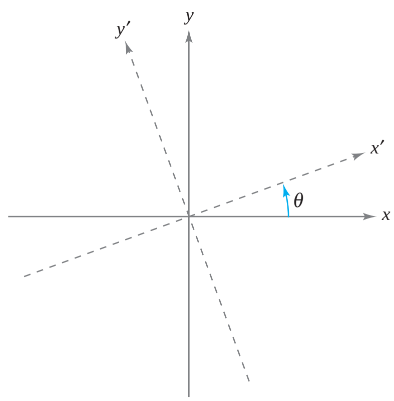
APPENDIX E

Rotation and the General Second-Degree Equation

Rotation of Axes • Invariants Under Rotation



After rotation of the x- and y-axes counterclockwise through an angle θ, the rotated axes are denoted as the x'-axis and y'-axis.
Figure E.1

Rotation of Axes

In Section 9.1, you learned that equations of conics with axes parallel to one of the coordinate axes can be written in the general form

$$Ax^2 + Cy^2 + Dx + Ey + F = 0.$$ Horizontal or vertical axes

Here you will study the equations of conics whose axes are rotated so that they are *not* parallel to the x-axis or the y-axis. The general equation for such conics contains an xy-*term*.

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$$ Equation in xy-plane

To eliminate this xy-term, you can use a procedure called **rotation of axes.** You want to rotate the x- and y-axes until they are parallel to the axes of the conic. (The rotated axes are denoted as the x'-axis and the y'-axis, as shown in Figure E.1.) After the rotation has been accomplished, the equation of the conic in the new $x'y'$-plane will have the form

$$A'(x')^2 + C'(y')^2 + D'x' + E'y' + F' = 0.$$ Equation in $x'y'$-plane

Because this equation has no $x'y'$-term, you can obtain a standard form by completing the square.

The following theorem identifies how much to rotate the axes to eliminate an xy-term and also the equations for determining the new coefficients A', C', D', E', and F'.

THEOREM A.1 Rotation of Axes

The general equation of the conic

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0,$$

where $B \neq 0$, can be rewritten as

$$A'(x')^2 + C'(y')^2 + D'x' + E'y' + F' = 0$$

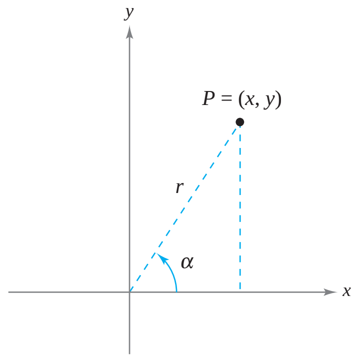
by rotating the coordinate axes through an angle θ, where

$$\cot 2\theta = \frac{A - C}{B}.$$

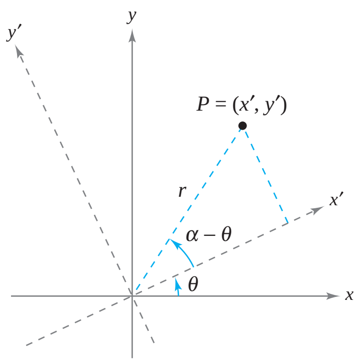
The coefficients of the new equation are obtained by making the substitutions

$$x = x' \cos \theta - y' \sin \theta$$

$$y = x' \sin \theta + y' \cos \theta.$$



Original: $x = r\cos\alpha$
$y = r\sin\alpha$



Rotated: $x' = r\cos(\alpha - \theta)$
$y' = r\sin(\alpha - \theta)$

Figure E.2

Proof To discover how the coordinates in the xy-system are related to the coordinates in the $x'y'$-system, choose a point $P = (x, y)$ in the original system and attempt to find its coordinates (x', y') in the rotated system. In either system, the distance r between the point P and the origin is the same, and thus the equations for x, y, x', and y' are those given in Figure E.2. Using the formulas for the sine and cosine of the difference of two angles, you obtain

$$\begin{aligned} x' = r\cos(\alpha - \theta) &= r(\cos\alpha\cos\theta + \sin\alpha\sin\theta) \\ &= r\cos\alpha\cos\theta + r\sin\alpha\sin\theta = x\cos\theta + y\sin\theta \\ y' = r\sin(\alpha - \theta) &= r(\sin\alpha\cos\theta - \cos\alpha\sin\theta) \\ &= r\sin\alpha\cos\theta - r\cos\alpha\sin\theta = y\cos\theta - x\sin\theta. \end{aligned}$$

