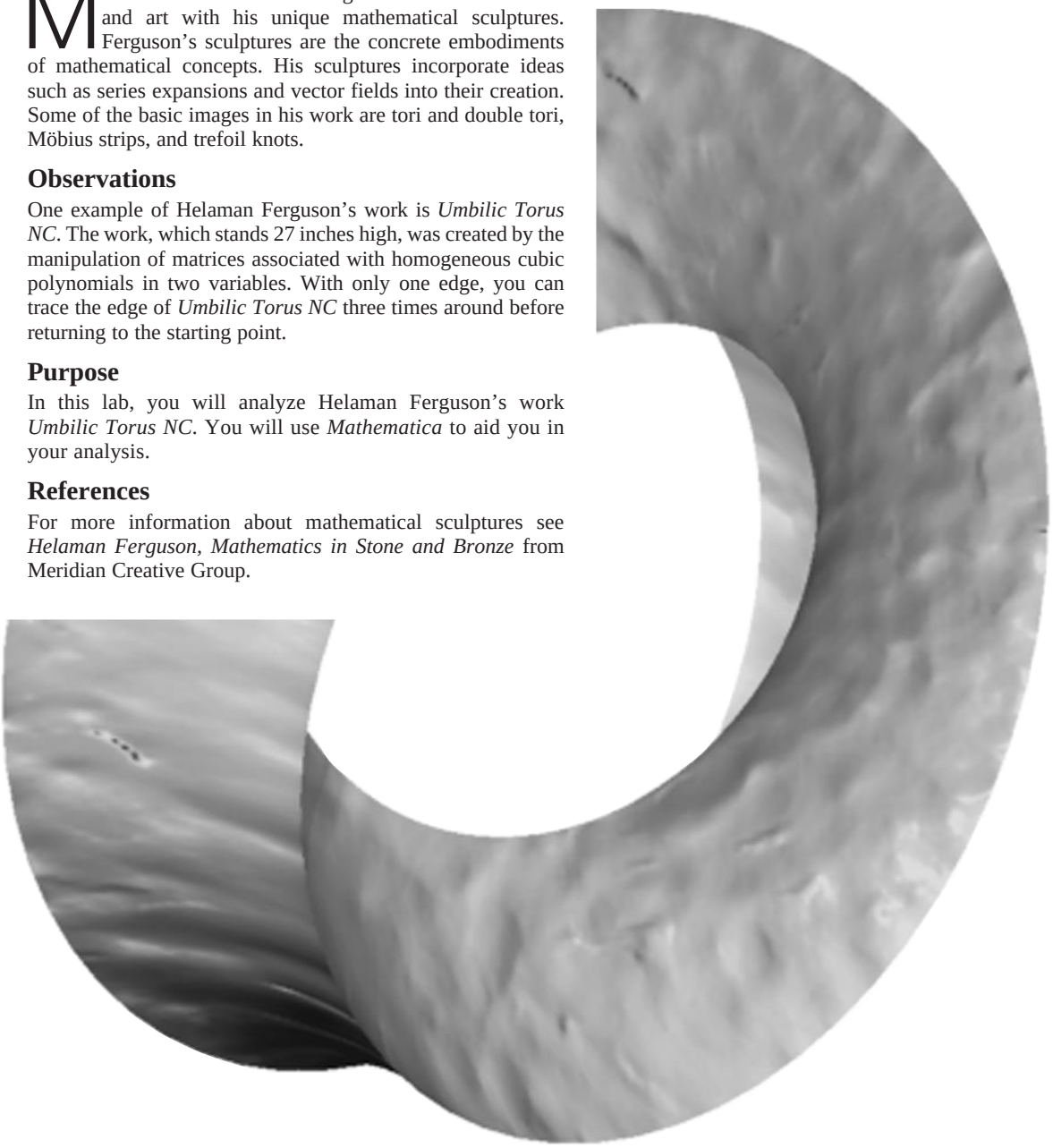
LAB 19

MATHEMATICAL SCULPTURES

Parametric Surfaces

Mathematician Helaman Ferguson combines science and art with his unique mathematical sculptures. Ferguson's sculptures are the concrete embodiments of mathematical concepts. His sculptures incorporate ideas such as series expansions and vector fields into their creation. Some of the basic images in his work are tori and double tori, Möbius strips, and trefoil knots.



Observations

One example of Helaman Ferguson's work is *Umbilic Torus NC*. The work, which stands 27 inches high, was created by the manipulation of matrices associated with homogeneous cubic polynomials in two variables. With only one edge, you can trace the edge of *Umbilic Torus NC* three times around before returning to the starting point.

