x + 1)} = \frac{1/4}{1 - (3/4)(x + 1)} = \sum_{n=0}^{\infty} ar^n = \sum_{n=0}^{\infty} \frac{1}{4}\left(\frac{3}{4}(x + 1)\right)^n$$

$$= \frac{1}{4} + \frac{3}{16}(x + 1) + \frac{9}{64}(x + 1)^2 + \cdots$$

Name ________________________________

Date ________________________________

Sample Questions

Show all your work on a separate sheet of paper. Indicate clearly the methods you use because you will be graded on the correctness of your methods as well as on the accuracy of your answers.

Multiple Choice

1. Find the radius of convergence of the series

$$\sum_{n=2}^{\infty} \frac{x^n n!}{2^n n^n}.$$

(a) $1/e$ (b) $2/e$ (c) e (d) $2e$ (e) ∞

2. What function is equal to the power series

$$\sum_{n=1}^{\infty} \frac{(2x)^n}{n!}?$$

(a) e^{2x} (b) $e^{2x} - 1$ (c) $e^{2x} + 1$ (d) $e^x - 1$ (e) $e^x + 1$

3. Use the Taylor polynomial of degree 7 for $f(x) = \arctan x$ about $x = 0$ to approximate $\arctan(-1)$.

(a) -0.86667 (b) -0.78540 (c) -0.74213 (d) -0.72571 (e) -0.72381

Free Response

Consider the function $f(x) = \sin x$.

(a) What is the Taylor series for f about $x = 0$?

(b) Graph the fifth degree Taylor polynomial for f together with the function $f(x) = \sin x$. What do you observe?

(c) What is the Taylor series for $\sin x^2$?

(d) Use the first 3 terms of the series in part (c) to approximate the integral

$$\int_0^1 \sin x^2 \, dx.$$

(e) Use the series from part (a) to find the Taylor series for $f(x) = \cos x$ about $x = 0$.

SOLUTIONS

Multiple Choice

1. Answer (d). Use the ratio test.

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{x^{n+1}(n+1)!}{2^{n+1}(n+1)^{n+1}} \Big/ \frac{x^n n!}{2^n n^n}\right| = \lim_{n\to\infty}\left|\frac{x}{2}\,\frac{(n+1)n!n^n}{(n+1)(n+1)^n n!}\right| = \lim_{n\to\infty}\left|\frac{x}{2}\left(\frac{n}{n+1}\right)^n\right|$$

$$= \lim_{n\to\infty}\left|\frac{x}{2}\,\frac{1}{(1+1/n)^n}\right| = \left|\frac{x}{2e}\right| < 1$$

So, the radius of convergence is $2e$.

2. Answer (b). Notice that the series begins at $n = 1$.

$$e^x = 1 + x + x^2/2! + \cdots = \sum_{n=0}^{\infty}\frac{x^n}{n!}$$

$$e^{2x} = 1 + (2x) + (2x)^2/2! + \cdots = \sum_{n=0}^{\infty}\frac{(2x)^n}{n!} = 1 + \sum_{n=1}^{\infty}\frac{(2x)^n}{n!}$$

So, $\displaystyle\sum_{n=1}^{\infty}\frac{(2x)^n}{n!} = e^{2x} - 1.$

3. Answer (e)

$$\arctan x = x - x^3/3 + x^5/5 - x^7/7 + \cdots$$

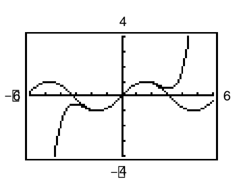
$$\arctan(-1) \approx -1 - (-1)^3/3 + (-1)^5/5 - (-1)^7/7 = -0.72381.$$

Free Response

(a) This Taylor series should be memorized.

$$\sin x = \sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{(2n+1)!}, \qquad -\infty < x < \infty$$

(b) The graphs are nearly identical for x-values near 0.



(c) $\displaystyle\sin x^2 = \sum_{n=0}^{\infty}(-1)^n\frac{(x^2)^{2n+1}}{(2n+1)!} = \sum_{n=0}^{\infty}(-1)^n\frac{x^{4n+2}}{(2n+1)!}$

(d) $\displaystyle\int_0^1 \sin x^2\,dx = \int_0^1\left(x^2 - \frac{x^6}{3!} + \frac{x^{10}}{5!}\right)dx = \left[\frac{x^3}{3} - \frac{x^7}{42} + \frac{x^{11}}{1320}\right]_0^1 = 1/3 - 1/42 + 1/1320 \approx 0.3102814$

This is very close to the calculator answer 0.3102683. Note that $\sin x^2$ does not have an elementary antiderivative.

(e) $\displaystyle\cos x = \frac{d}{dx}(\sin x) = \frac{d}{dx}\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{(2n+1)!} = \sum_{n=0}^{\infty}(-1)^n\frac{(2n+1)x^{2n}}{(2n+1)(2n)!} = \sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{(2n)!}$