Solving this system for x and y yields

$$x = x'\cos\theta - y'\sin\theta \qquad \text{and} \qquad y = x'\sin\theta + y'\cos\theta.$$

Finally, by substituting these values for x and y into the original equation and collecting terms, you obtain the following.

$$A' = A\cos^2\theta + B\cos\theta\sin\theta + C\sin^2\theta$$

$$C' = A\sin^2\theta - B\cos\theta\sin\theta + C\cos^2\theta$$

$$D' = D\cos\theta + E\sin\theta$$

$$E' = -D\sin\theta + E\cos\theta$$

$$F' = F$$

Now, in order to eliminate the $x'y'$-term, you must select θ such that $B' = 0$, as follows.

$$\begin{aligned} B' &= 2(C - A)\sin\theta\cos\theta + B(\cos^2\theta - \sin^2\theta) \\ &= (C - A)\sin 2\theta + B\cos 2\theta \\ &= B(\sin 2\theta)\left(\frac{C - A}{B} + \cot 2\theta\right) = 0, \quad \sin 2\theta \neq 0 \end{aligned}$$

If $B = 0$, no rotation is necessary, because the xy-term is not present in the original equation. If $B \neq 0$, the only way to make $B' = 0$ is to let

$$\cot 2\theta = \frac{A - C}{B}, \qquad B \neq 0.$$

Thus, you have established the desired results.

EXAMPLE 1 Rotation of a Hyperbola

Write the equation $xy - 1 = 0$ in standard form.

Solution Because $A = 0$, $B = 1$, and $C = 0$, you have (for $0 < \theta < \pi/2$)

$$\cot 2\theta = \frac{A - C}{B} = 0 \quad \Rightarrow \quad 2\theta = \frac{\pi}{2} \quad \Rightarrow \quad \theta = \frac{\pi}{4}.$$

The equation in the $x'y'$-system is obtained by making the following substitutions.

$$x = x' \cos\frac{\pi}{4} - y' \sin\frac{\pi}{4} = x'\left(\frac{\sqrt{2}}{2}\right) - y'\left(\frac{\sqrt{2}}{2}\right) = \frac{x' - y'}{\sqrt{2}}$$

$$y = x' \sin\frac{\pi}{4} + y' \cos\frac{\pi}{4} = x'\left(\frac{\sqrt{2}}{2}\right) + y'\left(\frac{\sqrt{2}}{2}\right) = \frac{x' + y'}{\sqrt{2}}$$

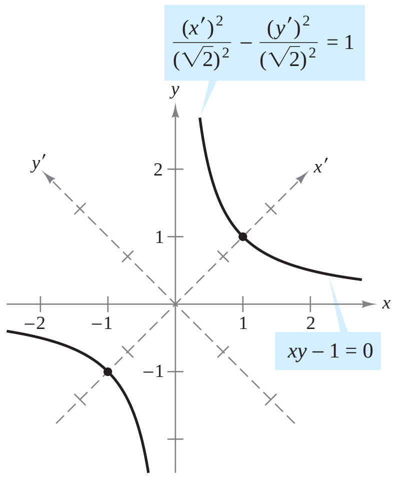
Substituting these expressions into the equation $xy - 1 = 0$ produces

$$\left(\frac{x' - y'}{\sqrt{2}}\right)\left(\frac{x' + y'}{\sqrt{2}}\right) - 1 = 0$$

$$\frac{(x')^2 - (y')^2}{2} - 1 = 0$$

$$\frac{(x')^2}{\left(\sqrt{2}\right)^2} - \frac{(y')^2}{\left(\sqrt{2}\right)^2} = 1. \qquad \text{Standard form}$$

This is the equation of a hyperbola centered at the origin with vertices at $\left(\pm\sqrt{2}, 0\right)$ in the $x'y'$-system, as shown in Figure E.3.