Purpose

In this lab, you will analyze Helaman Ferguson's work *Umbilic Torus NC*. You will use *Mathematica* to aid you in your analysis.

References

For more information about mathematical sculptures see *Helaman Ferguson, Mathematics in Stone and Bronze* from Meridian Creative Group.

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Data

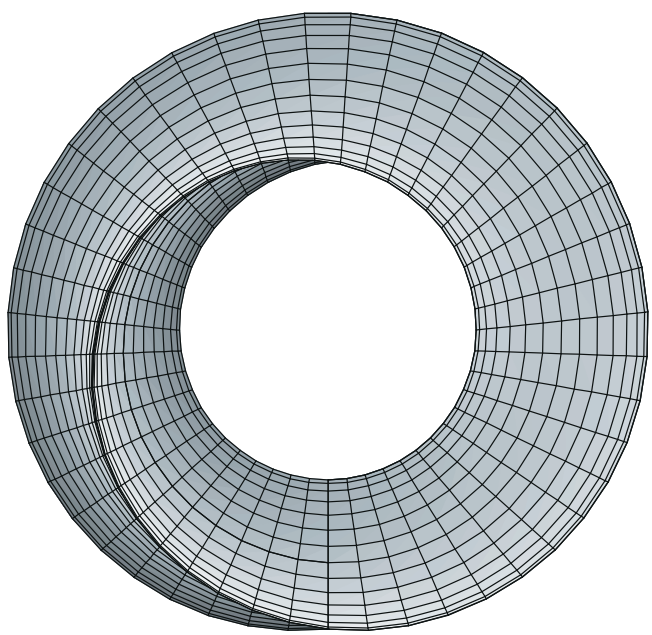
Helaman Ferguson's work *Umbilic Torus NC* is shown below. This form can be written as a parametric surface using the following set of parametric equations.

$$x = (\sin u)\left[7 + \cos\left(\frac{u}{3} - 2v\right) + 2\cos\left(\frac{u}{3} + v\right)\right]$$

$$y = (\cos u)\left[7 + \cos\left(\frac{u}{3} - 2v\right) + 2\cos\left(\frac{u}{3} + v\right)\right]$$

$$z = \sin\left(\frac{u}{3} - 2v\right) + 2\sin\left(\frac{u}{3} + v\right)$$

$$-\pi \le u \le \pi, -\pi \le v \le \pi$$



A graph of the parametric equations is stored in the *Mathematica* file called LAB19.NB.

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Exercises

Name _____

Date _____ Class _____

Instructor _____

1. ***Interchanging Equations.*** The view of the torus in this lab's Data is along the z-axis. Relative to the graph of the torus given in this lab's Data, describe the graph of the torus given by the parametric equations

$$x = (\cos u)\left[7 + \cos\left(\frac{u}{3} - 2v\right) + 2\cos\left(\frac{u}{3} + v\right)\right]$$

$$y = (\sin u)\left[7 + \cos\left(\frac{u}{3} - 2v\right) + 2\cos\left(\frac{u}{3} + v\right)\right]$$

$$z = \sin\left(\frac{u}{3} - 2v\right) + 2\sin\left(\frac{u}{3} + v\right)$$

with $-\pi \le u \le \pi$ and $-\pi \le v \le \pi$. Use *Mathematica* to confirm your description.

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2. ***Interchanging Equations, Part II.*** Repeat Exercise 1 for the following set of equations with $-\pi \le u \le \pi$ and $-\pi \le v \le \pi$.

$$x = \sin\left(\frac{u}{3} - 2v\right) + 2\sin\left(\frac{u}{3} + v\right)$$

$$y = (\cos u)\left[7 + \cos\left(\frac{u}{3} - 2v\right) + 2\cos\left(\frac{u}{3} + v\right)\right]$$

$$z = (\sin u)\left[7 + \cos\left(\frac{u}{3} - 2v\right) + 2\cos\left(\frac{u}{3} + v\right)\right]$$

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3. ***Interchanging Equations, Part III.*** Repeat Exercise 1 for the following set of equations with $-\pi \le u \le \pi$ and $-\pi \le v \le \pi$.