Vertices:
$\left(\sqrt{2}, 0\right), \left(-\sqrt{2}, 0\right)$ in $x'y'$-system
$(1, 1), (-1, -1)$ in xy-system
Figure E.3

EXAMPLE 2 Rotation of an Ellipse

Sketch the graph of $7x^2 - 6\sqrt{3}xy + 13y^2 - 16 = 0$.

Solution Because $A = 7$, $B = -6\sqrt{3}$, and $C = 13$, you have (for $0 < \theta < \pi/2$)

$$\cot 2\theta = \frac{A - C}{B} = \frac{7 - 13}{-6\sqrt{3}} = \frac{1}{\sqrt{3}} \quad \Rightarrow \quad \theta = \frac{\pi}{6}.$$

Therefore, the equation in the $x'y'$-system is derived by making the following substitutions.

$$x = x' \cos\frac{\pi}{6} - y' \sin\frac{\pi}{6} = x'\left(\frac{\sqrt{3}}{2}\right) - y'\left(\frac{1}{2}\right) = \frac{\sqrt{3}x' - y'}{2}$$

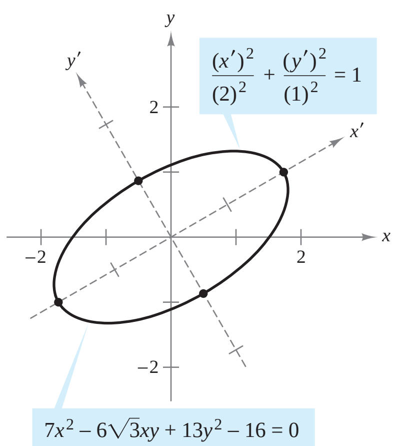
$$y = x' \sin\frac{\pi}{6} + y' \cos\frac{\pi}{6} = x'\left(\frac{1}{2}\right) + y'\left(\frac{\sqrt{3}}{2}\right) = \frac{x' + \sqrt{3}y'}{2}$$

Substituting these expressions into the original equation eventually simplifies (after considerable algebra) to

$$4(x')^2 + 16(y')^2 = 16$$

$$\frac{(x')^2}{(2)^2} + \frac{(y')^2}{(1)^2} = 1. \qquad \text{Standard form}$$

This is the equation of an ellipse centered at the origin with vertices at $(\pm 2, 0)$ and $(0, \pm 1)$ in the $x'y'$-system, as shown in Figure E.4.



Vertices:
$(\pm 2, 0), (0, \pm 1)$ in $x'y'$-system
$\left(\pm\sqrt{3}, \pm 1\right), \left(\pm\frac{1}{2}, \pm\frac{\sqrt{3}}{2}\right)$ in xy-system
Figure E.4

In writing Examples 1 and 2, we chose the equations such that θ would be one of the common angles 30°, 45°, and so forth. Of course, many second-degree equations do not yield such common solutions to the equation

$$\cot 2\theta = \frac{A - C}{B}.$$

Example 3 illustrates such a case.

EXAMPLE 3 Rotation of a Parabola

Sketch the graph of $x^2 - 4xy + 4y^2 + 5\sqrt{5}y + 1 = 0$.

Solution Because $A = 1$, $B = -4$, and $C = 4$, you have

$$\cot 2\theta = \frac{A - C}{B} = \frac{1 - 4}{-4} = \frac{3}{4}.$$

The trigonometric identity $\cot 2\theta = (\cot^2 \theta - 1)/(2 \cot \theta)$ produces

$$\cot 2\theta = \frac{3}{4} = \frac{\cot^2 \theta - 1}{2 \cot \theta}$$

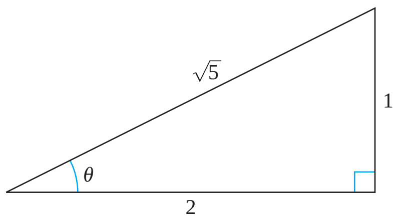


Figure E.5

from which you can obtain the equation

$$6 \cot \theta = 4 \cot^2 \theta - 4 \quad \Rightarrow \quad 4 \cot^2 \theta - 6 \cot \theta - 4 = 0$$
$$(2 \cot \theta - 4)(2 \cot \theta + 1) = 0.$$

Considering $0 < \theta < \pi/2$, it follows that $2 \cot \theta = 4$. Thus,

$$\cot \theta = 2 \quad \Rightarrow \quad \theta \approx 26.6°.$$

From the triangle in Figure E.5, you can obtain $\sin \theta = 1/\sqrt{5}$ and $\cos \theta = 2/\sqrt{5}$. Consequently, you can write the following.

$$x = x' \cos \theta - y' \sin \theta = x'\left(\frac{2}{\sqrt{5}}\right) - y'\left(\frac{1}{\sqrt{5}}\right) = \frac{2x' - y'}{\sqrt{5}}$$

$$y = x' \sin \theta + y' \cos \theta = x'\left(\frac{1}{\sqrt{5}}\right) + y'\left(\frac{2}{\sqrt{5}}\right) = \frac{x' + 2y'}{\sqrt{5}}$$

Substituting these expressions into the original equation produces

$$\left(\frac{2x' - y'}{\sqrt{5}}\right)^2 - 4\left(\frac{2x' - y'}{\sqrt{5}}\right)\left(\frac{x' + 2y'}{\sqrt{5}}\right) + 4\left(\frac{x' + 2y'}{\sqrt{5}}\right)^2 +$$
$$5\sqrt{5}\left(\frac{x' + 2y'}{\sqrt{5}}\right) + 1 = 0$$

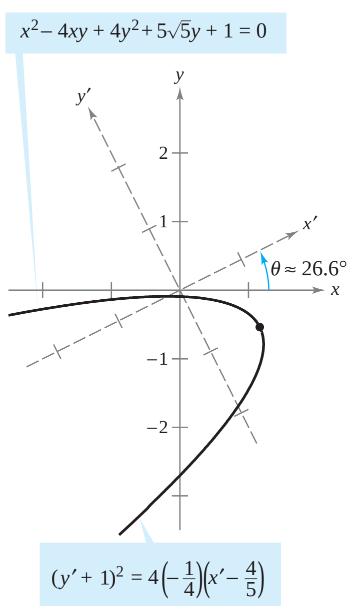
which simplifies to

$$5(y')^2 + 5x' + 10y' + 1 = 0.$$

By completing the square, you can obtain the standard form

$$5(y' + 1)^2 = -5x' + 4$$
$$(y' + 1)^2 = 4\left(-\frac{1}{4}\right)\left(x' - \frac{4}{5}\right). \quad \text{Standard form}$$

The graph of the equation is a parabola with its vertex at $\left(\frac{4}{5}, -1\right)$ and its axis parallel to the x'-axis in the $x'y'$-system, as shown in Figure E.6.



Vertex: $\left(\frac{4}{5}, -1\right)$ in $x'y'$-system

$\left(\frac{13}{5\sqrt{5}}, -\frac{6}{5\sqrt{5}}\right)$ in xy-system

Figure E.6

Invariants Under Rotation

In Theorem A.1, note that the constant term $F' = F$ is the same in both equations. Because of this, F is said to be **invariant under rotation.** Theorem A.2 lists some other rotation invariants. The proof of this theorem is left as an exercise (see Exercise 34).

THEOREM A.2 Rotation Invariants

The rotation of coordinate axes through an angle θ that transforms the equation $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$ into the form

$$A'(x')^2 + C'(y')^2 + D'x' + E'y' + F' = 0$$

has the following rotation invariants.

1. $F = F'$

2. $A + C = A' + C'$

3. $B^2 - 4AC = (B')^2 - 4A'C'$

You can use this theorem to classify the graph of a second-degree equation *with* an xy-term in much the same way you do for a second-degree equation *without* an xy-term. Note that because $B' = 0$, the invariant $B^2 - 4AC$ reduces to

$$B^2 - 4AC = -4A'C' \qquad \text{Discriminant}$$

which is called the **discriminant** of the equation

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0.$$

Because the sign of $A'C'$ determines the type of graph for the equation

$$A'(x')^2 + C'(y')^2 + D'x' + E'y' + F' = 0$$

the sign of $B^2 - 4AC$ must determine the type of graph for the original equation. This result is stated in Theorem A.3.