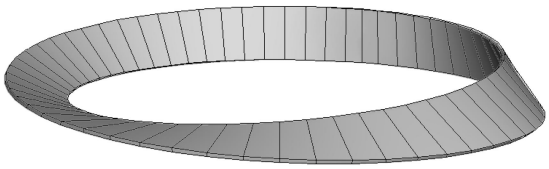
$$x = (\sin u)\left[7 + \cos\left(\frac{u}{3} - 2v\right) + 2\cos\left(\frac{u}{3} + v\right)\right]$$

$$y = \sin\left(\frac{u}{3} - 2v\right) + 2\sin\left(\frac{u}{3} + v\right)$$

$$z = (\cos u)\left[7 + \cos\left(\frac{u}{3} - 2v\right) + 2\cos\left(\frac{u}{3} + v\right)\right]$$

4. ***An "Elliptical" Torus.*** When viewed along the z-axis, the torus appears nearly circular. Is it possible to alter this appearance so that the shape of the torus is more like that of an elongated ellipse? If so, explain how this could be done and use *Mathematica* to graph an "elliptical" torus.

5. ***Comparing Graphs.*** Explain how the umbilic torus shown in this lab's Data is similar to the Möbius strip shown below.



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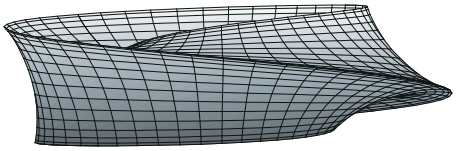
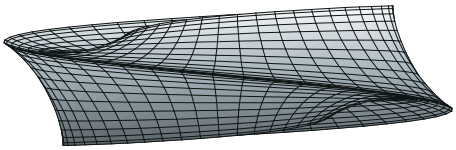
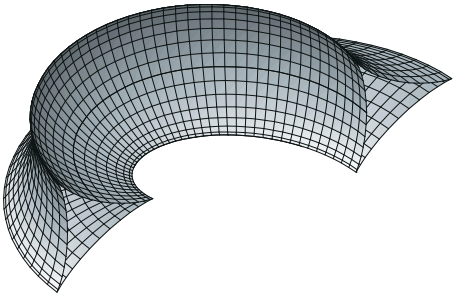
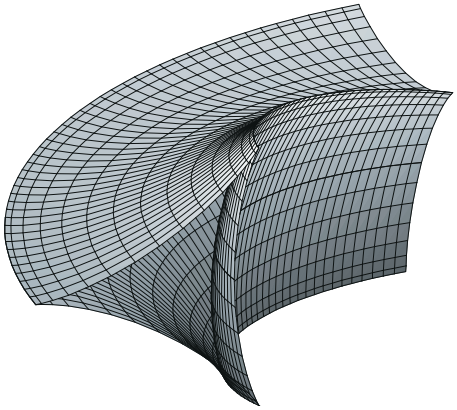
6. ***Graphing Portions of the Torus.*** Use *Mathematica* to recreate each of the following graphs. Explain your strategy to recreate each graph. Also, write the parametric equations and u- and v-ranges you used to recreate each graph.









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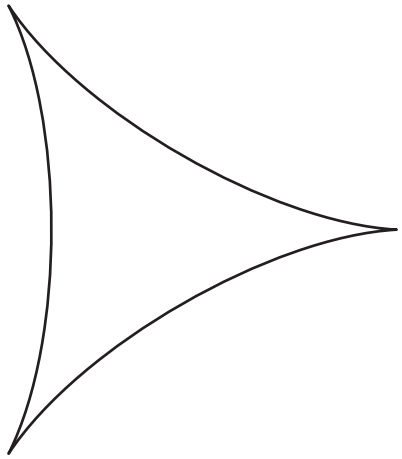
7. *Cross Section.* An example of a cross-section of *Umbilic Torus NC* is shown below.



The cross-section is a hypocycloid with three arches. A hypocycloid is a curve generated by a point on the circumference of a circle rolling within the circumference of a larger circle. The parametric equations of a hypocycloid in the *xy*-plane are

$$x = (a - b)\cos\theta + b\cos\left[\frac{(a - b)\theta}{b}\right]$$

$$y = (a - b)\sin\theta - b\sin\left[\frac{(a - b)\theta}{b}\right]$$

where $0 \le \theta \le 2\pi$. One possible set of parametric equations of the hypocycloid shown above is as follows.

$$x = 2\cos\theta + \cos 2\theta$$

$$y = 2\cos\theta + \cos 2\theta$$

Use *Mathematica* to graph a hypocycloid with four arches and a hypocycloid with five arches. What values of *a* and *b* did you use to create your graphs? What is the relationship between *a* and *b* for a hypocycloid with three arches? What is the relationship between *a* and *b* for a hypocycloid with four arches? What is the relationship between *a* and *b* for a hypocycloid with *n* arches?

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