THEOREM A.3 Classification of Conics by the Discriminant

The graph of the equation

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$$

is, except in degenerate cases, determined by its discriminant as follows.

1. *Ellipse or circle* $\quad B^2 - 4AC < 0$

2. *Parabola* $\quad B^2 - 4AC = 0$

3. *Hyperbola* $\quad B^2 - 4AC > 0$

EXAMPLE 4 Using the Discriminant

Classify the graph of each of the following equations.

a. $4xy - 9 = 0$

b. $2x^2 - 3xy + 2y^2 - 2x = 0$

c. $x^2 - 6xy + 9y^2 - 2y + 1 = 0$

d. $3x^2 + 8xy + 4y^2 - 7 = 0$

Solution

a. The graph is a hyperbola because

$$B^2 - 4AC = 16 - 0 > 0.$$

b. The graph is a circle or an ellipse because

$$B^2 - 4AC = 9 - 16 < 0.$$

c. The graph is a parabola because

$$B^2 - 4AC = 36 - 36 = 0.$$

d. The graph is a hyperbola because

$$B^2 - 4AC = 64 - 48 > 0.$$

EXERCISES FOR APPENDIX E

In Exercises 1–12, rotate the axes to eliminate the xy-term. Give the resulting equation and sketch its graph showing both sets of axes.

1. $xy + 1 = 0$
2. $xy - 4 = 0$
3. $x^2 - 10xy + y^2 + 1 = 0$
4. $xy + x - 2y + 3 = 0$
5. $xy - 2y - 4x = 0$
6. $13x^2 + 6\sqrt{3}xy + 7y^2 - 16 = 0$
7. $5x^2 - 2xy + 5y^2 - 12 = 0$
8. $2x^2 - 3xy - 2y^2 + 10 = 0$
9. $3x^2 - 2\sqrt{3}xy + y^2 + 2x + 2\sqrt{3}y = 0$
10. $16x^2 - 24xy + 9y^2 - 60x - 80y + 100 = 0$
11. $9x^2 + 24xy + 16y^2 + 90x - 130y = 0$
12. $9x^2 + 24xy + 16y^2 + 80x - 60y = 0$

In Exercises 13–18, use a graphing utility to graph the conic. Determine the angle θ through which the axes are rotated. Explain how you used the utility to obtain the graph.

13. $x^2 + xy + y^2 = 10$
14. $x^2 - 4xy + 2y^2 = 6$
15. $17x^2 + 32xy - 7y^2 = 75$
16. $40x^2 + 36xy + 25y^2 = 52$
17. $32x^2 + 50xy + 7y^2 = 52$
18. $4x^2 - 12xy + 9y^2 + \left(4\sqrt{13} + 12\right)x - \left(6\sqrt{13} + 8\right)y = 91$

In Exercises 19–26, use the discriminant to determine whether the graph of the equation is a parabola, an ellipse, or a hyperbola.

19. $16x^2 - 24xy + 9y^2 - 30x - 40y = 0$
20. $x^2 - 4xy - 2y^2 - 6 = 0$
21. $13x^2 - 8xy + 7y^2 - 45 = 0$
22. $2x^2 + 4xy + 5y^2 + 3x - 4y - 20 = 0$
23. $x^2 - 6xy - 5y^2 + 4x - 22 = 0$
24. $36x^2 - 60xy + 25y^2 + 9y = 0$
25. $x^2 + 4xy + 4y^2 - 5x - y - 3 = 0$
26. $x^2 + xy + 4y^2 + x + y - 4 = 0$

In Exercises 27–32, sketch the graph (if possible) of the degenerate conic.

27. $y^2 - 4x^2 = 0$
28. $x^2 + y^2 - 2x + 6y + 10 = 0$
29. $x^2 + 2xy + y^2 - 1 = 0$
30. $x^2 - 10xy + y^2 = 0$
31. $(x - 2y + 1)(x + 2y - 3) = 0$
32. $(2x + y - 3)^2 = 0$

33. Show that the equation $x^2 + y^2 = r^2$ is invariant under rotation of axes.
34. Prove Theorem A.2.

The symbol indicates an exercise in which you are instructed to use graphing technology or a symbolic computer algebra system. The solutions of other exercises may also be facilitated by use of appropriate technology.